LONGMANS' SENIOR ARITHMETIC

T. F. G. DEXTER
AND
A. H. GARLICK

LONGMANS' SENIOR ARITHMETIC

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LONDON, NEW YORK, BOMBAY, AND CALCUTTA

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FOR

SCHOOLS AND COLLEGES

BY

T. F. G. DEXTER, B.A., B.Sc.

AND

A. H GARLICK, B.A.

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PREFACE

As this work is meant to be a Senior Arithmetic, the elementary portions of the subject have been dealt with briefly, but it is believed sufficiently.

The Exercises are formed almost exclusively from Examination Questions, and include examples from about thirty different sets of examination papers. By this arrangement both variety and quality have been secured. The Exercises have also been carefully graded, the numerous examples contained in them being grouped into classes. Should a student be weak in any particular type of problem, the teacher will thus be able to put his hand upon a group of such problems for instruction and practice. Experience has shown the value of this.

Sound teaching demands a knowledge of principles as well as practice; of the science, as well as the art of arithmetic. For this reason the theory portion of the subject has been treated more fully than usual, and nearly every set of Exercises is preceded by some questions on theory. As far as possible, these questions should be worked before the Exercises are attempted. The student should then be able to approach the Exercises intelligently.

The aid of algebra and geometry has not been eschewed. Both subjects are necessary auxiliaries if arithmetic is to be progressive.

The book is more than usually comprehensive in its range. A glance at its "Contents" will reveal this. The Test Papers at the end are for Class Test or Examination purposes. The Miscellaneous Examples in the last exercise are for a final revise. The student who can work them need fear little in the way of ordinary arithmetic.

No oral exercises have been given. After some experience, we are somewhat sceptical as to the value of *printed* oral exercises, especially when answers are given. Oral exercises, to be of any value, should be prepared by the teacher and presented fresh to the class. The teacher then has some assurance that other faculties of the mind are working besides the memory.

The book is the result of a combined and varied experience extending over many years, and it is hoped that the methods employed will be found clear, up to date, and largely self-explanatory.

T. F. G. D.
A. H. G.

April, 1904.

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LIST OF EXAMINATION PAPERS FROM WHICH EXAMPLES ARE TAKEN

C.P.	College of Preceptors.
C.M.	Teachers' Certificate (Men).
C.M.S.	„ „ „ Scotland.
C.S.	Civil Service.
C.W.	Teachers' Certificate (Women).
C.W.S.	„ „ „ Scotland.
C.PR.	Cambridge Previous.
C.J.	Cambridge Junior Local.
C.S.L.	Cambridge Senior Local.
C.H.L.	Cambridge Higher Local.
I.I.	Irish Intermediate.
K.S.	King's Scholarship.
K.S.S.	„ „ Scotland.
L.M.	London Matriculation.
O.J.	Oxford Junior Local.
O.S.L.	Oxford Senior Local.
O. & C.S.C.	Oxford and Cambridge School Certificate.
O.R.	Oxford Responsions.
P.P.	Professional Preliminary.
P.E.A.	Preliminary Entrance to Army.
P.T.	Pupil Teacher.
R.M.A.	Royal Military Academy.
R.M.C.	Royal Military College.
R.N.	Royal Navy.
R.U.I.	Royal University of Ireland.
S.K.	South Kensington.
S.A.	Society of Arts.

ARITHMETIC.

CHAPTER I.

NUMERATION, NOTATION, THE FOUR SIMPLE RULES.

Arithmetic is both a science and an art. It is a science because it investigates the properties of numbers and establishes general principles based on those investigations. It is an art insomuch as it deals with various methods of using numbers. Viewed as a science, arithmetic is a branch of mathematics; regarded as an art, arithmetic carries out for practical purposes certain *rules* respecting numbers, paying, however, little or no regard to the *principles* upon which those rules are based.

Algebra differs from arithmetic inasmuch as it deals with *symbols* (which, however, often stand for numbers), whereas arithmetic operates on numbers themselves.

The basis of all calculation in arithmetic is the unit: 1, one.

A Unit is (*a*) a single thing, as *one* book; *or* (*b*) a whole class of things, as *one* class.

A Number is composed of one or more units, as—

7, 17, 170

A Digit is the name given to any one of the signs—

1, 2, 3, 4, 5, 6, 7, 8, 9.

1. Some authorities assert that 0 is not a digit, others include it among the digits, and call the nine digits given above *significant* digits.

2. 0 is variously called nought, naught, cipher, or zero; it should never be called "ought" or "aught."

3. Ten is not a digit.

4. The term *figure* is frequently used for the term *digit*. The

latter is preferable. A triangle is a figure, but not a *digit*. Long usage, however, sanctions the use of digit and figure as interchangeable in arithmetic.

An Abstract Number is a number which does not indicate the nature of the unit. It is a number which stands alone, and merely denotes *repetition* of the unit, and nothing more. 7, 17, 170 are abstract numbers.

A Concrete Number is a number which indicates the nature of the unit. It is used in conjunction with the name of some "thing;" as 7 horses, 17 dogs.

1. It would be better to use the phrase *concrete expression*, or *concrete term*. Strictly speaking, a *number* can never be concrete.

2. Note that a child has—

First, (*a*) *concrete ideas;* as brother, sister.

Next, (*b*) *composite ideas;* as 2 brothers, 3 sisters.

And lastly, (*c*) *abstract ideas;* as 2, 3.

Hence in the teaching of arithmetic, "concrete arithmetic" should precede "abstract arithmetic."

Notation is the art of expressing numbers by means of digits or figures. It concerns itself with the *writing* of figures.

Numeration is the art of expressing in *words* numbers already expressed in figures. It concerns itself with the *reading* of figures.

Connect *numeration* with *enumerate; notation* with *note-taking.*

The system of notation universally employed among civilized mankind is the Decimal System.

1. The decimal system is based on human anatomy. Man learned to count by tens because his earliest efforts at enumeration were made by counting on his fingers (hence the term digit).

2. There are traces of other systems:

(*a*) *The vigesimal*, i.e. one with 20 as a "base," due probably to the toes as well as the fingers being called into requisition. Cf. English, *score;* Latin, *duodeviginti;* French, *quatre vingts.*

(*b*) *The duodecimal*, i.e. one with 12 as a "base." Cf. English, *dozen, gross.*

The number of units in any order or denomination which is taken to form a unit of the next higher order is called the "base" of the system.

The root-words which we have to express numbers are very few.

1. *Those expressing numbers of the first order:* one, two, three, four, five, six, seven, eight, nine.

2. *Those expressing numbers of higher order:* ten, hundred, thousand, million.

All others are derivative.

The question now arises, How is it that with so few root-words we can express numbers up to any amount?

The answer can be given very simply: Because every digit has two kinds of value—

1. Its *intrinsic* value, *i.e.* its real or absolute value.

2. Its *local* value, *i.e.* its relative value due to the potentiality of its position.

H.	T.	U.
		7
	7	
7		

7 moved one place to the left becomes 7 *tens;* moved again one place to the left becomes 7 *hundreds.* The *intrinsic* value of the digit is 7 units; owing to the potentiality of its position, it assumes a *local* value 10 or 100 times greater than its intrinsic value.

The invention of the cipher made "place-value" easy, accurate, and reliable. Just as the *wheel* may be regarded as the greatest invention in mechanics, so the *nought* may be regarded as the greatest invention in mathematics. The inventors of both are quite unknown.

Carefully note the following:—

1. Writing 0 at the *beginning* (or left) of a number makes no difference to its value.

2. Writing 0 anywhere in the *middle* of a number changes it completely.

3. Writing 0 at the *end* (or right) of a number increases the value of every digit tenfold.

The nine digits, which are written in the first place on the right, are called *Units of the First Order;* the units denoting tens, written in the second place to the left, are called *Units of the Second Order;* similarly, "hundreds" are called *Units of the Third Order;* "thousands," *Units of the Fourth Order*, etc.

There are two systems of notation.

1. **The "Long" System,** used in this country, in which sometimes three and sometimes six digits constitute a "group." The digits as far as millions are marked off in sets of three; after millions they are marked off in sets of sixes: hence the term "long."

Thus—

Billions.	Millions.	Thousands.	Units.
123456	789123	456	789

would be read as one hundred and twenty-three thousand four hundred and fifty-six billions, seven hundred and eighty-nine thousand one hundred and twenty-three millions, four hundred and fifty-six thousand, seven hundred and eighty-nine.

2. **The "Short" System,** used on the Continent and in the United States, in which *three* digits invariably constitute a group.

The method of reading numbers in this system is shown below, and should be contrasted with that given above.

Quadrillions.	Trillions.	Billions.	Millions.	Thousands.	Units.
123	456	789	123	456	789

In the Long System a billion = a *million* millions.
In the Short System a billion = a *thousand* millions.
The Short System is symmetrical and preferable to the Long System, which is gradually falling into disuse.

Since we rarely wish to express numbers above a million, and since both systems agree to that point, the differences between the two systems are of academic rather than practical importance.

Preliminary Remarks on the Decomposition and Resolution of Numbers.

A number like 123 may be decomposed or resolved in several ways, some of which are given below—

H.	T.	U.
	12	3
		123
1	1	13
1	0	23

Some Remarks on the Four Simple Rules.

Simple Addition is the operation of finding one number which shall be exactly equal to two or more numbers taken together.

The numbers must be of the same "order;" *e.g.* we cannot at once add together 7 tens and 27 units. We can decompose the 7 tens into 70 units, and, adding these to 27 units, obtain 97 units.

Or we can resolve the 27 units into 2 tens and 7 units, and, adding these to the 7 tens, obtain 9 tens and 7 units.

Terms and Signs involved in Addition.—*The addends* are the quantities to be added. *The sum, sum total*, or *amount* is the result obtained by addition.

The sign +, called "plus," is the sign of addition.

"Carrying" in addition.

H.	T.	U.
7	2	9
	3	2
4	6	5
12	2	6

The sum of the units' column is 16. The old formula, "Put down the right-hand figure and 'carry' the left," is meaningless. The true explanation is as follows: Resolve 16 units into 1 ten and 6 units. Place the 6 units in its proper place, viz. the units' column (as part of the sum total), and the 1 ten in *its* proper place, viz. the tens' column, ready to be reckoned with the other addends.

The sum of the tens' column is 12 tens. Resolve into 1 hundred and 2 tens. Place the 2 tens in the tens' column (as part of the sum total), and the 1 hundred as an addend in the hundreds' column, and so on.

Subtraction is the operation of finding what number is left when a smaller number is taken away from a greater of the same kind.

Terms and Signs employed.—The greater number is called the *minuend*, the smaller the *subtrahend*. The excess of the minuend over the subtrahend is called the *difference*, or *remainder*.

The sign of the operation is −, called the *minus* sign, so that 9 − 7 means 9 less (or diminished by) 7.

The sign $\frown$ denotes that the smaller of the quantities is to be taken from the larger, as $a \frown b$, when the relative values of a and b are not known.

There are three recognized methods of working a subtraction sum.

1. The Method of Decomposition.

Example.—757 − 289.

6	14	17
~~7~~	~~5~~	~~7~~
2	8	9
4	6	8

9 cannot be taken from 7. Decompose 57 into 4 tens and 17 units. 9 units from 17 units leave 8 units.

8 tens cannot be taken from 4 tens. Decompose 7 hundreds and 4 tens into 6 hundreds and 14 tens. 8 tens from 14 tens leave 6 tens. 2 hundreds from 6 hundreds leave 4 hundreds.

Or we may write thus—

$$\begin{array}{lll} 757 = 700 + 50 + 7 = 600 + 140 + 17 \\ 289 = 200 + 80 + 9 = 200 + 80 + 9 \\ \hline 468 = & & 400 + 60 + 8 \end{array}$$

Note that the minuend only is decomposed.

2. The Method of Equal Additions depends on the following truth: The difference between two unequal quantities is unaltered if equal quantities are added to each.

Example.—757 − 289.

7	5	7
2	8	9
4	6	8

9 cannot be taken from 7. Add 10 *as ten units* to the 7 in the minuend, and 10 *as* 1 *ten* to the 8 tens in the subtrahend. 9 units from 17 units leave 8 units. 9 tens cannot be taken from 5 tens. Add 100 as 10 tens to the minuend and as 1 hundred to the subtrahend. 9 tens from 15 tens leave 6 tens. 3 hundreds from 7 hundreds leave 4 hundreds.

Or we may represent the method thus—

$$\begin{array}{lll} 757 = 700 + 50 + 7 = 700 + 150 + 17 \text{ (110 added)} \\ 289 = 200 + 80 + 9 = 300 + 90 + 9 \text{ (110 added)} \\ \hline 468 = & & 400 + 60 + 8 \end{array}$$

The process of "borrowing and paying back" is a perverted form of the method of equal additions. The ten "borrowed" from the tens' column is really ten *added* to the minuend, and the one "paid back" to the tens' column in the subtrahend is really one ten *added* to the subtrahend.

3. **The Method of Complementary Additions.**—"The difference between a number and a unit of the next higher order of that number is called the complement of that number."

Thus 3 is the complement of 7, 28 is the complement of 72, etc.

The essential difference between "Ordinary" subtraction and the method of complementary additions may be illustrated by the following: Suppose we have to take 3 from 7. Working by ordinary subtraction, we say, 3 from 7 leaves 4. Working by complementary additions we say, "What number added to 3 will give 7?" The answer is manifestly 4.

Suppose that I go into a shop, purchase an article worth 9*d.*, and present a shilling. If the shopman in giving the change counts thus: "ten pence, eleven pence, one shilling," he is performing his subtraction by the method of complementary additions.

In working subtraction by the method of complementary additions, the minuend is regarded as the result, and the object is to find a complementary number which, when added to the subtrahend, will exactly equal the minuend.

Example.—874 − 689.

874
689
185

9 is greater than 4. The unit of the next higher order is *one* ten or 10 units; adding this to the 4, we get 14. 9 and 5 (the complement) make 14. The 1 (representing one ten) from the 14 is "carried" forward to the tens' column, just as in addition.

8 and 1 are 9. 9 tens are greater than 7 tens. The unit of the next higher order is *one* hundred or 10 tens. Adding these 10 tens to 7 tens, we get 17 tens. 9 tens and 8 tens (the complement) make 17 tens. The 1 (representing one hundred) is "carried" forward to the hundreds' column, just as in addition. 6 hundreds and 1 hundred are 7 hundreds. 7 hundreds and *one* hundred (the complement) make 8 hundreds.

This method is particularly useful in working long division by the Italian method.

Multiplication is the process of finding the sum of a given number of repetitions of the same number.

The sign of multiplication is given thus: ×.

The multiplier is the number which expresses how often the repetition is to take place.

The multiplicand is the number to be repeated or multiplied.

The product is the result (or sum) of the repetitions. Thus, in the statement 27 × 7 = 189, 27 is the multiplicand, 7 the multiplier, and 189 the product.

Continued Product.—If the product of two numbers is multiplied by a third number, the result is the continued product of the three numbers.

"**Carrying**" **in multiplication**, like "carrying" in addition, resolves itself into a case of resolution of numbers.

Example.—35 × 4.

T.	U.
3	5
	4
14	0

4 times 5 units are 20 units. Some say, "Put down the 0 and 'carry' the 2." A more correct explanation is, Resolve 20 units into 2 tens and 0 units. Put the 0 in its proper place, viz. the units' column, and place the 2 in *its* proper place, viz. the tens' column, ready for the next operation.

4 times 3 tens are 12 tens, which, with the 2 tens obtained above, make 14 tens, which are resolved into 1 hundred and 4 tens.

The omission of the ciphers at the right of the partial products in Multiplication.

Example.—2736 × 678.

2 7 3 6
6 7 8

2 1 8 8 8 = 8 times 2736
1 9 1 5 2 [0] = 70 „ „
1 6 4 1 6[0][0] = 600 „ „

1 8 5 5 0 0 8 = 678 „ „

The noughts enclosed in brackets are always omitted, because they make no difference to the value of the product. Care must, however, be taken that the first right-hand digit of each partial product is placed in its proper position, viz. under the digit of the multiplier by which it is obtained. The reason for this is obvious. In the above example the 2 in the second line is placed under the 7 because the 7 is 7 tens, and 6 times 7 tens produces 42 tens, resolvable into 4 hundreds and 2 tens, the 2 tens being placed in their appropriate place, the tens' column.

Division may be defined as the process—

1. Of finding how many times one quantity is contained in another; *or*

2. Of finding one of the factors of a given product when the other factor is given; *or*

3. Of separating a given number into a certain number of equal parts.

Signs. $-$ or $\div$; as $\frac{6}{2} = 3$, or $6 \div 2 = 3$.

Terms. Taking definition (2) given above—

(*a*) The given factor is termed the *divisor.*

(*b*) The required factor is termed the *quotient.*

(*c*) The product is termed the *dividend.*

The difference between the dividend and the nearest product of the divisor less than the dividend is called the *remainder.*

The *divisor* is that which divides.

The *dividend* is that which must be divided.

The *quotient* signifies how many times the divisor is contained in the dividend.

Examine the following cases:—

(*a*) $\frac{20}{5} = 4$, or $\frac{\text{abstract number}}{\text{abstract number}} = \text{abstract number.}$

(*b*) $\frac{20 \text{ feet}}{5} = 4 \text{ feet}$, or $\frac{\text{concrete quantity}}{\text{abstract number}} = \text{concrete quantity.}$

(*c*) $\frac{20 \text{ feet}}{5 \text{ feet}} = 4$, or $\frac{\text{concrete quantity}}{\text{concrete quantity}} = \text{abstract number.}$

(*d*) $\frac{20}{5 \text{ feet}}$ gives no result, or $\frac{\text{abstract number}}{\text{concrete quantity}}$ gives no result.

From these examples we learn that the divisor may be—

1. An abstract number, in which case the quotient is the same name as the dividend; *e.g.* (*a*) and (*b*) above.

2. A concrete quantity of the same kind as the dividend, in which case the quotient is abstract: *e.g.* (*c*) above.

There are really **two kinds of division**—

1. *Quotition*, or arithmetical division, in which both divisor and dividend are of the same name; *e.g.* (*a*) and (*c*) above.

2. *Partition*, or physical division, in which the dividend is concrete, the divisor abstract, and the quotient concrete: *e.g.* (*b*) above.

In the definitions of division given on p. 7: (1) and (2) are definitions of quotition, (3) is a definition of partition.

Explanation of the ordinary method of working Division.

Example.—87564 ÷ 23.

```
23)87564(3000
  69000   800
  -----     7
  18564
  18400
  -----
    164
    161
    ---
      3
```

23 into 87000 "goes" 3000 times.
Multiply by 3000 and subtract.
23 into 18500 "goes" 800 times; and so on for the rest of the sum.

Answer.—3807 and 3 remainder; or $3807\frac{3}{23}$.

Never write: Answer 3807 + 3, for 3807 + 3 = 3810.

The Italian or contracted method of working Division.*

Example.—87564 ÷ 23.

```
23)87564(3807
  ----
   185
   ---
    164
    ---
      3
```

Explanation.—
3 times 3 = 9; 9 + 8 (put down) = 17. Carry 1.
3 times 2 = 6; 6 + 1 (carried) = 7.
7 + 1 (put down) = 8. Bring down 5.
8 × 3 = 24; 24 + 1 (put down) = 25. Carry 2.
8 × 2 = 16; 16 + 2 (carried) = 18. Bring down 6; and so on till end of sum.

The "Slip" method given on the next page is quite as short as the Italian method, and not liable to its defects.

* In working division the contracted method should always be followed.

(*a*) Write out a table of the multiples of the divisor on a long *narrow* strip of paper. Taking 23 (as above) as the divisor, we get—

1·23	2·46	3·69	4·92	5·115	6·138	7·161	8·184	9·207

(*b*) Place the slip at the right point under the dividend and subtract. There is no need to write the partial products, and, as in the Italian method, the results of subtraction only are shown.

Multiplication and Division by special methods.

To multiply by 5. Add 0 and divide by 2.

To multiply by 25. Add two 0's and divide by 4.

To multiply by 125. Add three 0's and divide by 8.

The reasons for these processes are based on the facts that 10 is twice 5; 100 is four times 25; 1000, eight times 125.

To divide by 5, 25, 125. Multiply by 2, 4, 8, and divide the product by 10, 100, 1000 respectively.

To multiply by 9, 99, 999, etc. Annex to the multiplicand as many ciphers as there are 9's, and from the result subtract the multiplicand.

By adding one, two, three . . . 0's to the multiplicand, we multiply by 10, 100, 1000, . . . By subtracting the multiplicand, we have 9 (10 − 1), or 99 (100 − 1), or 999 (1000 − 1) . . . times the multiplicand.

To divide by 9, 99, 999, etc.

(*a*) Cut off by a vertical line as many digits from the right hand of the dividend as there are 9's in the divisor.

(*b*) Take the digits to the left of the "cutting-off" line, and cut off as before as many digits as there are 9's in the divisor.

(*c*) Proceed so long as the number of digits in the dividend admits.

(*d*) Add the several lines together. The sum of the digits to the left of the "cutting-off" line represents the quotient; the sum of the digits to the right, the remainder.

(*e*) If in the addition a digit is carried from the remainder to the quotient, that digit must be added to the remainder. (The digit carried represents 10, or a certain number of 10's; these will contain 9, or a certain number of 9's, with one or more remaining.)

Example.—Divide 123456789 by 999.

123456	789
123	456
	123
123580	368

One was carried from remainder to quotient; hence answer is 123580 and 369 remainder.

Multiplication may sometimes be contracted by separating the multiplier into numbers, each of which is produced by multiplication from the lowest number.

Example.—Multiply 45989 *by* 864729 *in three lines.* (C.PR.)

We notice that the multiplier $= 9 + 720 + 864000 = 9 + (9 \times 80) + (1200 \times 720)$.

Consequently we may arrange in three lines, thus—

45989
9

413901 = 9 times 45989
33112080 = 80 times 413901, and ∴ 720 ,, 45989
39734496000 = 1200 ,, 33112080, and ∴ 864000 ,, 45989
39768021981 = 864729 ,, 45989

For division by factors the student is referred to Chapter III.

To find the difference of two squares.

Example.—Solve $(3794)^2 - (3754)^2$.
Let $3794 = a$, and $3754 = b$;

Then $(3794)^2 - (3754)^2 = a^2 - b^2 = (a + b)(a - b)$

Substituting, we have—

$$(3794 + 3754)(3794 - 3754) = (7548)(40) = 301920. \quad Ans.$$

From this we learn that the difference between the squares of two numbers is equal to the product of the sum and difference of those numbers.

Tests of accuracy.

The only methods requiring explanation are those of "casting out the 9's" and "casting out the 11's."

Casting out the Nines.

The process of "casting out the 9's" from a number is as follows:—

(*a*) Cross out all the 9's.

(*b*) Proceed to add up the remaining digits.

(*c*) Whenever the sum of two or more digits exceeds 9, subtract 9, and add the excess to the next digit.

(*d*) Proceed in this way till a single digit, called the remainder, is left.

Example.—Cast out the 9's from the multiplicand given on p. 9.

45989 (*a*) Cross out the 9's.
(*b*) 4 and 5 are 9. Cast out the 9.
(*c*) The remaining digit (8) is the remainder.

Application of the process to the proof of—

1. *Addition.*

(*a*) Cast out the 9's from all the addends.

(*b*) Cast out the 9's from the sum.

(*c*) If the remainder obtained from (*a*) is equal to that obtained from (*b*), the answer is probably correct.

Example.—Prove the accuracy of the addition in the multiplication sum given on p. 9.

(*a*) Casting out the 9's from the three addends, we obtain a remainder of 0.
(*b*) Casting out the 9's from the sum, we also obtain a remainder of 0.
(*c*) $0 = 0$, ∴ the addition is probably correct.

2. *Multiplication.*

(*a*) Cast out the 9's from the multiplicand, the multiplier, and the product. Three remainders will thus be obtained.

(*b*) Multiply the first two remainders.

(*c*) If necessary, cast out the 9's again and obtain another (a *fourth*) remainder.

(*d*) If this fourth remainder (or the product of the two remainders in (*b*)) is equal to the third remainder, the answer is probably correct.

Example.—Prove the accuracy of the multiplication on p. 9 by casting out the 9's.

(*a*) Remainders from multiplicand, multiplier, and product are respectively 8, 0, and 0.

(*b*) Multiply the first two remainders : $8 \times 0 = 0$.

(*c*) It is unnecessary to again cast out the 9's.

(*d*) The product of the two remainders (0) is equal to the third remainder (0). Hence the answer is probably correct.

The student will notice that the phrase, "the answer is *probably* correct," has appeared at the end of each demonstration in the last two examples.

Casting out the 9's will not detect an error—

(1) If the error happens to be a multiple of 9 ; *e.g.* if in a multiplication one of the partial products is put in the wrong place, for then the error is a multiple of 9.

(2) If the right digits have been used, but have been put in the wrong place ; *e.g.* if 72 has been written for 27.

Casting out the Elevens.—The process is as follows :—

(*a*) Commence at the units' digit, and add together the *alternate* digits until the extreme left-hand digit is reached.

(*b*) Return from left to right, add together the alternate digits, and subtract their sum from that obtained in (*a*).

(*c*) If the difference is less than 11, it is the difference required. If the difference is greater than 11, subtract 11's until a remainder less than 11 is reached. This remainder will be the one required.

The application of this method to addition and multiplication is exactly the same as that employed in casting out the 9's.

"Casting out the 11's" will not detect a mistake if the error amounts to 11 or a multiple of 11, *e.g.* if one of the partial products has been displaced *two* places ; for then the error is a multiple of 11.

For further explanation of these two methods, see the chapter on Factors.

NUMERATION, NOTATION, THE FOUR SIMPLE RULES.

(THEORY.)

1. Explain the ordinary system of notation. (C.PR.)

2. Define numeration. Distinguish between the *intrinsic* and *local* value of numbers. (C.W.)

3. Explain *local value* by the number 333, showing that the figure 3 has a different value according to its position. (C.P.)

4. What is notation? Explain what is meant by saying that "we employ a decimal system of notation." What other systems have been used? Compare the value of each of the three eights in the product of nine thousand and nine by nine thousand and ninety. (C.W.)

5. Distinguish numeration and notation. (C.M.)

6. What do numeration and notation respectively teach? Describe what is called the decimal notation. Express in words 2051600390. (C.P.)

7. Distinguish between abstract and concrete numbers. (P.T.) (K.S.)

8. What is the use of a cipher in the notation of numbers? (C.W.)

9. Write down the least possible number consisting of three digits.

10. What does the figure 6 represent? (C.W.)

11. What is meant by a number of the "second order"?

12. What is meant by a *figure*? what by a *number*? Can the same number be represented by different figures? If so, give examples. (S.A.)

13. What is meant by an *odd* number and an *even* number? Show

that it is only necessary to look at the digit in the units' place to ascertain if a number is odd or even. Several numbers have to be added together. What is the condition that their sum should be odd? (R.M.A.)

14. In what different ways can a child add 8 to 7? Which apply to adding 8 to 27? (C.M.)

15. Explain why the words "borrowing" and "carrying" are often misused in subtraction, and show, by the help of an illustration, what is the best way of making the meaning of that rule intelligible. (K.S.)

16. Define subtraction, and explain the process, taking for your example six millions four hundred and twelve, less by seventy thousand eight hundred and twenty-seven. (C.W.)

17. Describe two ways of making a sum in simple subtraction, and explain each method fully. (P.T.)

18. In subtraction how do you evade the difficulty of taking a greater digit from a less? Show how subtraction may be performed without "carrying." (I.I.)

19. Subtract 3948 from 6075, and explain each step clearly by the method of *equal addition*, or by that of *complementary addition*. (K.S.)

20. Explain the principle of the method of equal additions in subtraction. Apply this method in finding the difference between 201 and 186. (C.W.)

21. Prove that 4 times 5 equals 5 times 4. (C.W.)

22. How would you show that $5 \times 11 = 11 \times 5 = 55$? (C.W.)

23. How is it proved that 7 times 7 are 49, 7 times 8 are 56, 8 times 8 are 64? How do these results help us to obtain 78 times 87? State also what else we need to know in order to obtain this product. (S.A.)

24. Show that the continued product of any number of integers is independent of the order in which they are multiplied. (C.H.L.)

25. Prove that the product of any two numbers is the same as the sum of the products of the multiplicand and any two numbers which together make up the multiplier. (C.W.)

26. How would you show the truth of the following theorem: The product of any two numbers is the same as the sum of the products of the multiplier and any two or more numbers which together make up the multiplicand? (C.M.)

27. Explain as you would to a *blind* child, who knew the multiplication table, but nothing of *figures*, how to reckon 7 times 34. (C.M.)

28. Distinguish between the two meanings (*measuring* and *sharing*) of the word "division." (C.W.)

29. Give every meaning of *multiply* with which you are acquainted, with an example of each. (S.A.)

30. Explain the statement that short division is a contracted form of long division. (K.S.)

31. In division, if the quotient be added to the dividend for a new dividend, and the divisor be increased by 1, show from first principles that the quotient and remainder will be the same as before. Thence obtain the quotient and remainder when 678912 is divided by 601 from the quotient and remainder when the same dividend is divided by 600. (C.W.)

32. Explain the process called long division, taking as an example 102741 divided by 29. (C.W.)

33. Explain why a whole number is multiplied by 100 when you place two ciphers to the right of the figures which express it. State and prove the truth of the corresponding rule for dividing by 100. (P.E.A.)

34. Explain and correct the error in the statement "$66 \div 9 = 7 + 3$." (C.W.)

35. In what two senses do we use the word "divide" when we say "divide £100 by 4." and "divide £100 by £4"? (P.T.)

36. Divide 9173 by 39, and show, in the course of the working, as to a class of young children, how to find (*a*) The remainder after 39 has been subtracted from the dividend 200 times; (*b*) how many times 39 has been subtracted when the remainder is 203; (*c*) the product of 235 and 39. (K.S.)

37. (*a*) Multiply 368 by 40038 (*not* vice versâ); (*b*) in working the above sum, what ciphers do you omit without much risk of error if you begin by multiplying by the units figure? (*c*) in the least laborious way you can, divide (1) the actually largest partial product by the actually smallest, and (2) the complete product by 20019. (K.S.)

38. Take any number and multiply it by the two numbers next above it; take any other number and do the same to it; subtract one result from the other, and divide the difference obtained by the difference of the original numbers; add to the quotient the product of the original numbers; then the final result will be the product of some number by the number next above it. Prove this statement. (L.M.)

39. Ten numbers, each consisting of five figures, are to be (1) added together, (2) multiplied together. Explain carefully in each case what is the greatest possible number of figures in the result, and what the least number. (C.S.)

40. If to any whole number another number 1000 times its value be added, show that the result is always divisible by 143.

41. Write down the nine numbers which immediately precede 81 and the nine which immediately follow, and show that their sum is exactly divisible by 9. Would the sum still be divisible by 9, if instead of a group of nine numbers on either side of 81, a group of five, six, or any other amount were chosen? Give your reasons. (R.M.A.)

42. If from any number, *e.g.* 756860, any other number with the same digits in different order, *e.g.* 506879, be subtracted, the remainder will be exactly divisible by 9. Explain this. (C.M.)

43. Explain how the accuracy of arithmetical operations may be tested by the method of "casting out 9's." On what property of numbers does this method depend? Prove that if any number is divisible by 99, the number formed by the same figures in the reverse order is also divisible by 99. (C.S.)

44. Give instances of names of numbers in English not in accordance with the decimal system.

45. Is it possible for two numbers to have 100 for their sum and 3000 for their product? (L.M.)

46. Insert a cipher in 2345678, somewhere between the 2 and the 8—first, so as to make the greatest possible difference in the number; next, so as to make the least possible difference. (C.W.)

47. If we verify a process, say in multiplication or division, first by casting out the 9's, and next by casting out the 11's, and find by these tests that the work appears correct, it may yet be wrong, but the error can only be a multiple of 99. Prove this. (I.I.)

48. An addition sum is incorrectly performed, but the tests by casting out 9's and 11's both fail to detect the error. Show that the difference between the answer obtained and the correct answer must be a common multiple of 9 and 11. (C.S.)

EXERCISE I.

The Four Simple Rules.

1. Find the difference between five millions and twenty-two and three hundred and ninety-seven thousand and forty-nine. Express the result in words. (C.PR.)

2. Write down in figures ten millions and two and forty thousand and three. Express their difference in words. (C.PR.)

3. Taking the population of the German Empire at 45234061, if 28318592 are registered as of the Established Church, 124567 as members of other Protestant denominations, 561612 as Jews, and the remainder as Roman Catholics, how many Roman Catholics are there? (C.P.)

4. Add together two hundred and eight millions sixteen thousand and seventy, and forty millions nine hundred and five thousand four hundred, and divide the result by five hundred and eight thousand and three. (C.PR.)

5. Find the sum of all the numbers between 300 and 500 which are divisible by 23. (K.S.)

6. Multiply one million one thousand and one by one hundred thousand one hundred, and express the result in words. (R.M.A.)

7. Find the product of the sum and difference of two hundred and thirty-eight thousand four hundred and twenty-eight, and one hundred and five thousand and sixty-nine. (C.S.L.)

8. If 109 be multiplied by a certain number it is increased by 2071. Find the multiplier. (C.S.)

9. Multiply nine million nine thousand and nine by eleven, and write down in words the difference between the product and one hundred millions. (C.P.)

10. Find the continued product of 16, 64, 125, and 625. (C.H.L.)

11. Multiply 74908 by 96164, the multiplication to be done in three lines. (C.M.)

12. Multiply 477120 by 525, the multiplication to be done in two lines.

13. Multiply 771214 by 216636, using only three partial products.

14. Multiply 298765 by 336567, the multiplication to be done in three lines. (C.W.)

15. Multiply 45988 by 864729 in three lines.

16. Multiply 43002073252 by 133112111 in three lines. (C.PR.)

17. Multiply 78927 by 72864 in three lines.

18. In a multiplication sum the multiplier is a two-figure number with no cipher. The first partial product is 86415; the total product is 456765. State the sum. (C.W.)

19. Find a number such that, if it be added 27 times to 47577, the sum will be 50520. (C.P.)

20. Add together the squares of the four numbers 7666, 1298, 570, and 430. Express the result in words, and divide it by 6648. (C.PR.)

21. $(3794)^2 - (3744)^2$. (C.W.)

22. $(164369)^2 - (164119)^2$. (C.W.)

23. In a sum in division the divisor is twenty times the quotient and five times the remainder. What is the dividend, if the remainder be 76? (C.S.)

24. Divide the difference of the squares of 9409 and 10609 by the sum of the squares of 97 and 103. (C.PR.)

25. If a number when divided by 323 leave a remainder 125, what will the remainder be when the same number is divided by 19? (K.S.)

26. What number must be added to 123456789 to give a sum exactly divisible by 567? (C.P.)

27. Find the least number which must be added to seven thousand and one million nine hundred and seven thousand and sixty-one, in order that it may be exactly divisible by seven hundred and nine thousand four hundred and eighty. (C.PR.)

28. In a division sum the divisor is 406, and the quotient is 717, the remainder being 345. What is the dividend? (C.P.)

29. The quotient arising from the division of 2338425 by a certain number is 346, and the remainder is 157. Find the divisor. (P.P.)

30. In a division sum the dividend was 134135, and the quotient 253. Show that the divisor must have been one or other of two numbers, and find these numbers and the corresponding remainders. (C.H.L.)

31. If from two hundred and seventy-nine thousand nine hundred and three we deduct thirty-nine, the remainder is exactly divisible by eight times that number, what is the quotient? (C.H.L.)

32. State in words the least number which must be added to forty-six million seven thousand two hundred and sixty in order that the sum may be exactly divisible by nine hundred thousand nine hundred and nine. (C.J.)

33. By what number must 695 be multiplied so that, when the product is subtracted from one million, the result is 507245? (C.P.)

34. In a long-division sum the first subtrahend is 3900 (written, as usual, 390), and the second is 260; and the remainder after this second subtraction is 11. Show that the divisor must be one of two numbers, and that there can be one dividend only. What is the dividend? (C.M.)

35. In a long-division sum the dividend is 529565, and the successive remainders from the first to the last are 246, 222, 542. Find the divisor and quotient. (C.S.)

36. Divide four million seven hundred and twenty-two thousand three hundred and ninety-seven by one hundred and eight, employing the method of short division. Express the result in words. (C.PR.)

37. A certain number is divided by 165 by short division. The

quotient is 262, the first remainder is 2, the second is 4, and the third is 8. Find the number. (C.M.)

38. Divide 285651 by 77, using only short division. Obtain the remainder correctly, and explain briefly how you obtain it. (S.K.)

39. Divide 47863 by 72 by short division, and explain the process of obtaining the remainder. A certain number when divided by 8 has a remainder 5, and the quotient divided by 9 has a remainder 7: what will be the respective remainders when the order of division is reversed? (C.H.L.)

40. Divide 3790603808 by 378, making use of the factors 6, 7, and 9, and show clearly how you get the remainder. (C.P.)

41. Complete division given below by filling in numbers omitted in first and second lines:—

```
5 |
  9 |        remainder 2
      1260   remainder 6   (P.E.A.)
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42. The number of scholars in primary schools in England in 1893 was five millions one hundred and thirty-seven thousand and two, and the number in each school was on the average two hundred and sixty-one. Find the number of schools. (C.J.)

43. By what number less than 1000 must 463759 have been multiplied if the last three figures on the right of the product are 757, and by what number less than 1000 must 30107 have been divided if the remainder is 567? (C.P.)

44. A boy multiplied a number by 467 and obtained the product 1925817. He was told that the figures 9 and 7 were wrong, and the other figures right. What was the correct product? (C.P.)

45. A boy had to divide 76428 by 123. He copied a figure wrong in the divisor, and obtained as his quotient 611 with remainder 53. What mistake did he make? (P.T.)

46. The remainder when 5368 is divided by a certain number is 60 less than the remainder when 1691 is divided by the same number. What is the number? (C.P.)

47. Given that the divisor is three times the remainder and one-fifth of the quotient, also that the difference between the quotient and remainder is 868, find the dividend. (C.P.)

48. A certain number is divided by 3, the quotient by 5, and that quotient again by 7. The final quotient is 329, and the partial remainders are 1, 2, and 1 respectively. What was the number divided? (C.W.)

49. Multiply 38761492 by 59997. Two numbers when divided by a certain divisor have remainders 12482 and 9619 respectively; when the two numbers are added together and their sum divided by the same divisor, the remainder is 845. What is the divisor? (C.H.L.)

50. At a recent Cambridge election plumpers (*i.e.* single votes) were given, 23 for A, 31 for B, 6 for C, and 3 for D. Split votes (*i.e.* one each for two candidates) were given, 64 to A and B, 2281 to A and C, 18 to A and D, 33 to B and C, 1875 to B and D, and 6 to C and D. How many votes did each candidate receive, and how many persons voted? (C.PR.)

CHAPTER II.

THE COMPOUND RULES.

The following tables are given for reference:—

Money.

4 farthings = 1 penny (*d.*)
12 pence = 1 shilling (*s.*)
20 shillings = 1 pound (£)

Weights and Measures.

Avoirdupois Weight

(*for all common goods*).

16 drams = 1 ounce
16 ounces = 1 pound
28 pounds = 1 quarter
4 quarters (112 lbs.) = 1 hundredweight
20 hundredweight = 1 ton

14 lbs. = 1 stone

Troy Weight

(*for gold, silver, jewellery*).

24 grains = 1 pennyweight
20 pennyweights = 1 ounce
12 ounces = 1 pound

1 lb. Troy weighs 5760 grs.
1 lb. avoirdupois contains 7000 grs. Troy.

Apothecaries' Weight.

(*May* be used for drugs sold by retail.)

437½ grains troy = 1 ounce
16 ounces = 1 pound

Also, but not recognized by the British Pharmacopœia—
20 grains = 1 scruple
3 scruples = 1 dram
8 drams = 1 ounce

Lineal Measure.

12 inches = 1 foot
3 feet = 1 yard
220 yards (40 poles) = 1 furlong
8 furlongs (1760 yards) = 1 mile
3 miles = 1 league

4 inches = 1 hand
6 feet = 1 fathom

5½ yards = 1 pole
22 yards (4 poles) = 1 chain

"Gunter's Measure."

100 links = 1 chain
10 chains = 1 furlong

Square Measure.

144 square inches = 1 square foot
9 square feet = 1 square yard
30¼ square yards = 1 square pole
40 poles = 1 rood
4 roods = 1 acre
640 acres = 1 square mile

10000 square links = 1 square chain
10 square chains = 1 acre

Solidity (Measure of Capacity).

1728 cubic inches = 1 cubic foot
27 cubic feet = 1 cubic yard

Measure of Capacity.

4 gills = 1 pint
2 pints = 1 quart
4 quarts = 1 gallon
2 gallons = 1 peck
4 pecks = 1 bushel
8 bushels = 1 quarter

5 quarters = 1 load
1 gallon = 6 reputed quarts

36 gallons = 1 barrel
54 gallons = 1 hogshead (beer)
63 gallons = 1 hogshead (wine)

Time.

60 seconds = 1 minute
60 minutes = 1 hour
24 hours = 1 day
7 days = 1 week
4 weeks = 1 lunar month
365 days (52 weeks) = 1 year

Roughly speaking, $365\frac{1}{4}$ days = 1 year. These numbers connecting days, weeks, and years are, however, not accurate.

Angular.

60 seconds (″) = 1 minute (′)
60 minutes (′) = 1 degree (°)
90 degrees (°) = 1 right angle

Some important points in these tables should be noted, viz.—
£1 = 240 pence = 960 farthings.
1 mile = 1760 yards = 5280 feet.
640 acres = 1 square mile.
10 square chains = 1 acre.
220 yards = 1 furlong.
1 ton = 2240 pounds.
1 pound avoirdupois = 7000 grains.
1 pound Troy = 5760 grains.
1 ounce avoirdupois = $437\frac{1}{2}$ grains.
1 ounce Troy = 480 grains.
1 gallon of water weighs 10 pounds.

1 cubic foot of water weighs $62\frac{1}{2}$ pounds, or approximately 1000 ounces.

1 cubic foot measures $6\frac{1}{4}$ gallons.

"A pint of pure water weighs a pound and a quarter."

Note carefully—

1. Dram in avoirdupois weight is not the same as dram or drachm in apothecaries' weight.

2. Ounce in avoirdupois weight is not the same as ounce in Troy weight.

3. Pound „ „ „ „ „ pound Troy.

But that—

4. Grain apothecaries' is the same as grain Troy.

5. Ounce „ „ „ ounce avoirdupois.

6. (*a*) A *quart* = 2 pints.
   (*b*) A quarter = 8 bushels.
   (*c*) A quarter = 28 lbs.

The Standard Units employed in the various tables are—

1. Money. The sovereign (or pound sterling).
2. Weight. The pound.
3. Length. The yard.
4. Surface. The square yard.
5. Solidity. The cubic yard.
6. Capacity. The gallon.
7. Time. The day.

The standard pound (avoirdupois) is the weight of a piece of platinum deposited with the Board of Trade.

The standard yard is the distance (at 62° Fahr.) between two marks on a bronze bar deposited with the Board of Trade.

The standard gallon is the volume occupied by 10 lbs. of pure water at 62° Fahr. and 30 inches barometric pressure.

Notes.—(1) The occurrence of the word *pound* in its application to both *money* and *weight* points to the primitive system of *weighing* and not coining money. A pound sterling was originally the value of a bar of silver weighing one pound Troy, which was divided into 240 equal parts, each termed a pennyweight.

(2) The symbols £. *s. d.* are abbreviations of the names of three Roman coins—*libra, solidus, denarius.*

(3) *Avoirdupois* = Old French *avoirs de pois* = things that sell by weight and not by measurement.

(4) *Troy* = Troyes, a town south-east of Paris, where this weight was first used.

(5) *Long measure,* like the decimal system, is based on human anatomy. The *foot* is derived from the length of a man's foot; the *yard* is the length of the entire arm; the *mile* (Lat. *millia*, a thousand) is the length of a *thousand* paces.

(6) The word *inch* is derived from Latin *uncia*, a twelfth part.

(7) The word *ounce* is also derived from *uncia*, because the pound was originally divided into 12 ounces.

(8) A pound of feathers is heavier than a pound of gold, because the former contains 7000 grains, and the latter 5760 grains. An ounce of feathers is lighter than an ounce of gold, because the former contains $\frac{7000}{16}$ grains, and the latter $\frac{5760}{12}$ grains.

Diagram showing the connection between—

$5\frac{1}{2}$ yds. = one rod, pole, or perch (long measure)
and $30\frac{1}{4}$ sq. yds. = one sq. rod, pole, or perch (square measure)

Cape of Good Hope Weights and Measures.

The Cape of Good Hope Weights and Measures, which are still in general use throughout South Africa, differ from the English in some respects.

Avoirdupois Weight.

100 lbs. = 1 Cape hundredweight
(112 lbs. = 1 English do.)
2000 lbs. = 1 Cape ton
(2240 lbs. = 1 English ton)
Hence 20 Cape hundredweight = 1 Cape ton

Lineal Measure.

12 Cape inches = 1 Cape foot
12 Cape feet = 1 Cape rood

Since 1000 Cape feet = 1033 English feet (see below), it follows that the Cape foot and Cape inch are slightly greater than that of the English.

Surface, Square Measure.

144 Cape square feet = 1 Cape square rood
600 Cape square roods = 1 morgen

Approximate Connections of the Two Tables.

100 Cape feet = 1033 English feet
426 Cape roods = 1 English mile
$2\frac{1}{9}$ acres (English) = 1 morgen
$302\frac{1}{2}$ morgen = 1 square mile (English)

Although for practical purposes these equivalents are sufficiently correct, it should be noticed that but few of them are absolutely accurate.

Indian Measures of Weight.

Imperial.

1 re = 1 tola, or 180 grs.
5 tolas = 1 chatak
4 chataks = 1 powa
4 powas = 1 seer (= $2\frac{2}{35}$ lb. av.)
5 seers = 1 panch seer
40 seers = 1 maund (= $82\frac{2}{7}$ lb. av.)

THEORY.

1. Compare the *pound* avoirdupois with the *pound* Troy, and the *ounce* avoirdupois with the *ounce* Troy.

2. "Long measure, like the decimal system, is based on human anatomy." Explain this.

3. What inference do you derive from the occurrence of the term "pound" both in Troy weight and the money table?

4. What are the standard units of weight, length, and capacity? Trace as far as you can some connection between them.

5. How would you demonstrate the connection between long and square measure? (C.W.)

6. "Our system of weights and measures is unscientific and misleading." Exemplify this.

7. Explain why it is that, although you can divide one sum of money by another, you cannot multiply one sum of money by another. (C.W.)

8. Comment very briefly on the answers to the following question: "Divide half a pound by half a pound." (1) "It cannot be done;" (2) "a quarter of a pound;" (3) "a pound;" (4) "one whole number;" (5) "one." (C.W.)

9. Given that a linear pole is equal to $5\frac{1}{2}$ yards, prove that a square pole is equal to $30\frac{1}{4}$ square yards. (P.E.A. and C.M.)

10. Assuming that there are 3 lineal feet in a yard, show by means of diagrams that there must be 9 square feet in a square yard, and 27 cubic feet in a cubic yard. (P.E.A.)

11. A linear mile containing 1760 yards, prove that there are 640 acres of land in a square mile. (C.P.)

EXERCISE II.

Compound Rules.

Money.

1. Multiply £8241 9*s.* $11\frac{1}{2}$*d.* by 945, and divide the result by 319. (C.H.L.)

2. Multiply £871 19*s.* 9*d.* by 420, and divide £1058 15*s.* 9*d.* by 142. Also ascertain how often £18 12*s.* is contained in £1864 13*s.* (O.R.)

3. Divide three hundred and twenty-seven millions three hundred and five thousand and four pounds among 315 persons in equal shares. How many poor persons, in addition, may receive one pound ten shillings each from the remainder? (C.P.)

4. Into how many parcels of £1 4*s.* $3\frac{3}{4}$*d.* each can £700 be divided? How much more would be required to make up one more parcel? (C.P.)

5. To how many persons may the sum of £19 19*s.* 11*d.* be paid out of an estate worth £10,000? How much each may 25 other persons receive from the residue? (C.P.)

6. In a chest were 13 drawers, in each drawer were 7 divisions, and in each division 3 purses, each purse containing £6 in gold, 9 half-crowns, and 142 pennies. How much money was in the chest? (C.W.)

7. What is the cost of telegraphing to a distance of 3800 miles a message consisting of 300 English and 150 Persian words, at the cost of $\frac{1}{2}$*d.* per mile for every 50 English, and $\frac{3}{4}$*d.* per mile for every 30 Persian words? (C.J.)

8. The Mint buys silver for the purpose of coining seventy-four millions eight hundred and sixty-seven thousand five hundred florins; but when three-fourths of the florins have been finished, the Mint is

ordered to coin the rest of the silver into half-crowns. How many half-crowns will there be? (C.J.)

9. Supposing a million and a half of French bronze coins to be exchanged for English pence, at the rate of thirteen for a shilling, what will be the loss to those who had taken them for pence? (C.P.)

10. The National Debt, on March 31, 1890, amounted to six hundred and ninety million six hundred and sixty-four thousand pounds, and has since then been gradually reduced at the average rate of seven million nine hundred and sixty-five thousand pounds a year. If this rate be maintained till March 31, 1900, state in words the amount of the debt at that date. (S.L.C.)

11. One hundred and seventy-four men are employed on a piece of work, and each of them receives 4*s.* 4*d.* per day. If eighteen of them fall ill, and their daily work and wages are divided among the others, what will be the daily wages of each of the men who do the work? (C.W.)

12. A man whose weekly earnings are 17*s.* 6*d.* saves a fifth part of that sum every fortnight. Find how many weeks it will take him to save as much as he spends in forty weeks. (C.W.S.)

13. In a workshop a certain number of men receive £1 a week each, three times as many receive £1 6*s.* 9*d.* each, and seven times as many receive 15*s.* 3*d.* each. The total amount of their wages for a week is £31 1*s.* How many workmen are there? (C.W.)

14. A sum of £657 9*s.* 9½*d.* is divided among 68 persons, so that one receives four times as much as each of the others. Find the shares. (K.S.)

15. A bill of £43 6*s.* 1½*d.* was paid with an equal number of pounds, shillings, pence, and halfpence. How many were there of each? (P.T.)

16. Divide £449 6*s.* 2*d.* among A, B, and C, giving B £21 5*s.* 4*d.* more than C, and A £7 6*s.* 3*d.* more than B. (C.S.)

17. A party consisting of 27 persons set out to travel. The fare for each was to be £1 16*s.* 5*d.*, but some of them having no money, the others agreed to pay their fares in equal proportions, and so had to pay £2 2*s.* 9*d.* each. Find how many of the party did not pay. (C.W.S.)

18. Divide £300 between A, B, and C, so that A may have £20 more than B, and B twice as much as C. (O.S.L.)

19. Divide £320 2*s.* 10½*d.* among 11 persons, so that 6 of them shall each have £2 5*s.* 9*d.* more than the others. (O.R.)

20. A sum of £1020 is to be divided between 10 men, 32 women, and 48 children. If each man's share is to be equal to the share of two women, and if the 32 women are to have twice as much as the 48 children, how much will the several individuals receive? (C.W.)

21. If £456 0*s.* 6*d.* be divided among 16 men, 10 women, and 7 boys, so that two men get as much as three women, and two women as much as three boys, what is the share of each woman? (C.PR.)

22. The sum of 7½*d.* was divided among three persons, A, B, and C, in such proportion that A received 1½*d.* more than C, and B 2¼*d.* less than C. Had a sovereign been divided among them in the same proportion, what would each have received? (C.W.)

23. Divide £50 12*s.* 4½*d.* among 5 men, 7 women, and 9 children, giving a woman twice as much as a child, and giving a man half as much again as a woman and child together. (O.S.L.)

24. Divide a sum of £370 7*s.* 6¾*d.* among 78 claimants, half of which number, being males, are to receive twice as much as the remainder, being females. What will be the share of a male, and what of a female? (K.S.)

25. A sum of money is divided between A, B, and C; C gets twice as much as A; A and B together get £50; B and C together get £60. Find the sum of money, and how much each person gets. (S.K.)

26. A certain sum of money was divided among A, B, and C; A and B received £18 19*s.*; A and C, £17 13*s.* 6*d.*; B and C, £15 12*s.* 6*d.* How much did each receive? (C.W.S.)

27. A man owes three bills, of which one could be paid by a certain number of florins, another by twice that number of half-crowns, and the third by six times that number of shillings. The bills amount in all to £7 3*s.* What are the several amounts? (S.K.)

28. A farmer selling wheat at 34*s.* a quarter would just pay his expenses; he has, however, to sell half his crop at 33*s.* 8*d.*, but waits then, and sells the remainder at 35*s.* 10*d.*, making altogether a profit of £31 17*s.* 6*d.* How many quarters of wheat did he sell? (C.PR.)

29. A grocer mixes 11 lbs. of tea at 2*s.* 6*d.* with 15 lbs. at 1*s.* 10*d.*, and 14 lbs. at 4*s.* 2*d.* per pound. If he sells the whole at 3*s.* per pound, what does he gain per pound, and what is his total gain? (P.T.)

30. A ship's cargo is unloaded by 27 barges of equal size, each barge taking three loads, but if each barge had contained 170 tons more, then it would have had only one load. Find the weights of the cargoes of ship and barge respectively. (K.S.)

31. One-half of the subscriptions to a certain fund are a guinea each, and one-third half a guinea each. The fund is completed by half-crown subscriptions amounting to £6. How many subscribers were there at each rate, and what was the whole amount subscribed? (C.P.)

32. A dealer has bought 9 oxen and 5 sheep for £186 2*s.* 6*d.*, and would lose £2 by exchanging 2 oxen for 11 sheep. What is the price of an ox? (O. and C.S.)

33. The number of boys in a school is increased from forty to fifty, and a saving of £2 5*s.* a head is thus effected. If the increase in the total expenditure be £525, find the cost of maintaining each of the forty boys. (C.S.L.)

34. To bring goods from Zanzibar to Unyamwezi costs £200 per ton, and, in consequence, calico which costs 1½*d.* a yard in Zanzibar costs 5*d.* a yard in Unyamwezi. How many yards are there to the ton? (S.A.)

35. Large needles cost 1*d.* per score more than small needles. A girl who buys equal quantities of both finds that on the whole she gets 8 for a penny. What is the price per score of each kind? (P.T.)

36. The number of passengers in the second-class carriages of a train was three times the number in the first class; and the number in the third class exceeded the other two together by 36; the fares for the several classes were at the rate of 3*d.*, 2*d.*, and 1*d.* per mile; the average distance travelled by each of the first-class passengers was 120 miles, by each of the second class 50 miles, and by each of the third class 25 miles; and the whole of the fares amounted to £390 1*s.* 8*d.* How many passengers were there of each class? (C.S.)

Avoirdupois Weight.

37. How many pounds of tea at 1*s.* $4\frac{3}{4}$*d.* per pound are worth as much as 2 tons 18 cwt. 3 lbs. of coffee at $11\frac{3}{4}$*d.* per pound? (P.P.)

38. If 3 cwt. 2 qrs. 22 lbs. of coffee cost as much as 2 cwt. 1 qr. 1 lb. of tea, how much tea should be given in exchange for 9 lbs. of coffee? (C.W.)

39. If 121 tons 17 cwt. 2 qrs. of coal cost £137 2*s.* $2\frac{1}{4}$*d.*, what is its price per ton? At what price per ton do the poor pay for it, buying at the rate of 7 lbs. for a penny? (C.P.)

40. A mixture consists of fifteen parts of coffee, purchased at £4 4*s.* per cwt., and one part of chicory, purchased at £1 16*s.* per cwt. It is sold at 1*s.* 4*d.* per pound. What profit would be made on the sale of 2 cwt. 24 lbs.? (C.S.L.)

Troy Weight.

41. Seven parts by weight of gold are mixed with eight parts by weight of silver, and the total weight is 250 ozs. What is the value of the mixture, gold and silver being worth £3 17*s.* 9*d.* and $40\frac{1}{2}$*d.* per ounce respectively? (C.P.)

42. Knowing that 5760 grains make a pound Troy, and 7000 grains make a pound avoirdupois, find how many pounds and ounces Troy will equal 3 qrs. 12 lbs. avoirdupois. (C.P.)

43. A nugget of gold is worth £150 at £3 17*s.* $10\frac{1}{2}$*d.* per ounce. What is the value of an equal weight of silver at 54*s.* 6*d.* per pound? (C.P.)

44. If 1869 sovereigns are coined out of 40 Troy pounds' weight of standard gold, what is the weight of a sovereign in grains? and what is the value of an ounce of standard gold? (S.K.)

45. A sovereign consists of twenty-two parts by weight of pure gold to two parts of alloy, and it weighs 123·274 grains Troy. Neglecting the value of the alloy, what is the value of pure gold per ounce Troy to the nearest penny? (S.K.)

Long Measure.

46. Show clearly how to reduce linear yards to poles, taking as an example the sum: Reduce 1384 linear yards to poles. (C.W.)

47. How many more yards are there in 100,188 inches than in $3\frac{1}{2}$ furlongs? (K.S.)

48. From one mile take 7 fur. 39 po. 5 yds. 1 ft. 5 in. Explain the difficulty. (P.T.)

49. How many telegraph poles will be required for wires extending 73 mi. 3 fur. 16 rods, supposing the interval between any two poles to be 99 ft.? (C.P.)

50. How many times round a garden, 96 ft. long and 80 ft. wide, will make a walk of 3 miles? (P.T.)

51. From Dover to Calais the distance is 1271160 in., from Folkestone to Boulogne 9289 po. Which is the greater, and by how much (in miles, etc.)? (C.H.L.)

52. A chimney is protected by a lightning-conductor made up of 17 lengths of copper rod, which overlap each other 3 in. at each joint. The three uppermost rods are 7 ft. each in full length, the next eight 11 ft. each, the lowest six 13 ft. each. The conductor projects 6 ft. above the top of the chimney, and enters the ground to a depth of 9 ft. What is the height of the chimney? (C.W.)

Wheels.

53. A carriage wheel is 8 ft. 3 in. in circumference. How many turns does it make in driving 7 mi. and 1331 yds.? (S.K.)

54. How many times will a bicycle wheel 12 ft. in circumference revolve between two places, A and B, distant 12 mi. 1 fur. 12 po. from each other? (O.R.)

55. The circumference of a carriage wheel is 2 yds. 2 ft. 2 in. How many times will it revolve during a journey of 27 mi. 578 yds. 2 ft. 10 in.? (O.R.)

56. The circumference of the large and small wheels of a bicycle are 176 in. and 48 in. respectively. How many more turns will the latter have made than the former in going a distance of 15 mi.? (C.PR.)

57. The wheels of a cart go round 6600 and 3520 times respectively in a journey; the larger wheel is $13\frac{1}{2}$ ft. in circumference. Determine (1) the distance in miles, (2) the circumference of the smaller wheel. (C.P.)

58. Assuming the diameter of a circle to be to its circumference as 7 : 22, how many times will a bicycle wheel, which is 28 in. across, revolve in $4\frac{1}{2}$ min., when going at the rate of 16 mi. an hour? (C.P.)

59. A four-wheeled carriage travels a distance of 10 mi. along a hilly road; a hind wheel, with circumference 4 ft. $1\frac{1}{3}$ in., is locked in going down two hills, one of which is 760 yds. long, and the other 2003 yds. long. How often in the journey does this wheel turn round? (P.T.)

Square Measure.

60. How many acres are there in 15681600 sq. in.? (O.R.)

61. A cricket club purchased the freehold of their ground, which contained 10 ac. 3 ro. 38 po., for the sum of £17500. What was the cost per acre? (C.H.L.)

62. A chain being 22 yds. long, how many square chains are contained in 3 ac. $2\frac{1}{2}$ ro. of land? (C.P.)

63. A farm of 234 ac. contains 40 ac. 2 ro. 17 po. of arable, and 93 ac. 1 ro. 18 po. of pasture land. The remainder is made up of coppices and garden ground. If the coppice land contains $15\frac{1}{4}$ ac. more than the garden ground, find how much coppice land there is, and what it is worth at £13 7*s.* 6*d.* per acre. (C.P.)

64. Determine the number of square links in a piece of land $3\frac{1}{4}$ acres in extent, a surveyor's chain of 100 lks. being 22 yds. long. (C.P.)

Cubic Measure and Volumes.

65. Divide by 17 lin. ft. (*a*) 2023 lin. ft., (*b*) 2023 sq. ft., (*c*) 2023 cub. ft. State carefully what the quotient is in each case, and give reasons for your answers. (P.E.A.)

66. How many cubic yards, etc., are there in 1458725 cub. in.? (C.P.)

67. Reduce 35469099 cub. in. to cub. ft. by short division, explaining how the remainder is obtained, and the reason of the rule. (C.P.)

68. Copper weighs 550 lbs., and tin 462 lbs. to the cubic foot. What will be the weight of a cubic foot of a mixture of 6 parts copper to 5 parts tin? (S.K.)

69. A milkman buys milk at 10*d.* per gallon, and sells on an average 57 gals. daily, Sundays included. At what price per quart does he sell it if he makes an annual profit of £346 15*s.*? (C.H.L.)

70. A farmer, who had nine cows, sold to an agent all the milk they gave at 1*s.* 0½*d.* per gallon. The agent retailed it at 4*d.* per quart, and in one week his profit was £1 16*s.* 9*d.* What was the average quantity of milk given per day by each cow during that week? (C.P.)

71. If a cubic foot of water weighs 1000 ozs., how many cubic yards of water will weigh 337½ tons? (O.J.)

72. In the second week of February the Excise returns of wheat sold were 61047 qrs. 5 bush. 3 pks., at an average price of £1 5*s.* 11*d.* per quarter. What was the value of the wheat? (O. and C.H.C.)

Time.

73. A millionaire's income is said to be at the rate of 25*s.* a minute. What is it for a year of 365¼ days? (C.P.)

74. Find the number of hours from noon on May 1, 1895, to noon on March 1, 1896. (K.S.)

75. A candidate for an examination was born at 1 a.m. on December 15, 1872. What was his age in hours at 9 a.m. on December 15, 1891? (C.S.L.)

76. A man had 25*s.* in hand on Thursday, January 1, and on every subsequent Thursday he received £1 2*s.* 6*d.* If he spent 3*s.* 4*d.* daily, on what day would he first be in debt? (P.E.A.)

77. A young clerk is offered £50 a year, with his choice of an increase of £2 per quarter or of £35 per year. Compare the sum total of his earnings at the end of five years, according as he adopts one or the other alternative. (P.T.)

78. If Neptune be 2706920000 miles distant from the Earth, and if light travels at the rate of 185000 miles a second, in what time will it traverse the space from one planet to the other? (C.P.)

79. A railway passenger counts the telegraph posts on the line as he passes them. If they are 58 yds. apart, and the train is going 48 miles per hour, how many will he pass per minute? (C.PR.)

80. The s.s. *Ruahine* left Plymouth at 6 a.m. on June 28, and reached Auckland at 9 a.m. on August 14. She called at Teneriffe, Cape Town, and Hobart, occupying just 2¼ days at those places. Taking the whole distance to be 12,500 miles, what was her average speed per hour? (C.P.)

CHAPTER III.

FACTORS AND PRIME NUMBERS.

Factors.—Take as an example, $3 \times 5 = 15$.

The numbers 3, 5, when multiplied together, *make* a result or *product* 15. Hence the numbers 3, 5 are called **factors**, and the result 15 is termed the **product.**

A Factor is a number which, when multiplied by one or more numbers, yields a product.

Measures.—Examine the following :—

$$(a)\ 3 \times 5 = 15 \qquad (b)\ \tfrac{15}{3} = 5 \qquad (c)\ \tfrac{15}{5} = 3$$

The explanation of (*a*) has been given above. From (*b*) we see that 3 is contained exactly 5 times in 15, *i.e.* 3 *measures* 15 exactly five times; hence 3 is said to be a **measure** of 15. Similarly, from (*c*) we see that 5 is also a measure of 15.

A Measure is a number contained in another number an exact number of times.

The terms Measure and Factor are practically synonymous. A slight distinction can, however, be made between them.

Difference between a Factor and a Measure.—The term *measure* is connected rather with *division* and with producing a quotient; *factor* is connected with *multiplication* and with a product. Thus—

$$\frac{\text{a given number}}{\text{a measure}} = \text{a quotient}$$

$$\text{a factor} \times \text{a factor} = \text{a product}$$

Aliquot Part.—5*s.* is contained four times in £1; hence 5*s.* is a measure of £1. However, when we deal with *concrete* numbers we use the term Aliquot Part.

An Aliquot Part is a measure of a concrete quantity.

Resolution into Factors.

Resolve the numbers from 1 to 12 into factors.

$1 = 1 \times 1$
$2 = 1 \times 2$
$3 = 1 \times 3$
$4 = 1 \times 4$ *or* 2×2
$5 = 1 \times 5$
$6 = 1 \times 6$ *or* 2×3
$7 = 1 \times 7$
$8 = 1 \times 8$ *or* 2×4 *or* $2 \times 2 \times 2$
$9 = 1 \times 9$ *or* 3×3
$10 = 1 \times 10$ *or* 2×5
$11 = 1 \times 11$
$12 = 1 \times 12$ *or* 2×6 *or* 3×4 *or* $2 \times 2 \times 3$

On examination, we observe—

(*a*) That each number has at least two factors—itself and unity.

(*b*) That some numbers (*e.g.* 1, 2, 3, 5, 7, 11) have these two factors only. These are called **prime numbers.**

(*c*) That the other numbers (*e.g.* 4, 6, 8, 9, 10, 12) have other factors besides the two mentioned in (*a*). These are called **composite numbers.**

A Prime Number (or a Prime) is a number which has no factor except itself and unity.

It is a common error to say that a prime number has *no* factor. Every number has at least *two* factors—itself and unity.

A Composite Number is a number which has other factors in addition to itself and unity.

Prime Factors are those factors of a number which are primes.

1, 2, 3 are prime factors of 12; 4 and 6 are factors, but not prime factors, of 12.

Numbers prime to each other.—Numbers which have no common factor greater than unity are said to be prime to each other.

Thus 21 and 22 are prime to each other, because they have no common factor greater than unity. But observe that neither 21 nor 22 is a prime number.

Tests of Divisibility.—The following rules will be of considerable service in the determination of factors.

A number is divisible by—

2, if its last digit is divisible by 2.

Thus 328 is divisible by 2, because its last digit (8) is divisible by 2.

$$328 = 320 + 8 = (32 \times 10) + 8$$

Since 10 is divisible by 2, any multiple of 10 is divisible by 2.
Therefore 32×10 is divisible by 2.
And since 8 is divisible by 2, the whole expression $(32 \times 10) + 8$, or 328, is divisible by 2.

4, if the last two digits are divisible by 4.

Thus 328 is divisible by 4, because its last two digits (28) are divisible by 4.

$$328 = 300 + 28 = (3 \times 100) + 28$$

Since 100 is divisible by 4, any multiple of 100 is divisible by 4.
Therefore 3×100 is divisible by 4.
And since 28 is divisible by 4, the whole expression $(3 \times 100) + 28$, or 328, is divisible by 4.

8, if the last three digits are divisible by 8.

Thus 4328 is divisible by 8, because its last three digits are divisible by 8.

$$4328 = 4000 + 328 = (4 \times 1000) + 328$$

1000 is divisible by 8; therefore 4000 is divisible by 8, and since 328 is also divisible by 8, the whole expression 4328 is divisible by 8.

5, if the last digit is 5 or 0.

Any digit but the last must represent 10 or some multiple of 10, and hence is divisible by 5; and if the last digit is a 5 or a 0, the whole expression is divisible by 5.

10, if the last digit is 0. This is obvious.

3 or 9, if the sum of the digits is divisible by 3 or 9.

Take 2745 as the number. It is divisible by 9 (or 3) because, the sum of its digits $2 + 7 + 4 + 5$, or 18, is divisible by 9 (or 3).

$$2745 = 2000 + 700 + 40 + 5$$

		A	B
$2000 = 2 \times 1000$	$= 2 \times (999 + 1)$	$= \overbrace{2 \times 999}$	$+ 2$
$700 = 7 \times 100$	$= 7 \times (99 + 1)$	$= 7 \times 99$	$+ 7$
$40 = 4 \times 10$	$= 4 \times (9 + 1)$	$= 4 \times 9$	$+ 4$
$5 =$			5

The expressions in column A are multiples of 9 (or 3), therefore their sum is divisible by 9 (or 3), and the digits in column B are the digits of the whole number. Since these are divisible by 9 (or 3), the whole number is divisible by 9 (or 3).

11, when the difference between the sum of the digits in the even places, and the sum of the digits in the odd places is either 0 or is divisible by 11.

Take 2178 as the number. We see that it is divisible by 11, because the difference between the sum of the digits in the even places $7 + 2$ (or 9) and the sum of the digits in the odd places $8 + 1$ (or 9) is equal to 0.

The nearest number to 10 divisible by 11 is 11
" " 100 " 11 is 99
" " 1000 " 11 is 1001
" " 10000 " 11 is 9999
etc.

$$2178 = 2000 + 100 + 70 + 8$$

$$\overbrace{2000 = 2 \times 1000 = 2 \times 1001 - 2}^{\text{A}} \quad \text{B}$$
$$100 = 1 \times 100 = 1 \times 99 + 1$$
$$70 = 7 \times 10 = 7 \times 11 - 7$$
$$8 = +8$$

The expressions in column A are divisible by 11, and the -2 and -7 (or -9) is the sum of the digits in the even places, while $1 + 8$ (or 9) is the sum of the digits in the odd places, and $9 - 9 = 0$, therefore the whole number is divisible by 11.

The above are the chief of the simple factors.

Many tests have been suggested for divisibility by 7, but none of them is shorter than actual trial. The student should now re-read the paragraphs on "casting out the 9's" and "casting out the 11's," pp. 10 and 11.

Other Tests of Divisibility, the "proofs" for which are nearly or quite self-evident.

A number is divisible by—

6, if it is divisible by 3 and 2
12, " " 4 " 3
15, " " 5 " 3
16, if the last *four* digits are divisible by 16
25, " *two* " " " 25
125, " *three* " " " 125

Resolution of a number into its Prime Factors.

Example.—*Separate* 7812 *into prime factors.* (C.P.)

Try the prime numbers in succession, beginning with 2. When the quotient is no longer divisible by 2, try 3, and so on.

$$\begin{aligned} 7812 &= 2 \times 3906 \\ &= 2 \times 2 \times 1953 \\ &= 2 \times 2 \times 3 \times 651 \\ &= 2 \times 2 \times 3 \times 3 \times 217 \\ &= 2 \times 2 \times 3 \times 3 \times 7 \times 31 \end{aligned}$$

2, 3, and 7 are prime factors (p. 30), and 31 will be shown to be a prime factor in the next paragraph.

To determine whether a given number is a Prime.—Take 31 from the preceding example; divide by the prime numbers in succession.

$\frac{31}{2}$ = 15 and 1 over
$\frac{31}{3}$ = 10 and 1 over
$\frac{31}{5}$ = 6 and 1 over
$\frac{31}{7}$ = 4 and 3 over

We have now obtained a quotient less than the divisor. Leaving

31 and 1 out of consideration, if 31 contained a factor less than 7, we should have found it as a divisor; if it contained a factor greater than 7, we should have found it as a quotient. With the exception of itself and unity, 31 has no factor greater than 7 or less than 7, *i.e.* it has no other factor. 31 is therefore a prime number.

Those students who know what square root means will see that the trial divisor need not go beyond the approximate square root.

Hence the following rules :—

Commence with 2 and divide by the prime numbers in succession until you obtain a quotient less than the divisor. If you have not by this time obtained a factor, you may conclude that the number is a Prime ; or, more briefly—

If a number has no factor equal to or less than its square root, then the number must be a prime.

No process of ascertaining whether a number is or is not a prime has been discovered except the process of successive divisions by primes explained above.

Method of forming a Table of Prime Numbers.

1. Write down the numbers 1, 2, 3 and all the odd numbers up to the limit required.
2. Eliminate—

(*a*) Every *third* number after 3
(*b*) „ *fifth* „ „ 5
(*c*) „ *seventh* „ „ 7

3. Determine whether the remaining numbers are primes as shown above.

To find the number of different divisors of any Composite Number

1. Find the prime factors of the number.
2. Add unity to the highest index of each factor.
3. The product of the sums will be the number of different divisors.

Example—*How many different divisors (excluding unity) has* 9009 ? (L.M.)

$$9009 = 3^2 \times 7 \times 11 \times 13$$

Number of divisors including unity $= (2+1)(1+1)(1+1)(1+1) = 24$
Number of divisors excluding unity $= 24 - 1 = 23$ *Ans.*

Reason for the rule.

$9009 = 3^2 \times 7 \times 11 \times 13$

3^2	is divisible by	1	and	3	and 3^2
7	„	„	1	„	7
11	„	„	1	„	11
13	„	„	1	„	13

Each term in the product (1 and 3 and 3^2)(1 and 7)(1 and 11)(1 and 13) is a factor of 9009, or $3^2 \times 7 \times 11 \times 13$.

The above product gives 24 terms, therefore there are 24 divisors including unity, or 23 excluding unity.

Powers and Indices.—Examine the following :—

$$(a) \quad 2 \times 2 = 4 \text{ or } 2^2$$
$$(b) \quad 2 \times 2 \times 2 = 8 \text{ or } 2^3$$
$$(c) \quad 2 \times 2 \times 2 \times 2 = 16 \text{ or } 2^4$$

(*a*) 4 is the product of *two* 2's; is called the **second power** of 2, or, more briefly, the **square** of 2.

(*b*) 8 is the product of *three* 2's; is called the **third power** of 2, or, more briefly, the **cube** of 2.

(*c*) 16 is the product of *four* 2's, and is called the **fourth power** of 2.

Two itself may be called the **first power** of 2.

Indices.—Suppose we have to indicate the ninth power of 4. We *could* write our expression thus :

$$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$$

This method would be laborious, especially with high powers, and consequently the following device is adopted: The factor is written once, and a small digit which indicates the number of times the factor is to be used in multiplication is put at the top right-hand corner of the factor.

Thus 4 to the ninth power is written 4^9, and 4 to the hundredth power is written 4^{100}.

The digit 9 is called an **index** (plural *indices*), because it *indicates* or points out the number of times 4 is to be used in multiplication.

Observe the indices used in (*a*), (*b*), and (*c*) in the last paragraph.

Two common errors.

(1) 3^2 does *not* mean 3×2, or 6. It means 3×3, or 9.

(2) 3^2 does *not* mean that 3 is to be multiplied by itself twice—a very common mistake.

3 multiplied by itself twice $= 3 \times 3 \times 3 = 27$, *i.e.* 3^3.

3^2 simply means that *two threes are to be multiplied together*; 4^9 means that nine fours are to be multiplied together.

Roots.—Observe that each of the products in (*a*), (*b*), and (*c*), above, is obtained by multiplying a factor into itself a certain number of times. This factor is called the **root.**

Thus 2 is called the *square root* of 4, and is written thus : $\sqrt[2]{4} = 2$.
,, 2 is called the *cube root* of 8, and is written thus : $\sqrt[3]{8} = 2$.
,, 2 is called the *fourth root* of 16, and is written thus : $\sqrt[4]{16} = 2$.

Extraction of a Root by resolution into Prime Factors.

Example.—*Find by resolution into factors the square root of* 7056. (P.T.)

$$\begin{aligned} 7056 &= 2 \times 3528 \\ &= 2 \times 2 \times 1764 \\ &= 2 \times 2 \times 2 \times 882 \\ &= 2 \times 2 \times 2 \times 2 \times 441 \\ &= 2 \times 2 \times 2 \times 2 \times 3 \times 147 \\ &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 49 \\ &= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7 \times 7 \end{aligned}$$

Arranging these factors into *two* equal groups, we have—

$$(2 \times 2 \times 3 \times 7) \times (2 \times 2 \times 3 \times 7)$$

hence square root $= 2 \times 2 \times 3 \times 7 = 84$ *Ans.*

Example.—*Find by resolution into factors the cubic root of* 10648. (C.S.)

$$10648 = 2 \times 2 \times 2 \times 11 \times 11 \times 11$$

Arranging these factors into *three* equal groups—

$$(2 \times 11) \times (2 \times 11) \times (2 \times 11)$$

hence the cubic root $= 2 \times 11 = 22$ *Ans.*

Division by Factors.

Example.—*Divide* 57085 *by* 216, *using factors, and explain clearly how you obtain the remainder.* (C.W.)

4 | 57085 units
6 | 14271 groups of 4 and 1 unit remainder
9 | 2378 groups of 24 and 3 groups of 4 remainder
264 groups of 216 and 2 groups of 24 remainder

Remainder = 2 groups of 24 + 3 groups of 4 + 1 unit
= (2 × 24) + (3 × 4) + 1 unit
= 48 + 12 + 1
= 61 units

Answer.—*Quotient* = 264; *remainder* = 61.

We may now rearrange the above, thus:

4 | 57085
6 | 14271 . . . 1 . . . 1
9 | 2378 . . . 3 . . . 12
264 . . . 2 . . . 48

61 remainder

From the above and similar examples, the following rule may be deduced :—

Multiply each remainder by the product of all the preceding divisors except its own. Add these products and the first remainder together. The sum will be the true remainder.

THEORY.

1. Define a prime number, and write down all the prime numbers between 100 and 120. Find whether 401 is a prime number. (C.W.)

2. Explain what is meant by a *prime* and a *composite* number. Show how you may resolve any number into its *prime factors.* (C.P.)

3. Taking the word "number" to mean "whole number," define a *prime* number, and prove what is the largest divisor (in the series 2, 3, 4,

5, etc.) that need be tried in order to be sure that 293 is a prime number. Prove that 293 is a prime number. (C.H.L.)

4. Define factor, common factor, prime number, and odd number. How many numbers between 1000 and 10,000 are exactly divisible by 29? (K.S.)

5. When is one number prime to another? (R.M.A.)

6. Can two numbers be prime to each other without being prime numbers? (K.S.)

7. What is a prime factor? Write down all the prime factors of 10296. (P.E.A.)

8. Is it universally true that the sum of two prime numbers cannot be a prime number? If not, amend the statement, giving your reasons. (C.S.)

9. Show that the difference between any number and another a million times as great is always divisible by 7, 13, and 37. (C.M.)

10. What is a *measure*, and what are the *factors* of any given number? (C.P.)

11. Explain how to tell by inspection when a number is divisible by 8, and when it is divisible by 18. (C.W.)

12. A whole number is notated by 63 fives: is it a multiple of 45? Give a reason for your answer. (C.W.)

13. How can you see at a glance whether a number is divisible (1) by 5, (2) by 8? Give a reason for your answers in each case. (P.E.A.)

14. State how it can be seen by inspection whether a number is or is not divisible by 9 or by 11, and prove the first of the rules referred to. (C.H.L.)

15. State and prove the rules for ascertaining whether a given number is divisible by 11. Show that if a number be divided into pairs of digits beginning with the units' figure, the number is divisible by 99 if the sum of the numbers formed by each pair of digits is divisible by 99. (S.A. and S.L.C.)

16. Define the following: continued product, square, cube, index, root. Write all the numbers between 1 and 100 that are perfect squares; then all that are perfect cubes.

17. Explain why a prime number can be converted into a composite number by the addition of unity. What is the exception to this statement? (C.S.)

18. Examine and criticize the following: (*a*) All prime numbers are odd numbers; (*b*) all odd numbers are prime numbers; (*c*) the number 17 has no factors.

19. If the sum of the digits of a number be divisible by 9, prove that the number itself is also divisible by 9. (R.M.A.)

20. From 98765 subtract any other smaller number consisting of the same five digits in a different order, and divide the remainder by 9. Why must the remainder be divisible by 9? (P.E.A.)

21. Prove that the difference of any two numbers composed of the same digits is divisible by 9. (C.S.)

22. Prove that a factor of any number is also a factor of any multiple of that number. (C.W.)

23. When is one number said to be a factor of another? Show that if two numbers have a common factor their sum contains the same factor. Show that 337 and 663 have no common factors. (P.T.)

24. Show that the sum of any numbers (each less than 100) will be exactly divisible by 11, if the difference between the sum of the units-digits and the sum of the tens-digits vanishes, or is exactly divisible by

11. Take any set of numbers—35, 67, 89, 8, 32, for instance—and base your reasoning upon them. (C.S.)

25. Show how you would adapt the tests by casting out nines and elevens (usually applied to multiplication sums) to simple addition and subtraction sums. An addition sum is incorrectly performed, but the tests by casting out nines and elevens both fail to detect the error. Show that the difference between the answer obtained and the correct answer must be a common multiple of 9 and 11. (C.S.)

26. When division by a composite number is performed by the successive division by its prime factors, state the rule for finding the complete remainder. (C.S. and C.W.)

27. Show that if in any number with an even number of digits, say 8, the last digit be taken away and placed in front, the sum of the number thus formed and the original number will be divisible by 11. (C.S.)

EXERCISE III.

1. Find whether the following numbers are exactly divisible by 2, 3, 4, 5, 6, 7, 8, 9, 10, 11: (*a*) 288; (*b*) 1050; (*c*) 3563; (*d*) 166648; (*e*) 123456789.

2. Find the prime factors of (*a*) 45; (*b*) 105; (*c*) 365; (*d*) 1760; (*e*) 1331.

3. Write down the prime numbers between (*a*) 111 and 150; (*b*) 550 and 600; (*c*) 950 and 1000.

4. Find the nearest prime number to the following: (*a*) 189; (*b*) 434; (*c*) 981; (*d*) 1000.

5. Resolve the following into prime factors, and hence find the square root: (*a*) 625; (*b*) 11664; (*c*) 128164; (*d*) 883600.

6. Resolve the following into prime factors, and hence find the cube root: (*a*) 216; (*b*) 21952; (*c*) 35287552.

7. Add a digit to the right of each of the following numbers so as to make the whole number divisible by 11: (*a*) 31221; (*b*) 4756657; (*c*) 426486082.

8. Write out the integral number of pence which are measures of (*a*) 5*s.*; (*b*) 11*s.*; (*c*) £1 1*s.*

9. Obtain the prime factors of 496, and show that the sum of all the different numbers which will exactly divide 496 (including unity, but excluding 496) is 496. (C.P.)

10. What number is the same multiple of 7 that 43,120 is of 55? (C.W.)

11. Reduce the following numbers to their most elementary factors: 28, 45, 135, and 243. (C.P.)

12. Find the prime factors of 273, 2717, and 5382. (C.H.L.)

13. Ascertain whether 2779 is a prime number or not. Also resolve 842673 into prime factors. (S.A.)

14. Separate 249984 into prime factors, and show that it is the continued product of three consecutive numbers. (C.P.)

15. Resolve 58744125 into its prime factors. (S.A.)

16. If 180621 be separated into two factors, show that the lesser of the two cannot be greater than a certain number. What is that number? and what are the factors most nearly equal to it? (S.A.)

17. Separate 3150 into two factors, each consisting of more than one digit, in every way that you can. (S.A.)

18. Arrange the digits 1, 2, 3, 5, 9, 9, so as to form a number divisible by 1375, explaining how you arrive at your result. (Factorize 1375.) (C.P.)

19. There are four prime numbers; the product of the first three is 2431; that of the last three 4199; that of the first, second, and fourth 2717; and that of the first, third, and fourth 3553; determine the four numbers. (S.A.)

20. Find the number which, when divided by 264 by short division (taking its factors 3, 8, 11), gives a quotient 3562, and 2, 3, 6, as successive remainders. (C.M.)

21. Find a number which, when divided by 231 by short division, taking the factors, beginning with the smallest in order of magnitude, will have 2 for the first remainder, 5 for the second, and 9 for the third, and 21 for the quotient. (C.W.)

22. Show that the number 1321 has no divisors. (C.PR.)

23. Write down four prime numbers, all greater than 150, which differ only in the units' place. (C.S.)

24. Find how many different integers, besides unity, divide 9009 without remainder. (L.M.)

25. How many different integers, besides unity, will divide 576000 without a remainder?

CHAPTER IV.

MEASURES: COMMON MEASURE, GREATEST COMMON MEASURE.

Measures.—Resolve 60 and 84 into prime factors.

$$60 = 2 \times 2 \times 3 \times 5$$
$$84 = 2 \times 2 \times 3 \times 7$$

The factors or measures we have obtained are 2, 3, 5, and 7 (see p. 29).

Common Measures.—Of these, 2 and 3 are measures of *both* 60 and 84; hence 2 and 3 are said to be *common* measures of 60 and 84. But the expression 2×2 is also a common measure of 60 and 84, and it is greater than either of the measures 2 and 3; hence 2×2 *might* be termed a greater common measure of 60 and 84, but the term greater common measure is never used. Similarly, 2×3 might be so termed.

The Greatest Common Measure.—Further inspection will show that $2 \times 2 \times 3$ is also a common measure of 60 and 84. We have thus found five common measures of 60 and 84, viz.—

$$2$$
$$3$$
$$2 \times 2$$
$$2 \times 3$$
$$2 \times 2 \times 3$$

and of these $2 \times 2 \times 3$ is the *greatest*, so it is termed the *Greatest Common Measure.*

The following definitions will now be clear:—

A Measure of a given number is a number which is contained in that given number an exact number of times.

A Common Measure of two or more numbers is a number which is contained in the two or more given numbers an exact number of times.

The Greatest Common Measure of two or more numbers

is the greatest number which is contained in the two or more given numbers an exact number of times.

Observe the following points in these definitions: a measure is a measure of *one* number; a *common* measure is a measure of *two* or *more* numbers. We do not say *the* common measure, because in all probability there are two or more common measures of two or more numbers. If there is only *one* common measure, then *the* common measure is also the greatest common measure.

The Highest Common Factor.

The expressions 2, 3, 2×2, 2×3, $2 \times 2 \times 3$ given on p. 39 are often termed the *common factors* of 60 and 84, and $2 \times 2 \times 3$ is then termed the *highest common factor*.

The Highest Common Factor of two or more numbers is thus the highest number which, when multiplied by one or more numbers, produces the numbers of which it is the highest common factor.

The term "highest common factor" has the advantage of being applicable both in algebra and in arithmetic, whereas the term "greatest common measure" cannot always be correctly employed in algebra.

x and x^2 are both common factors of x^3 and x^4, and x^2 is the highest common factor. But if x happens to be less than unity, x^2 is not *greater* than x; therefore x^2 cannot be properly termed the *greatest* common measure. The *highest* factor is not necessarily the *greatest*.

Sometimes the greatest common measure is called the highest common divisor.

G.C.M. is frequently written as a contraction of the term greatest common measure. Similarly, H.C.F. stands for highest common factor, and H.C.D. for the highest common divisor.

To find the G.C.M. (or the H.C.F.) of two numbers.

First Method. The Method of Prime Factors.—Resolve the given numbers into prime factors, and the product of the *common* prime factors is the G.C.M. (or H.C.F.).

This rule is evident from the preceding paragraphs.

Example.—*Find the G.C.M. (or H.C.F.) of* 420 *and* 2310. (C.P.)

$$420 = 2 \times 2 \times 3 \times 5 \times 7$$
$$2310 = 2 \times 3 \times 5 \times 7 \times 11$$
$$\text{the G.C.M.} = 2 \times 3 \times 5 \times 7 = 210$$

Second Method. The Method of Division.—(*a*) Divide the greater of the two numbers by the less; (*b*) divide the divisor by the remainder; (*c*) repeat process (*b*) till no remainder is left. The last divisor is the G.C.M. (or H.C.F.).

Example.—*Find the G.C.M. of* 1702 *and* 1998. (C.S.L.)

```
1702)1998(1
     1702
     ----
 296)1702(5
     1480
     ----
  222)296(1
      222
      ---
   74)222(3
      222
      ---
      ...  G.C.M. = 74  Ans.
```

Or, preferably, the following contracted form should be employed:—

```
5)1702 | 1998(1
 1480  | 1702
-------|-----
3)222  |  296(1
 222   |  222
-------|-----
 ...   |   74
```

The rule given above is based upon the following principles:—

(i.) *Whatever measures a number measures any multiple of that number.*

3 measures 15 five times
Then 3 ,, 15×8 five × eight times.

(ii.) *Whatever measures each of two numbers measures both their sum and difference.*

7 measures 49 and 21
Then 7 ,, 70, *i.e.* the *sum* of 49 and 21
And 7 ,, 28, *i.e.* the *difference* between 49 and 21

Applying these principles—

The G.C.M. of 1702 and 1998
= ,, G.C.M. of 296 ,, 1702 for 296 is the difference between 1998 and and 1702 (Principle ii.)
= ,, G.C.M. of 222 ,, 296 for 222 is the difference between 1702 and five times 296 (Principles i. and ii.)
= ,, G.C.M. of 74 ,, 222 for 74 is the difference between 296 and 222 (Principle ii.)
and ,, G.C.M. of 74 ,, 222 is 74, for 222 is a multiple of 74 (Principle i.)

To find the G.C.M. of three or more numbers.

First Method.—Apply the method of prime factors.

Second Method.—Find the G.C.M. of the first and second numbers, as in the method of division. Then find the G.C.M. of the G.C.M. just found and the third number. This process may be continued for four or more numbers.

Contracted methods of finding the G.C.M.

Contracted methods that may be employed in working the method of division in the ordinary way.

1. The principles underlying the first process is that of finding two smaller numbers having the same G.C.M. Hence—

(*a*) If one of the given numbers has a prime factor *not contained in the other*, it may be discarded at once.

(*b*) A factor which is *common* to both numbers may be taken out and set aside for multiplication into the last divisor, for, being a factor of both numbers, it must be a factor of the G.C.M.

Either rule may be used at any stage of the process.

(*c*) If we find that any two consecutive remainders have no common measure except unity, we need not proceed farther; in that case the numbers are prime to each other.

Example.—*Find the G.C.M. of* 3085345 *and* 45386655. (C.S.)

5 | 3085345 5 | 45386655

617069) 9077331(14
617069

2906641
2468276

5 | 438365

87673)617069(7
613711

2 | 3358

1679)87673(52
8395

3723
3358

73 | 365 73 | 1679

5 23

Divide each by 5, *which will be one of the factors of the G.C.M.* See (*b*) above.

Proceed with the G.C.M. by usual method.

Divide by 5. See (*a*) above.

Again proceed as in G.C.M.

Divide by 2. See (*a*) above.

Again proceed as in G.C.M.

A factor of 365 is 73.

Divide 365 and 1679 each by 73, *which will be another factor of the G.C.M.* See (*b*) above.

5 and 23 are prime to each other. See (*c*) above.

The two factors of the G.C.M. are thus 5 and 73.

G.C.M. $= 5 \times 73 = 365$ *Ans.*

For comparison, the student should work by the ordinary method. He will find that he employs nearly twice as many figures.

2. The second contracted method is the application of the Italian method of division (see p. 8), and may be illustrated by the example above.

5 | 3085345 5 | 45386655

617069 9077331(14

2906641

5 | 438365

87673)617069(7

2 | 3358

1679)87673(52

3723

73 | 365 73 | 1679

5 23

The student who cannot follow this contracted method is recommended to re-study p. 8.

G.C.M. $= 5 \times 73 = 365$ *Ans.*

Problems solved by the application of G.C.M.

Example.—*Two bills, one of* £6 7*s.* 6*d. and the other of* £9 17*s.* 6*d., are to be paid in coins of one kind. What is the largest coin that can be used?* (C.PR.)

The largest coin is the G.C.M. of the two quantities. But in order to find the G.C.M. of two concrete quantities, they must first be reduced to the same unit.

£6 7 6 = 255 sixpences
£9 17 6 = 395 sixpences
255 = 5 × 3 × 17
395 = 5 × 79
G.C.M. = 5
the largest coin = one containing 5 sixpences = 2*s.* 6*d. Ans.*

Example.—*What is the greatest number which, when used as the divisor of* 68130 *and* 107275, *will leave remainders* 27 *and* 49 *respectively?* (C.S.)

Since, on dividing 68130, a remainder of 27 is left, the "greatest number" required must divide 68130 − 27 (or 68103) exactly. Similarly, it must divide 107275 − 49 (or 107226) exactly.

The problem is now to find the G.C.M. of 68103 and 107226, which is 1449, the number required.

THEORY.

1. Define the following: measure, common measure, greatest common measure. (C.PR.)

2. Why is the term "highest common factor" considered by many to be preferable to the term "greatest common measure"?

3. When are two numbers said to be "prime to each other"? Need two such numbers be prime numbers? (C.M.)

4. Comment on the following: "The numbers 17 and 23 have no greatest common measure."

5. Considering only integers, what is the greatest and what the least measure of any number?

6. What is the meaning of the word "common" in the expression "greatest common measure"? (C.M.)

7. On what arithmetical truths is the ordinary rule for finding the greatest common measure of two or more numbers based? Express each of those truths in words, or in a symbolical form, and give a numerical example of each. (C.W.)

8. How would you find the greatest common measure of two numbers which can easily be resolved into simple factors (*e.g.* 1848 and 2079)? Explain why your answer is the *greatest* common measure. (K.S.)

9. How do you proceed to find the greatest common measure of three or more numbers? Prove the correctness of your method. (C.H.L.)

10. Define a measure of a number. (*a*) Of how many two-figure numbers is 2 a measure? (*b*) Is it their greatest common measure? Give your reason. (P.T.)

11. One of the two principles on which the "rule" for finding the greatest common measure of two numbers depends is this: If a number C measures A and also measures B, it measures the sum of A and B, and measures the difference of A and B. What is the other principle? (P.T.)

12. Prove that the numbers formed by adding their greatest common measure to each of two odd numbers, have a greatest common measure at least twice the original greatest common measure. (S.A.)

13. The ordinary process of finding the greatest common measure of two numbers can be simplified when a factor common to the two is evident, or when an obvious factor of one is clearly not a factor in the other. Explain how you would proceed in each case. (I.I.)

14. The process of finding the greatest common measure of two numbers may sometimes be abbreviated by the consideration of factors, which inspection shows to be (i.) common, (ii.) not common, at any stage of the work. Explain the principles upon which such factors must be dealt with. (C.S.)

EXERCISE IV.

Greatest Common Measure.

1. Find by the method of prime factors the G.C.M. of: (*a*) 46 and 138; (*b*) 135 and 225; (*c*) 384 and 1296; (*d*) 900 and 3474; (*e*) 94248 and 504900. (C.M.)

2. Find the G.C.M. of: (*a*) 348 and 1024; (*b*) 1379 and 2401; (*c*) 10166 and 46189 (O.S.L.); (*d*) 46835 and 82824 (C.P.); (*e*) 202587 and 281517; (*f*) 13547, 17081, and 20677 (C.H.L.); (*g*) 116039, 122067, and 137137 (S.K.); (*h*) 464321 and 683111; and hence find all the common measures of the last two numbers.

3. I have two pieces of string, one 12 inches, the other 8 inches long. How shall I divide them so that the pieces of each piece of string are of the same size and as large as possible?

4. What is the G.C.M. of £36 0*s.* 4½*d.*, £48 14*s.* 7½*d.*, and £861 7*s.* 4½*d.*?

5. Find the highest common divisor of the number of halfpence in £4 6*s.* 7½*d.*, and the number of feet in 1 mi. 7 fur. 30 po.

6. Find the greatest length of which both 48 po. 1 ft. and 55 po. 4½ yds. 2 ft. are multiples. (C.W.)

7. What is the greatest length of which 18 yds. 1 ft. 3 in. and 20 yds. 1 ft. 9 in. are multiples? (K.S.)

8. Find the longest period of time which is contained an exact number of times in 629 days 4 hrs. 16 min. 28 sec., and also in 486 days 4 hrs. 23 min. 38 sec. (C.H.L.)

9. Three persons have respectively £275, £150, and £225, with which they agree to purchase horses at the highest price per head which will allow each to invest all his money. What will be the price of each horse?

10. A farmer wishes to put 29 bushels of wheat, 403 bushels of oats, and 182 bushels of barley into the largest bags of equal size that will exactly hold each kind. How many bushels must each bag hold?

11. A schoolmaster divided his scholars, consisting of 221 boys and 143 girls, into the largest possible equal classes, so that each class of boys should number the same as each class of girls. Find the number of classes.

12. In one factory the wages are 13202 pence, and in another 146083 pence. If every man receives the same wages, how many men were employed in each factory?

13. A merchant with 12028 apples and 12772 oranges wishes to pack them in boxes so that each box shall contain the same number of apples or of oranges. What is the largest number of apples or oranges he can pack into a box?

14. A number of casks, all of one size, and that size an integral number (more than one) of gallons, are exactly filled from a vessel holding 323 gals. of water. Other casks of the same size are exactly filled from a vessel holding 255 gals. The vessels are then empty. Find the total number of casks. (P.T.)

15. 990 gals. of beer and 3474 gals. of stout have to be put into an exact number of barrels, all of the same size and as large as possible, without mixing the beer and stout together. Find how much each barrel must hold.

16. Two army corps contain 12,028 men and 12,772 men respectively. What is the largest size of regiment if each corps be divided into an exact number of regiments of equal size?

17. Find the largest number which will divide 4352 and 9039, leaving a remainder of 9 in each case.

18. What is the greatest number such that if 13850 and 17030 be divided by it the remainder in each case is 17? (C.S.L.)

19. What is the greatest number which will divide 2000 and 2708, and leave remainders 11 and 17 respectively? (C.P.)

20. Find the greatest number which is such that, when 12288, 19139, and 28200 are divided by it, the remainders are all the same. (C.S.L.)

21. Find the largest number which is such that, when 142408, 153599, and 166402 are divided by it, the remainders are all alike. (C.H.L.)

22. A boy has two pieces of tape. Their lengths are 416 yds. and 611 yds., but the boy does not know this, and has no means of finding it out. How can he mark off on one of these tapes a piece which is the greatest common measure of the two tapes? (P.T.)

23. What is the length of the largest-sized square slab that could be used in paving a courtyard 77 ft. 1 in. by 56 ft. 3 in.?

24. A courtyard measures 96 ft. long and 45 ft. broad. What is the largest square slab with which it could be paved?

25. A man has a triangular piece of ground, the sides of which are 96, 84, and 104 ft. long respectively. He wishes to enclose this field with hurdles of the greatest possible uniform length. What is the length of each hurdle?

26. Standard gold is worth £3 17*s.* 10½*d.* per ounce. What is the least number of ounces that can be coined into an exact number of sovereigns? (C.W.)

27. In finding the G.C.M. of two numbers, the last remainder is 35 and the quotients are 1, 2, 1, 3. Find the numbers. (C.S.)

28 In finding the G.C.M. of two numbers, the last divisor is 49 and the quotients 17, 3, 2. Find the numbers. (C.S.)

29. Of all the odd numbers intermediate between 1000 and 2000, which two have the greatest common measure, and what is the common measure? (L.M.)

CHAPTER V.

MULTIPLES: COMMON MULTIPLES, LEAST COMMON MULTIPLES.

TAKE as an example—

$$6 \times 5 = 30$$

We have learned that 6 and 5 are *factors* of 30, and that 30 is the *product* of 6 and 5. But 30 is sometimes termed a *multiple* of 6 (or 5), because when 6 (or 5) is *multiplied* by some other number the multiple 30 is obtained.

Hence the following definition :—

A Multiple of a given number is a number which is obtained by multiplying that given number by some other number.

Or, viewing the definition from the aspect of division rather than of multiplication, we have—

A Multiple of a given number is a number which is divisible by that given number.

Thus 30 is a multiple of 6 because 30 is divisible by 6.

By "divisible" is meant divisible without a remainder, *i.e.* exactly divisible.

The terms *Measure* and *Multiple* are correlative. Thus 12 being a *multiple* of 3, 3 is a *measure* of 12.

The term *Sub-Multiple* is sometimes used for measure.

Next write out the multiples of 2, 3, and 4 which are less than 25.

2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24
4	4	8	12	16	20	24

Examining this table, we see that 4 is a multiple of both 2 and 4. Hence 4 is said to be a *common* multiple of 2 and 4. Further examination shows that 6 is a common multiple of 2 and 3; 8 a common multiple of 2 and 4, etc.

If we look for common multiples of 2, 3, and 4 we find 12 and 24,

and of these 12 is the least: hence 12 is said to be the *least* common multiple of 2, 3, and 4.

The following definitions can now be drawn up:—

A Common Multiple of two or more given numbers is a number which is divisible by each of the two or more given numbers.

The Least Common Multiple of two or more given numbers is the least number which is divisible by each of the two or more given numbers.

Note.—(*a*) The number of common multiples of two or more numbers is infinite, but all of them are multiples of the least common multiple.

(*b*) The letters L.C.M. denote Least Common Multiple—never Least Common *Measure*.

To find the L.C.M. of two or more numbers.

1. *By the method of Prime Factors.*

Example.—Find the L C.M. of 14, 24, and 30.

Resolve into factors—

$$14 = 2 \times 7$$
$$24 = 2^3 \times 3$$
$$30 = 2 \times 3 \times 5$$

Bearing in mind our definitions, we can see by inspection that—

The L.C.M. of 14, 24, and 30 must contain 2^3
" " 14, 24, and 30 " " 3
" " 14, 24, and 30 " " 5
" " 14, 24, and 30 " " 7
∴ " " 14, 24, and 30 " " 2^3, 3, 5, 7
Hence L.C.M. $= 2^3 \times 3 \times 5 \times 7 = 840$ *Ans.*

From the above we may deduce the following rule:—

To find the L.C.M. of two or more numbers.

(*a*) Find the prime factors of the numbers.
(*b*) Pick out all the different factors found in all the numbers.
(*c*) Raise each factor to the highest power in which it is found in any of the numbers.
(*d*) Multiply the factors found in (*c*) together.
The product is the L.C.M.

2. *By the method of finding the G.C.M.*

Suppose we have to find the L.C.M. of 48 and 60. If we multiply 48 by 60 we obtain *a* common multiple (2880), but it is not the *least* common multiple. If we divide by the G.C.M. (12) we remove from the product (2880) all the factors of the second number (60) which

are contained in the first number (48), and the result (240) is the L.C.M. Instead of dividing the product by the G.C.M. we may divide one of the numbers by the G.C.M., and multiply the other number by the quotient thus obtained. Hence the following rule :—

To find the L.C.M. of two numbers.

Divide one of the numbers by their G.C.M., and multiply the quotient by the other number.

Example.—Find the L.C.M. of 144 and 320.

G.C.M. = 16

L.C.M. = $\frac{144}{16} \times 320 = 9 \times 320 = 2880$ *Ans.*

So far, the above method is applicable to two numbers only. To find the L.C.M. of three or more numbers by this method we might find the L.C.M. of any two, then the L.C.M. of the L.C.M. just obtained and the third number, and so on, till all the numbers are exhausted. In practice, however, the L.C.M. is generally found—

3. *By the method of division by Prime Factors.*

(*a*) Set down the given numbers side by side, separating them by commas; (*b*) divide by any prime number which will exactly divide two at least of the numbers; (*c*) set down side by side the quotients and the numbers not exactly divisible by the divisor; (*d*) proceed as in (*b*) and (*c*) till all the numbers obtained are prime to one another; (*e*) multiply together all these remaining numbers and all the divisors used. The product is the least common multiple of all the numbers.

Note.—Any one of the numbers which is exactly contained in any other of the numbers may be omitted in finding the L.C.M.

Example.—*Find the L.C.M. of* 8, 10, 15, 16, 18, 20.

8 and 10 may be struck out because they are contained in 16 and 20 (see *Note* above). Therefore, whatever is the L.C.M. of 15, 16, 18, 20 is also the L.C.M. of 8, 10, 15, 16, 18, 20.

Divide by 2 (see (*b*) above). The L.C.M. of 8, 10, 15, 16, 18, 20 is the same as the (L.C.M. of 15, 8, 9, 10) × 2.

Divide by 3. The L.C.M. of 8, 10, 15, 16, 18, 20 is the same as the (L.C.M. of 5, 8, 3, 10) × 2 × 3.

Strike out 5 because it is contained in 10. The L.C.M. of 8, 10, 15, 16, 18, 20 is the same as the (L.C.M. of 8, 3, 10) × 2 × 3.

Divide by 2. The L.C.M. of 8, 10, 15, 16, 18, 20 is the same as the (L.C.M. of 4, 3, 5) × 2 × 3 × 2.

2 | ~~8~~, ~~10~~, 15, 16, 18, 20
3 | 15, 8, 9, 10
2 | ~~5~~, 8, 3, 10
4, 3, 5

But 4, 3, 5 are prime to each other and so their L.C.M. will be the product of 4, 3, 5 (see (*e*) above).

∴ L.C.M. of 8, 10, 15, 16, 18, 20 = 2 × 3 × 2 × 4 × 3 × 5 = 729 *Ans.*

The Relationship of G.C.M. to L.C.M.

On p. 47 we noticed that the L.C.M. could be determined by finding the product of the two numbers and dividing by their G.C.M. Putting this truth into equational form, we have—

$$\text{L.C.M.} = \frac{\text{one number} \times \text{other number}}{\text{G.C.M.}}$$

$$\therefore \text{L.C.M.} \times \text{G.C.M.} = \text{one number} \times \text{other number}$$

i.e. *the product of two numbers is equal to the product of the L.C.M. and G.C.M.*, a truth which the student will find of value in solving some problems connected with G.C.M. and L.C.M.

Rewriting our equation above in another form, we have—

$$\frac{\text{G.C.M.}}{\text{one number}} = \frac{\text{other number}}{\text{L.C.M.}}$$

Or, stating this in the form of a proportion, we get—

$$\text{G.C.M.} : \text{one number} :: \text{other number} : \text{L.C.M.}$$

The following proof of the above theorem will be of interest. To show that—

$$\text{L.C.M.} \times \text{G.C.M.} = \text{product of the two numbers.}$$

Suppose x and y are the two numbers, and a their G.C.M.

$$\begin{aligned} \text{Then } x &= a \times b \\ y &= a \times c \\ \text{and } xy &= a \times b \times a \times c \\ &= a \times bac \end{aligned}$$

But a is the G.C.M., and bac is the L.C.M. of $a \times b$, and $a \times c$, *i.e.* of x and y.

$\therefore$ the product of two numbers = L.C.M. $\times$ G.C.M.

The G.C.M. and L.C.M. of fractions.

The G.C.M. of several fractions is the greatest fraction which is contained in each of them an exact number of times. Thus $\frac{1}{12}$ is the G.C.M. of $\frac{1}{3}$ and $\frac{1}{4}$, because it is the greatest fraction contained in each of them an exact number of times.

The L.C.M. of several fractions is the least fraction or integer containing all of them an exact number of times. Thus 1 is the L.C.M. of $\frac{1}{3}$ and $\frac{1}{4}$, because it is the least quantity containing both fractions an exact number of times.

To find the G.C.M. of several fractions.

Example.—*Find the G.C.M. of* $\frac{5}{6}$, $\frac{3}{8}$, $\frac{9}{20}$.

Bring them all to their least common denominator—

$$\frac{100, 45, 54}{120}$$

The fraction $\frac{1}{120}$ is contained an integral number of times in them all, and is thus a *measure* of them all. If it is not the *greatest* common measure, then some multiple of $\frac{1}{120}$ must be the G.C.M.

The problem thus becomes: Have 100, 45, 54 any common measure greater than unity? No. Then $\frac{1}{120}$ is the G.C.M. Hence the following rule:—

To find the G.C.M. of two or more fractions.

(1) Express the quantities as fractions in their lowest terms.
(2) Find the L.C.M. of the denominators for a new denominator.
(3) And find the G.C.M. of the numerators for a new numerator.

Or, more briefly—

$$\text{The G.C.M. of several fractional quantities} = \frac{\text{G.C.M. of numerators}}{\text{L.C.M. of denominators}}$$

To find the L.C.M. of several fractions.

Example.—*Find the L.C.M. of* $\frac{5}{6}, \frac{3}{8}, \frac{9}{20}$.

Find the least common *numerator*—

$$\frac{45}{54}, \frac{45}{120}, \frac{45}{100}$$

Forty-five evidently contains all the fractions an integral number of times. It is a *multiple* of the fractions. The question is: Is it the *least* common multiple? Expressing 45 as $\frac{45}{1}$, we can see that the greater value we can give to the denominator, the less will be the value of the whole fractions, *i.e.* the more we shall approximate to the *least* common multiple of the fractions. And if we can find the *greatest* value of the denominator, we shall have found the *least* common multiple of the fractions. But this greatest denominator must divide into 54, 120 and 100 an integral number of times, *i.e.* it must be the G.C.M. of these numbers. The G.C.M. of 54, 120 and 100 is 2.

Therefore the L.C.M. of the fractions is $\frac{45}{2} = 22\frac{1}{2}$.

Hence the following rule:—

To find the L.C.M. of several fractions.

(1) Express the quantities as simple fractions in their lowest terms.
(2) Find the L.C.M. of the numerators for a new numerator.
(3) And the G.C.M. of the denominators for a new denominator.

Or, more briefly—

$$\text{The L.C.M. of several fractional quantities} = \frac{\text{L.C.M. of the numerators}}{\text{G.C.M. of the denominators}}$$

Types of problems in L.C.M.

Example.—*What is the smallest number which, when divided by* 16, 24, 30, *and* 32, *leaves a remainder of* 5? (C.W.)

The smallest number which, when divided by 16, 24, 30, and 32, leaves *no* remainder is the L.C.M. of those numbers, which is 480.
∴ the smallest number which leaves a remainder of 5 is—

$$480 + 5 = 485 \quad \textit{Ans.}$$

Example.—*Find the least number of days in which an exact number of half-guineas can be earned, a day's wage being* 2*s.* 11*d.* (L.M.)

The L.C.M. of 10*s.* 6*d.* and 2*s.* 11*d.* = L.C.M. of 126*d.* and 35*d.* = 630*d.*

Number of days in which 630*d.* can be earned at 35*d.* per day $= \frac{630}{35} = 18$ days. *Ans.*

Example.—*The L.C.M. of two numbers is* 3060288, *and their G.C.M. is* 168. *One of them is* 12096. *Find the other.* (C.S.)

G.C.M. × L.C.M. = one number × the other number
168 × 3060288 = 12096 × the other number

$$\frac{168 \times 3060288}{12096} = \text{the other number}$$

42504

42504 = the other number. *Ans.*

Example.—*Three men start together from the same point, and walk continually in the same direction round a circle* $\frac{3}{4}$ *of a mile in circumference in* 10, 18, 90 *minutes respectively. Find the least time in which they will be together, and the number of miles each travels.* (C.S.)

First man in 10 min. walks $\frac{3}{4}$ mile
then „ „ 60 „ „ $\frac{3}{4} \times 6$ miles $= 4\frac{1}{2}$ miles
second man in 18 „ „ $\frac{3}{4}$ mile
then „ „ 60 „ „ $\frac{3 \times 60}{4 \times 18} = 2\frac{1}{2}$ miles
third man in 90 „ „ $\frac{3}{4}$ mile
then „ „ 60 „ „ $\frac{3 \times 60}{4 \times 90} = \frac{1}{2}$ mile

First man gains on the third at the rate of 4 miles in 60 min.

then „ „ „ „ $\frac{3}{4}$ mile in $\frac{60 \times 3}{4 \times 4} = 11\frac{1}{4}$ min.

first „ „ second at the rate of 2 miles in 60 min.

then „ „ „ „ $\frac{3}{4}$ mile in $\frac{60 \times 3}{2 \times 4} = 22\frac{1}{2}$ min.

The least time in which they will be together is the L.C.M. of $11\frac{1}{4}$ and $22\frac{1}{2} = 22\frac{1}{2}$ min.

Distance first man goes in $22\frac{1}{2}$ min. $= \frac{3 \times 22\frac{1}{2}}{4 \times 10} = 1\frac{11}{16}$ miles

„ second „ „ $22\frac{1}{2}$ min. $= \frac{3 \times 22\frac{1}{2}}{4 \times 18} = \frac{15}{16}$ mile

„ third „ „ $22\frac{1}{2}$ min. $= \frac{3 \times 22\frac{1}{2}}{4 \times 90} = \frac{3}{16}$ „

Least time $= 22\frac{1}{2}$ min.
First man travels $1\frac{11}{16}$ miles
second „ „ $\frac{15}{16}$ „
third „ „ $\frac{3}{16}$ „
} *Ans.*

THEORY.

1. Define the term *multiple*. (C.J. and C.W.)

2. What is a sub-multiple? Give the sub-multiples of 24.

3. Define the following terms: *A* common multiple; *the* least common multiple. Account for the use of the article in each case. How does a multiple differ from a measure?

4. What is the effect upon the sum, difference, product, quotient, G.C.M., and L.C.M. of two numbers of multiplying each by 5? (S.A.)

5. What is the L.C.M. of two or more numbers? (P.P.)

6. Write the following contractions in full: L.C.M., H.C.F., G.C.M., and H.C.D.

7. Explain the method by which the L.C.M. of numbers is found. (C.W.)

8. Prove that the L.C.M. of two numbers is the product of the numbers divided by their G.C.M. (L.M. and C.S.)

9. Show that if the product of any two numbers is a square, the product of their L.C.M. and G.C.M. is also a square. (C.W.)

10. What is meant by the expression "the least common multiple of 6 and 9"? Prove that every multiple of two or more numbers is a multiple of their L.C.M. (C.W.)

11. Define a *measure*, a *multiple*, a *common measure*, and a *common multiple*. (C.S.)

12. Is the G.C.M. of two numbers always a factor of the L.C.M.? Give reasons for your answer. (K.S.)

13. Indicate clearly the *reasoning* which leads to the conclusion that 127 is the G.C.M. and 40,513 the L.C.M. of 1397 and 3683. (C.S.)

14. Prove that the product of any two numbers is equal to the product of their G.C.M. and their L.C.M. (C.H.L.)

15. State the arithmetical rule for finding the L.C.M. of three or more numbers. (P.E.A.)

16. Prove that the G.C.M. of two numbers is the L.C.M. of all their common measures. (C.S.L.)

17. What do you mean by the G.C.M. and the L.C.M. of two numbers? Show from your definition that the product of the two numbers 72 and 99 is the product of their L.C.M. into their G.C.M. (P.P.)

18. "The least common multiple of two numbers is the quotient of their product divided by their greatest common measure." Explain this

as fully as you can. Is it possible to extend this proposition to three or more numbers? Give your reasons. (P.E.A.)

19. Given a series of fractions all in their lowest terms, show that their G.C.M. is the fraction whose numerator is the G.C.M. of their numerators, and whose denominator is the L.C.M. of their denominators. Show also that their G.C.M. will be a fraction in its lowest terms. (C.S.)

20. I have to find the L.C.M. of 28, 9, 10, 12, and the L.C.M. of 28, 9, 12. In each case I take 4 as a divisor. I write down 7, 9, 10, 3, and 7, 9, 3. I strike out 3, and declare the L.C.M's to be $4 \times 7 \times 9 \times 10$ (or 2520) and $4 \times 7 \times 9$ (or 252). How is it that the results are not both right? Explain fully. (P.T.)

21. Show that the least integer which contains two or more fractions an exact number of times is the L.C.M. of their numerators. (C.M.)

22. Give the rules for finding the L.C.M. and the G.C.M. of fractions. (R.M.A.)

EXERCISE V.

Least Common Multiple.

1. Find the L.C.M. of (*a*) 3127, 3551, and 3953 (C.S.L.); (*b*) 156, 168, 208, and 432 (L.M.); (*c*) 2059 and 4189 (C.H.L.); (*d*) 1517, 1739, and 1927 (C.H.L.); (*e*) 1404, 3042, and 663 (L.M.); (*f*) the first ten numbers; (*g*) the first twelve odd numbers; (*h*) the even numbers from two to twenty.

2. Find the smallest number of which 8, 12, 22, and 25 are divisors.

3. Split 135, 126, 432, 255, and 1207 into their prime factors; and hence find their L.C.M. (P.T.)

4. Resolve 11466 and 10725 into their prime factors; and thence find their L.C.M., *which need not be multiplied out.* (C.P.)

5. Find the prime factors of 2652, 4147, and 5742; and write down their L.C.M. (C.S.L.)

6. Find the prime factors of 1066, 1815, and 1896; and hence write down the factors of their L.C.M. (C.P.)

7. Find the three numbers between 2500 and 3000 which are divisible by all the numbers 21, 24, and 28. (C.J.)

8. Find a number between 300000 and 310000 which is exactly divisible both by 79 and by 97. (C.S.L.)

9. What is the lowest number, which, when divided separately by 15, 20, 48, and 36, will in each case leave 9 as a remainder? (C.W.)

10. Find the least number greater than 6 which leaves the same remainder 6 when divided by 15, 35, or 42. (C.W.)

11. Find the G.C.M. of 1 hr. 7 min. 47 sec., and 7 hrs. 32 sec. (K.S.)

12. Find the smallest length which is an exact multiple of 6 yds. 5 in., 6 yds. 2 ft. 7 in., and 8 yds. 2 ft. 11 in. (C.W.)

13. What is the smallest sum of money which will exactly contain each of the following sums: 5*s.* 5*d.*, 7*s.* 1*d.*, 7*s.* 7*d.*, and 9*s.* 11*d.*? (C.W.)

14. Determine the smallest sum of money out of which a number of men, women, or children respectively may receive £1 7*s.* 6*d.*, 18*s.* 9*d.*, or 8*s.* 3*d.* each. (C.P.)

15. What is the smallest sum of money which contains an exact number of thalers of the value of 2*s.* 11$\frac{3}{4}$*d.* each, and of dollars of the value of 4*s.* 1$\frac{1}{2}$*d.* each? (C.P.)

16. Find the least sum of money payable in pounds or guineas.

17. Find the smallest number greater than 5, which, when divided by each of the numbers 121, 145, and 319, has 5 as a remainder.

18. What is the least number, which, when divided by 12, 20, 30, or 54, leaves a remainder of 4?

19. What is the least number which leaves 4 as a remainder when divided by 9, 99, 999, or 9999? (C.S.)

20. Four men can walk 105, 112, 126, and 168 miles in a week respectively. What is the least distance they can all walk in an exact number of weeks? (K.S.)

21. Two men step off together from the same spot, each walking 4 miles an hour; but one takes steps 2 ft. 10 in. long, and the other 2 ft. 8 in. When, and at what distance from the starting point, will they first again both step out together? How many more steps does one take than the other in 17 minutes? (C.P.)

22. In walking, A covers 3 ft. at each step, and B covers $2\frac{1}{2}$ ft. If both start together, and keep step until A has walked a mile, how far behind will B be, and if they keep abreast, how often will they be in step in a mile? (C.M.)

23. On a rock are three revolving lights: one revolves in 2 min. 12 sec., another in 3 min. 7 sec., the third in 3 min. 24 sec. I saw them all bright at midnight. When will they next be so? (K.S.)

24. Two cogged wheels work together, there being 32 cogs on one and 36 on the other; the larger wheel makes 64 revolutions per second. How often will the same cogs come in contact during 6 working days of 10 hours each? (C.S.)

25. Six bells are tolling, and they toll at intervals of 3, 5, 7, 8, 9, and 10 seconds respectively. What interval will elapse between their once tolling together and tolling together again? (O.R.)

26. There are four bells, each of which strikes respectively at intervals of 3, 7, 12, and 14 seconds. The four begin to strike at twelve o'clock. When will they next strike together, and how often will they strike in unison in 7 minutes? (C.P.)

27. If four clocks tick at one instant all together, how long will it be before they tick together again, their rates of ticking being respectively twice in a second, three times in two seconds, five times in two seconds, and four times in three seconds? (C.H.L.)

28. Eight bells, which toll at intervals of 1, 2, 3, 4, 5, 6, 7, and 8 seconds respectively, begin tolling all simultaneously with the clock striking; how many hours must elapse before they all toll simultaneously again with the clock striking? The clock is supposed to strike at the hour only. (R.M.A.)

29. Three bells toll at intervals of 12, 15, and 28 seconds respectively. A person at a distance from them can hear their sound only when two at least toll at the same time. If they begin together, how many times will he hear their sound in 7 minutes?

30. The fore wheel of a carriage is 4 ft. 4 in. round, and the hind wheel 7 ft. 7 in. round. How far must the carriage travel before each wheel shall have made a number of complete turns and have the same point downwards in each as at starting? How often will this happen in one mile?

31. Find in grains the least weight that can be expressed by an exact number of ounces in both troy and avoirdupois weight.

Motion round Circles.

32. The periods of three planets which move uniformly in circular orbits round the sun are respectively 200, 250, and 300 days. Supposing that their positions relative to one another and to the sun be given at any given moment, determine how many days must elapse before they have again exactly the same relative positions. (C.S. and L.M.)

33. Five horses start together and continue running round a circular course which is two miles in circumference. If they run respectively 18, 16, 15, 12, and 10 miles per hour, how soon will they be again together at the starting place? (C.S.)

34. Three men walk round a circular path 2640 yds. in circumference at the rate of 132 yds., 120 yds., and 110 yds. per minute respectively. If they start together, when will they be together again? (C.S.)

35. Three riders pass a given point on a circular track at the same instant, with velocities 10, 12, and 15 miles per hour respectively. How long, from that instant, must they continue to move round the track with those velocities before they next pass the same point abreast (the distance round the track for each rider being 704 yds.)? (C.P.)

36. A, B, and C start from the same place to go round an island. A makes the circuit in $\frac{2}{3}$ of a day, B in $\frac{3}{5}$ of a day, and C in $\frac{5}{6}$ of a day. How many circuits will each make before they are all together at the starting point? (R.U.I.)

37. A and B start together from the same point on a circular path, and walk in the same direction till they both again arrive together at the starting point. If A performs a circuit in 224 sec., and B in 364 sec., how many times will A have passed B? (C.S.)

38. A circular running path is 553 yds. round. Two men start back to back to run round, one running at the rate of $9\frac{1}{2}$ miles, and the other at the rate of $10\frac{1}{4}$ miles per hour. When and where will they meet for the first time? (C.W.)

39. Two men start together at the same instant and in the same direction along a circular path whose circumference is one-third of a mile. One walks at the rate of 5 miles an hour, the other runs at the rate of 13 miles an hour. How soon will the runner overtake the walker for the first time? Would the walker win his wager or not, if he has undertaken to walk 7 rounds before the runner completes 18 rounds? (C.H.L.)

40. A and B start from the same point to run in opposite directions round a circular racecourse 4324 ft. in circumference, A not starting till B has run 716 ft. They pass each other when A has run 1927 ft. Which will come first to the starting point, and what distance will they then be apart? (C.M.)

41. Two men run round a circle one mile in circumference; one runs at the rate of 220 yds., and the other at the rate of 176 yds. per minute. If they both start from the same point at the same time and run in the same direction, when will they first be together again at the starting point?

42. A and B start from the same point to run in opposite directions round a circular racecourse 9755 ft. in circumference, A not starting until B has run 105 ft. They pass each other when A has run 4850 ft. Which will first come round again to the starting point (their speeds being uniform throughout), and what distance will they then be apart? (R.M.A.)

43. Two bicyclists, X and Y, started at the same instant, and rode at uniform rates round a circular track the same way, and X gave Y a start of halfway round the track. When X reached Y's starting place for the second time, Y had 33 yds. to go to reach X's starting place for the second time, and X got level with Y just as Y completed his tenth round of the track. X rode at $17\frac{1}{2}$ miles per hour: how long did it take him to catch Y? (C.P.)

44. Suppose two bicyclists, A and B, to traverse *in opposite directions* the rim of a circle, the circumference of which measures 15 miles, A at the rate of 11, B at the rate of 13 miles an hour: (*a*) At what distances from their common starting point will they *first* meet? (*b*) If they continue, where and when will they *next* meet? (*c*) When will they *together* reach their original starting point, and what distance will each *then* have altogether travelled? (C.P.)

45. Three bicyclists, A, B, and C, ride at the rate of 639, 615, and 603 yds. per minute respectively. If they start together in the same direction round a circular track a quarter of a mile in circumference, find how long after the start and at what part of the track they will next be all together, and how often A will have passed C in the mean time. (C.S.)

EXERCISE VI.

G.C.M. AND L.C.M. OF FRACTIONAL QUANTITIES.

1. Find the G.C.M. of $5\frac{1}{2}$ and $8\frac{1}{4}$.

2. Find the L.C.M. of 6·4, 7·2, and 4·8. (C.M.)

3. What is the least integer that is a common multiple of $1\frac{194}{121}$, $22\frac{11}{49}$, and $1\frac{9}{16}$? (R.M.A.)

4. Find the G.C.M. of $\frac{2}{3}$ in., $\frac{3}{4}$ in., and $\frac{4}{5}$ in.; find also the smallest length which is a multiple of all three of them. (C.W.)

5. Two persons take steps of 2·21 and 1·95 ft.; if they start in step, how far will they walk before they are in step again? (C.W.)

6. Two friends start in step, and walk side by side, but each step of the one is $2\frac{13}{22}$ ft., and of the other $2\frac{7}{44}$ ft. How far will they have walked before they are in step again? (C.W.)

7. One pendulum oscillates in $\frac{57}{58}$ of a second, and another in $\frac{76}{87}$ of a second; if started simultaneously, how often will they tick together in an hour? (C.M.)

8. Four bells, which commence tolling together, toll at intervals of 1·2, 1·75, 1·8, and 2·1 sec. respectively; at what intervals will they toll together? (K.S.)

9. Five bells begin to toll together, and toll respectively at intervals of $7\frac{1}{2}$, $7\frac{1}{3}$, $6\frac{3}{4}$, $6\frac{2}{3}$, and $5\frac{1}{2}$ sec.; what time will elapse before they all toll together again? (C.W.)

10. Three girls began skipping at exactly the same moment. They jump at intervals of 0·6, 0·75, and 0·8 sec. respectively. How long will it be before they again jump simultaneously? (C.W.)

11. A person possesses a certain number of sovereigns sufficient to give four persons $\frac{1}{12}$, $\frac{3}{35}$, $\frac{3}{28}$, $\frac{2}{21}$ of that number respectively without obtaining change for a single sovereign. What number has he? (C.S.M.)

12. How often is each of the quantities $19\frac{1}{8}$, $19\frac{5}{6}$, and $21\frac{1}{4}$ contained in their L.C.M.? (R.M.A.)

13. Express the G.C.M. of $\frac{3}{7}$, $\frac{5}{14}$, $\frac{1}{21}$, $\frac{3}{35}$ as the fraction of the L.C.M. of the same fractions. (R.M.A.)

EXERCISE VII.

MISCELLANEOUS—FACTORS, G.C.M., AND L.C.M.

1. Find the L.C.M. of 5, 7, 16, 28, 48, and 76, and the G.C.M. of 1288, 1736, and 104. (O.R.)

2. Find the G.C.M. of 2233, 2639, and 4147, and the L.C.M. of 1311, 2346, and 3876. (C.S.L.)

3. Find the G.C.M. of 12321 and 54345, and the L.C.M. of 12, 18, 30, 48, and 60. (C.P.)

4. Obtain the least perfect cube which is exactly divisible by 42, 56, 154, 231, and 484. (Factorize the numbers.) (C.P.)

5. Find the prime factors of 1287 and 2145 and their G.C.M. (highest common factor), and the L.C.M. of 7, 9, 14, 24, 27 and 54. (C.P.)

6. Resolve into prime factors the numbers 112860 and 862125, and hence find their G.C.M. and L.C.M. (C.H.L.)

7. Find the G.C.M. and the L.C.M. of 7560, 27720, and 108108. (R.M.A.)

8. Find the greatest number which will exactly divide both 867502 and 1000369; find also the smallest number which is exactly divisible by both the numbers. (L.M.)

9. Find the greatest and least numbers of six digits which are exactly divisible by 789. (C.S.L.)

10. How many numbers greater than 100 and less than 1000 are divisible by 5? and what is the sum of these numbers? (L.M.)

11. Find the G.C.M. and the L.C.M. of 936 and 2925. (C.PR.)

12. Find the G.C.M. of 114135 and 213052, and the L.C.M. of 4, 14, 21, 27, 33. (O.R.)

13. Find the G.C.M. of 27781 and 23507, and show that no other number can be their G.C.M. (C.S.)

14. Find the G.C.M. and L.C.M. of $13 \times 15 \times 17$, $15 \times 17 \times 19$, $17 \times 19 \times 13$. (C.P.)

15. Find the G.C.M. and L.C.M. of 157 days 7 hrs. 4 min. 7 sec. and 243 days 2 hrs. 11 min. 49 sec. (C.S.)

16. In how many ways can a piece of string 12 in. long be cut, so that each piece may be of the same size, and contain a whole number of inches? (P.T.)

17. Express 364, 2520, and 5445 as products of powers of prime numbers.

18. Find the largest number that will divide both 123 and 147 without a remainder, and also find the smallest number which can be divided exactly by each of the above two numbers. (P.T.)

19. Find the least sum of money which contains 14*s*., 32*s*. 8*d*., and 21*s*. each an exact number of times. Find also the G.C.M. of these three sums. (L.M.)

20. Divide 1904 into two parts, so that the G.C.M. of the parts is 28, and the L.C.M. 32340. (O.S.L.)

21. If the H.C.F. of two numbers, one of which lies between 2000 and 2500, be 51, and the L.C.M. be 148461, find the numbers. (C.P.)

22. What is the least length which is a multiple of 11 ft., 14 ft. 8 in., and 16 ft. 6 in., the lengths of 3 bars of iron; also what is the least number of bars of the same length into which the 3 bars can be divided? (C.P.)

23. Find the three largest numbers such that their G.C.M. is 11, and their L.C.M. is 5005. (S.A.)

24. The driving-wheels of a locomotive are 7 ft. 8 in. in diameter, the small wheels 3 ft. 8 in. in diameter, and one spoke in each wheel is painted red. At the start all the painted spokes are upright and above the axle: how many more times in the first 5 miles will they all be upright and above the axle at the same instant? The circumference of a wheel is to be taken as $\frac{22}{7}$ of its diameter. (C.J.)

25. The circumference of the front wheel of a bicycle is 28 in., and of the back wheel 26 in. How far must the bicycle go before both wheels complete revolutions together? (C.PR.)

26. Find the G.C.M. of 4081, 7007, and 7392 by expressing them in their prime factors.

27. What is the least sum of money I could pay with half-crowns or florins or shillings or threepences?

28. A sum of £43 15*s.* is to be divided among 112 persons in such a way that some are to receive 8*s.* 9*d.* each and the rest 6*s.* $1\frac{1}{2}d.$ each. How many are to receive the larger sum? (S.A.)

29. A gardener was handed a number of shrubs to plant in rows; he tried 10 in each row, then 12 in each, afterwards 15 in each row, but had always 9 plants left. On trying 13 in each he had none left. What is the smallest number of shrubs which could have been given him? (C.S.)

30. A sum of £10000 is to be divided between several claimants so that some are to receive £80 12*s.* 10*d.*, and the others £80 12*s.* 11*d.* each. How many persons will receive each of these amounts? (S.L.C.)

31. Two fields, measuring respectively 24 and 28 ac., have to be divided into equal-sized plots as large as possible. Find the number of plots.

32. If a franc is worth $9\frac{1}{2}d.$ and a dollar 4*s.*, find the least number of francs that make an exact number of dollars.

33. Determine the circumferences of the different wheels which satisfy the following conditions: The circumference of each wheel must be an exact number of feet, not greater than 13, and each wheel must make an exact number of revolutions in travelling $1\frac{1}{2}$ mi. (C.P.)

34. Find the number between 30000 and 31000 which is exactly divisible by both 79 and 97. (P.T.)

35. Obtain the prime factors of 2431; also all the sums of money of which £2 10*s.* $7\frac{3}{4}d.$ is a multiple, and add together those sums that are less than 4*s.* (C.P.)

36. Find the least weight in avoirdupois weight which contains an exact number of pounds troy, or of pounds avoirdupois. (C.S.)

37. In any year that is not leap year show that the same day of the month in January and October will fall on the same day of the week. (C.S.)

38. Find two numbers having 1188 as an L.C.M. and 9 as a G.C.M. (P.P.)

39. A heap of pebbles can be exactly made up into groups of 55; but when made up into groups of 18, 27, and 30 there is always a remainder of 10. Find the least number of pebbles in the heap.

40. The population of a parish exceeds 3000 and is less than 4000. Whether the people are arranged in groups of 8, of 9, of 15, of 18, or of 25, 7 always remain over. Find the exact population.

41. Four numbers have one factor in common, and there is no other

factor which is common to any two of them. Their product is 65975910, and their least common multiple 30030. What is their common factor? (C.S.)

42. If the H.C.F. of two numbers is 7, and the L.C.M. is 420, find two numbers that satisfy the conditions, and show that there are three other pairs of numbers that satisfy them. (C.P.)

43. The G.C.M. of two numbers is 13, and the L.C.M. is 819. Find the product of the two numbers, and explain how the result is obtained. (K.S.)

44. Obtain the L.C.M. of 16, 18, 21, 33, and 42, and the greatest number of six digits, which, when divided by any one of those numbers, will leave a remainder 13. (C.P.)

45. £1 15*s.* 9*d.* is divided into equal portions, and then £28 18*s.* 6*d.* is divided into portions of just the same value. In how many ways can this be done, using no coin less than a penny?

46. If the product of two numbers is 2416700, and their G.C.M. is 65, what is their L.C.M.? (S.A.)

47. What numbers of 4 digits each can have 119 as their G.C.M. and 13923 as their L.C.M.? (C.S.L.)

48. A circular race-path is 1078 yds. round. Two men start, back to back, to run round it; the one runs at the rate of 12 mi., and the other 12½ mi. an hour. When and where will they first meet? (P.T.)

49. Three numbers, A, B, and C, have a common prime factor; also 2 is a common factor of A and B, 3 of B and C, and 7 of C and A. There is no other factor common to any two of the three numbers. The L.C.M. of A, B, and C is 903210, and their continued product 20067519780. Find the common prime factor, and write down one set of values of A, B, and C. (C.P.)

50. The first of a series of cogged wheels, working into each other in a straight line, has a certain number of teeth; the number of teeth in the second is to that of the first as 6 : 7; of the third to the second as 5 : 6; and of the fourth to the second as 2 : 3. If the wheels are set in motion, how many revolutions must each wheel make before they are simultaneously in their original positions? (R.M.A.)

CHAPTER VI.

VULGAR FRACTIONS.

A Fraction is a quantity which consists of one or more *equal* parts of a unit or whole.

"Fraction" is derived from the Lat. *frango, fractum*, I break. The word thus refers to the *breaking up* of the unit into a number of parts, which it should be observed must be *equal* parts.

Because the whole line AD is divided into three *equal* parts, we say that AB is one-third of the whole line. But insomuch as the line EH is broken up into three *unequal* parts, we cannot say that EF or FG, or GH is one-third of the whole EH.

The fraction $\frac{3}{4}$ denotes two things:—

1. That the unit is divided into *four* equal parts.
2. That *three* of those equal parts are taken.

Thus a fraction has two distinct terms:—

1. **The Denominator,** which shows into how many equal parts the unit is divided.

2. **The Numerator,** which shows how many of those equal parts have been counted or taken to form the fraction.

The *denominator* (Lat. *nominare*, to name) gives its *name* to the fraction. In the fraction three-fifths ($\frac{3}{5}$), the word "*fifths*" describes or *names* the "three" just as the word "hats" describes or names "*three*" in the expression "*three hats.*"

The *numerator* (Lat. *numerare*, to count) is the strictly numerical part of a fraction. The numerator is a *number*, whereas the denominator is a *name*.

$$\frac{\text{Numerator}}{\text{Denominator}} = \frac{\textit{Number} \text{ of parts to be taken}}{\textit{Name} \text{ of the parts into which the whole is divided}}$$

A fraction may be regarded as expressing the division of the numerator by the denominator. Thus the fraction $\frac{3}{4}$ may be viewed as representing that the number "three" is to be divided into four equal parts.

Kinds of Fractions.—The several varieties of fractions may be distinguished by formulating some *basis of classification.*

I. *Fractions classified according to the nature of their denominators.*

1. *Vulgar Fractions.*—Those with a denominator which is not a power of 10, as $\frac{1}{3}$, $\frac{2}{7}$, $\frac{5}{9}$.

2. *Decimal Fractions.*—Those with a denominator which is a power of 10, as $\frac{1}{10}$, $\frac{3}{100}$, $\frac{17}{1000}$.

The above distinction is seldom observed.

Generally such quantities as $\frac{1}{10}$, $\frac{3}{100}$, are termed *vulgar fractions*, and the term *decimal fraction* is reserved for such expressions as 0·25, etc. It would be perhaps better to view such expressions as part of our ordinary system of notation, and to call them simply *decimals*. (See Chap. VIII.)

Note that 10 is *a* power of ten—the *first* power.

Vulgar means usual, ordinary, or common.

Decimal means pertaining to tens or tenths.

II. *Fractions classified according to the relative value of their terms* (i.e. *of their numerators and denominators*).

1. *Proper Fractions.*—Those in which the numerator is *less* than the denominator, as $\frac{2}{5}$, $\frac{12}{17}$.

2. *Improper Fractions.*—Those in which the numerator is *equal* to or *greater* than the denominator, as $\frac{3}{3}$, $\frac{5}{3}$, $\frac{17}{12}$.

Such quantities are really not fractions at all; hence their name.

III. *Fractions classified according as their terms are simple whole numbers or fractional whole numbers.*

1. *Simple Fractions.*—Those in which both terms are simple whole numbers, as $\frac{5}{7}$, $\frac{14}{5}$.

2. *Complex Fractions.*—Those in which one or both terms are fractional numbers, as $\frac{2}{3\frac{1}{5}}$, $\frac{\frac{3}{5}}{2}$, $\frac{\frac{3}{5}}{3\frac{1}{5}}$.

The term *Compound Fraction* must be distinguished from the term *Complex Fraction*. A compound fraction is a fraction of a fraction, *e.g.* $\frac{3}{4}$ of $\frac{4}{5}$.

The term *Fraction* may be contrasted with the term *Integer*.

If a quantity consists of a whole number and a fraction, it is called a *Mixed Number*, *e.g.* $4\frac{5}{6}$.

Note that $4\frac{5}{6}$ means $4 + \frac{5}{6}$, not $4 \times \frac{5}{6}$: a point sometimes lost sight of in solving complex fractions.

Reduction of Fractions.

Investigations on the effects produced by altering the numerator or denominator of a fraction lead to some important results, which may be represented thus :—

Examples.	Denominator.	Numerator.	Effect on value of whole fraction.
$\frac{1}{7}$; $\frac{2}{7}$; $\frac{3}{7}$	Constant	Increased	Increased
$\frac{3}{5}$; $\frac{2}{5}$; $\frac{1}{5}$		Decreased	Decreased

Increasing / Decreasing the numerator means taking more / less of the equal parts named by the denominator, and thus must mean increasing / decreasing the value of the fraction.

Examples.	Numerator.	Denominator.	Effect on value of whole fraction.
$\frac{2}{7}$; $\frac{2}{9}$; $\frac{2}{11}$	Constant	Increased	Decreased
$\frac{2}{7}$; $\frac{2}{5}$; $\frac{2}{3}$	Constant	Decreased	Increased

Increasing / Decreasing the denominator means decreasing / increasing the magnitude of each of the equal parts into which the whole is divided, and consequently decreasing / increasing the value of the fraction.

From the above we can draw the following important deduction :—

The value of a fraction is not altered by multiplying / dividing Numerator and Denominator by the same number.

By multiplying / dividing the numerator by a given number, we increase / decrease the fraction a certain number of times, but by multiplying / dividing the denominator by the same number, we decrease / increase the fraction the same number of times. Hence the value of the fraction is unaltered.

Examples.—

$$\frac{3}{5} = \frac{3 \times 2}{5 \times 2} = \frac{6}{10}$$

$$\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$$

Cancelling.—The division of both numerator and denominator by a common factor is called cancelling.

The above examples deal with the effect produced by *multiplying* or *dividing* numerator and denominator by the same number. The effects produced by *adding* equal quantities to, or *subtracting* equal quantities from, numerator and denominator might be noticed.

Effects of Addition.—Take $\frac{2}{3}$: add unity both to numerator and denominator, and we obtain $\frac{3}{4}$, which is greater than $\frac{2}{3}$. If to $\frac{2}{3}$ we add 100 to both parts we get $\frac{102}{103}$, which is again greater than $\frac{2}{3}$.

In adding 100 to the denominator, we diminished by over 33 times the value

of the parts into which the whole is divided. In adding 100 to the numerator we increased the number of parts taken by over 50 times. Hence we have—

$$\left.\begin{array}{l}\text{an increase of over 50 times}\\ \text{a decrease of over 33 times}\end{array}\right\}\ \text{and a consequent } \textit{increase} \text{ in the value of the whole fraction.}$$

Effects of Subtraction.—By taking $\frac{102}{103}$ and subtracting 100 from numerator and denominator we obtain $\frac{2}{3}$, a decrease in the value of the fraction.

Reduction to a Common Denominator.

If we wish to add 17*s.* to 17*d.* we first reduce the two quantities to the same name, *i.e.* to a common denominator. Similarly, if we wish to add $\frac{1}{3}$ and $\frac{1}{5}$ we must reduce them to a common denominator.

$$\frac{1}{3} = \frac{1 \times 5}{3 \times 5} = \frac{5}{15} \qquad \frac{1}{5} = \frac{1}{5} \times \frac{3}{3} = \frac{3}{15}$$

$$\frac{1}{3} + \frac{1}{5} = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}$$

The two fractions have been reduced to equivalent fractions, having a common denominator 15.

Least Common Denominator (L.C.D.).—A common denominator must be a multiple of the denominator of each of the fractions. If the common denominator is as small as possible, it must be the *least common multiple* of the denominators of the given fractions.

Comparison of Fractions.—A series of fractions can be compared, *i.e.* the greatest and least can be pointed out or all the fractions may be arranged in order of magnitude—

(*a*) By reducing the fractions to a common denominator, and then comparing their new numerators.

(*b*) By reducing the fractions to a common numerator, and then comparing their new denominators.

(*c*) By converting the numerators to unity, and then comparing the new denominators.

Example.—*Arrange in order of magnitude:* $\frac{3}{8}, \frac{5}{24}, \frac{7}{10}, \frac{9}{32}, \frac{11}{48}$.

Method (*a*).—*By reduction to a common denominator.* Reducing to a common denominator we have—

$$\frac{180}{480}, \frac{100}{480}, \frac{336}{480}, \frac{135}{480}, \frac{110}{480}$$

Arranging these in order of magnitude (beginning with the smallest)—

$$\frac{100}{480}, \frac{110}{480}, \frac{135}{480}, \frac{180}{480}, \frac{336}{480}$$

and substituting the original fractions, we obtain —

$$\frac{5}{24}, \quad \frac{11}{48}, \quad \frac{9}{32}, \quad \frac{3}{8}, \quad \frac{7}{10} \quad \textit{Ans.}$$

Method (*b*).—*By reduction to a common numerator.* [It is advisable that the *least* common numerator should be found, *i.e.* the L.C.M. of the numerators 3, 5, 7, 9, 11, which is 3465.]

Reducing to the least common numerator, we have—

$$\frac{3465}{9240},\ \frac{3465}{16632},\ \frac{3465}{4950},\ \frac{3465}{12320},\ \frac{3465}{15120}$$

Arranging these in order of magnitude (beginning with the smallest), we have—

$$\frac{3465}{16632},\ \frac{3465}{15120},\ \frac{3465}{12320},\ \frac{3465}{9240},\ \frac{3465}{4950}$$

and substituting the original fractions, we obtain—

$$\frac{5}{24},\quad \frac{11}{48},\quad \frac{9}{32},\quad \frac{3}{8},\quad \frac{7}{10}\quad \textit{Ans.}$$

Method (*c*).—*By conversion of the numerators to unity.*

To do this, we divide both numerator and denominator by the numerator.

$$\frac{1}{2\frac{2}{3}},\ \frac{1}{4\frac{4}{5}},\ \frac{1}{1\frac{3}{7}},\ \frac{1}{3\frac{5}{9}},\ \frac{1}{4\frac{4}{11}}$$

Arranging these in order of magnitude (beginning with the smallest), we have—

$$\frac{1}{4\frac{4}{5}},\ \frac{1}{4\frac{4}{11}},\ \frac{1}{3\frac{5}{9}},\ \frac{1}{2\frac{2}{3}},\ \frac{1}{1\frac{3}{7}}$$

and substituting—

$$\frac{5}{24},\quad \frac{11}{48},\quad \frac{9}{32},\quad \frac{3}{8},\quad \frac{7}{10}\quad \textit{Ans.}$$

Order of Magnitude.—In the above examples the fractions are arranged so that each is *less* than the one that follows it, *i.e.* they are said to be in *ascending order*. Fractions are said to be in *descending order* when each is *greater* than the one that follows it.

Addition and Subtraction of Vulgar Fractions.—In addition and subtraction of vulgar fractions it is—

(1) *Necessary* to bring the fractions to a common denominator—preferably the least common denominator.

(2) *Unnecessary* to bring mixed numbers to improper fractions.

If $11\frac{3}{4}d.$ is to be added to $2\frac{1}{2}d.$, we should not think of reducing the quantities to farthings; yet the reduction of mixed numbers to improper fractions for purposes of addition or subtraction is just as unnecessary and absurd. The sum of the fractions should be found, also the sum of the whole numbers, and then the sum of the whole.

Example.—

$$\begin{aligned} 7\tfrac{3}{11} + 3\tfrac{5}{8} + 3\tfrac{5}{6} + 1\tfrac{19}{22} &= 7 + 3 + 3 + 1 + \tfrac{3}{11} + \tfrac{5}{8} + \tfrac{5}{6} + \tfrac{19}{22} \\ &= 14 + \frac{72 + 165 + 220 + 228}{264} \\ &= 14 + \tfrac{685}{264} \\ &= 14 + 2\tfrac{157}{264} \\ &= 16\tfrac{157}{264}\quad \textit{Ans.} \end{aligned}$$

Frequently the working can be considerably shortened by manipulating the fractions *in pairs*—

$$\begin{aligned} 7\tfrac{3}{11} + 3\tfrac{5}{8} + 3\tfrac{5}{6} + 1\tfrac{19}{22} &= 14 + \tfrac{6}{22} + \tfrac{19}{22} + \tfrac{15}{24} + \tfrac{20}{24} \\ &= 14 + \quad \tfrac{25}{22} \quad + \quad \tfrac{35}{24} \\ &= 16 + \quad \tfrac{3}{22} \quad + \quad \tfrac{11}{24} \\ &= 16 + \quad \frac{36 \quad + \quad 121}{264} \\ &= 16\tfrac{157}{264} \quad \textit{Ans.} \end{aligned}$$

In subtraction it sometimes happens that the fractional part to be subtracted is greater than the fractional part of the mixed number from which it is to be taken.

Example.—

$$\begin{aligned} 13\tfrac{39}{50} - 8\tfrac{74}{75} &= 5 + \frac{117 - 148}{150} \\ &= 4 + \frac{150 + 117 - 148}{150} \\ &= 4 + \frac{267 - 148}{150} \\ &= 4\tfrac{119}{150} \quad \textit{Ans.} \end{aligned}$$

Since the subtrahend 148 is greater than the minuend 117, we "borrow" one whole number (*i.e.* $\frac{150}{150}$). The similarity of this process to that of "borrowing" in simple and compound subtraction will be evident.

Note that there is no need whatever to reduce the mixed numbers to improper fractions.

Combined Addition and Subtraction.—The method of dealing with a sum in combined addition and subtraction is best shown by an example.

Example.—

$$5\tfrac{1}{5} - 2\tfrac{5}{6} - 3\tfrac{3}{10} + 1\tfrac{3}{2} - 16\tfrac{1}{4} + 3\tfrac{1}{12} + 8\tfrac{1}{9}$$

(*a*) Reducing the improper fraction to a mixed number—

$$= 5\tfrac{1}{5} - 2\tfrac{5}{6} - 3\tfrac{3}{10} + 6\tfrac{1}{2} - 16\tfrac{1}{4} + 3\tfrac{1}{12} + 8\tfrac{1}{9}$$

(*b*) Separating whole numbers from fractions—

$$= 5 - 2 - 3 + 6 - 16 + 3 + 8 + \tfrac{1}{5} - \tfrac{5}{6} - \tfrac{3}{10} + \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{12} + \tfrac{1}{9}$$

(*c*) Summing whole numbers and grouping fractions where possible—

$$= 22 - 21 + \tfrac{1}{5} - \tfrac{5}{6} - \tfrac{3}{10} + \tfrac{1}{4} + \tfrac{7}{36}$$

(*d*) Still further grouping fractions and bringing the groups to a common denominator—

$$\begin{aligned} &= 1 + \tfrac{2}{10} - \tfrac{3}{10} - \tfrac{30}{36} + \tfrac{9}{36} + \tfrac{7}{36} \\ &= 1 - \quad \tfrac{1}{10} \quad - \tfrac{14}{36} \\ &= 1 - \quad \frac{18 \quad + 70}{180} \\ &= 1 - \tfrac{88}{180} \\ &= \tfrac{92}{180} = \tfrac{23}{45} \quad \textit{Ans.} \end{aligned}$$

Note.—In actual working, most of the steps should be worked mentally.

Multiplication of Fractions.

In multiplication of fractions two principles must be borne in mind—

1. If the denominator of a fraction be constant, the greater the value of the numerator, the greater the value of the fraction (see p. 61).

2. If the numerator of a fraction be constant, the greater the value of the denominator, the less the value of the fraction (see p. 62).

We shall now apply these principles to the following :—

Example.—*Multiply* $\frac{2}{3}$ *by* $\frac{4}{5}$.

Step 1.—*To multiply* $\frac{2}{3}$ *by* 4. $\frac{2}{3} \times 4 = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{8}{3}$. The same result could be obtained by multiplying the numerator by 4, thus $\frac{2}{3} \times 4 = \frac{2 \times 4}{3} = \frac{8}{3}$. But 4 is just five times $\frac{4}{5}$, and we had to multiply by $\frac{4}{5}$, not by 4: hence we have used a multiplier which is 5 times too large. Thus the product $\frac{8}{3}$ is 5 times too large. (Principle 1.) Consequently to obtain a correct answer we must divide our product by 5.

Step 2.—*To divide* $\frac{8}{3}$ *by* 5. If we multiply the denominator by 5, we shall make the whole fraction one-fifth of its former value (Principle 2)—

$$\frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$$

Combining the two operations—

$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15} \quad Ans.$$

Hence—

To find the product of two or more fractions, multiply the two or more numerators for a new numerator, and the two or more denominators for a new denominator.

The meaning of the Multiplication Sign in Multiplication of Fractions.

The general idea of multiplication is that of *increase*, but this holds good only in the case of multiplication by numbers greater than unity. In multiplication by unity the product is *the same* as the multiplicand, and in multiplication by a quantity less than unity the product is *less* than the multiplicand. Evidently an extended meaning must be given to multiplication where fractions are concerned.

Explanation 1.—As the multiplier represents only a part of a unit, the multiplication of a quantity by it means that we take the multiplicand that number of *parts of times* represented by the multiplier.

Objection.—The expression $\frac{2}{3} \times \frac{1}{2}$ means, according to the above, that $\frac{2}{3}$ is to be taken *half a time*. We can take $\frac{2}{3}$ once, or any whole number of times more than once, but we cannot take it less than once. We must take it once or leave it altogether ; we cannot half take it and half leave it.

Explanation 2.—The ordinary conception of multiplication applies only to the case of an integral multiplier. "Multiplication" by a fraction includes *division* as well as multiplication. The multiplication of $\frac{3}{5}$ by $\frac{1}{2}$ is equivalent to dividing $\frac{3}{5}$ into two equal parts. Similarly the multiplication of $\frac{3}{5}$ by $\frac{4}{5}$ is equivalent to *dividing* $\frac{3}{5}$ by 5, and then multiplying by 4.

A quantity "multiplied" in this way is consequently *decreased* when the multiplier is a proper fraction. It is increased only when the multiplier is an *improper* fraction, and improper fractions are really not fractions at all (p. 61).

Division of Fractions.

Example.—$\frac{5}{8} \div \frac{3}{7}$.

Divide $\frac{5}{8}$ by 3, *i.e.* divide $\frac{5}{8}$ into 3 equal parts (such that the sum of those three equal parts shall be equal to $\frac{5}{8}$). Applying Principle 2, p. 66, we get—

$$\frac{5}{8 \times 3} = \frac{5}{24}$$

which is correct, for—

$$\frac{5}{24} + \frac{5}{24} + \frac{5}{24} = \frac{15}{24} = \frac{5}{8}$$

But 3 is just 7 times greater than $\frac{3}{7}$; hence in taking 3 as a divisor we have taken a divisor which is 7 times too large.

Hence the result ($\frac{5}{24}$) obtained above is *too small*, and since the divisor was 7 times too large the result ($\frac{5}{24}$) must be just 7 times too small.

To make the result 7 times larger, we must multiply by 7 (Principle 1, p. 66)—

$$\frac{5}{24} \times 7 = \frac{35}{24} = 1\frac{11}{24} \quad \textit{Ans.}$$

Reviewing what has been done, we see that $\frac{5}{8}$ has been multiplied by 7 and divided by 3.

Therefore $\frac{5}{8} \div \frac{3}{7} = \frac{5}{8} \times \frac{7}{3} = \frac{35}{24} = 1\frac{11}{24}$ *Ans.*

Hence the following rule:—

To divide one fraction by another, invert the divisor and proceed as in multiplication.

The Meaning of the Division Sign in Division of Fractions.

The quotient $1\frac{11}{24}$ obtained above is *greater* than the dividend $\frac{5}{8}$. Division has hitherto been connected with *decrease*, but when the divisor is unity the quotient is neither more nor less than the dividend, and when the divisor is less than unity, the quotient is greater than the dividend.

Hitherto we have attached two meanings to an ordinary division (*e.g.* 12 ÷ 4), viz.—

(1) How many times is 4 contained in 12?

(2) Distribute 12 into 4 equal parts.

But, applying meaning (1) to such an expression as $\frac{1}{2} \div \frac{3}{4}$, since $\frac{3}{4}$ is larger than $\frac{1}{2}$, it cannot be *contained* in it.

And, applying meaning (2) to the same expression, we find that $\frac{1}{2}$ cannot be *distributed* into $\frac{3}{4}$ equal parts.

The question now arises: *What meaning can we attach to division that shall be applicable both to fractions and integers?* We know that the following is always true:—

Divisor × quotient = dividend

Hence the interpretation we can give to such an expression as $12 \div 4$, and also to such an expression as $\frac{1}{2} \div \frac{3}{4}$, is—

By what quantity must we multiply 4 (or $\frac{3}{4}$) in order to produce 12 (or $\frac{1}{2}$)?

Concrete and Diagrammatic Representations of Some of the Principles involved in the Solution of Vulgar Fractions.

To change an Improper Fraction into a Mixed Number.

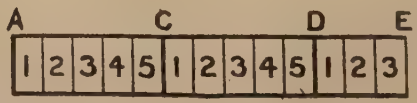
Example.—Convert $\frac{13}{5}$ into a mixed number.

I. *By cubes.*—The unit is divided into 5 equal parts, and 13 such parts are taken to form the fraction. Obviously there is more than one unit in the fraction.

A unit contains 5 equal cubes or parts. Take 13 such cubes and build up units from them. There are 2 units and 3 cubes left; *i.e.* 2 units and $\frac{3}{5}$ of a unit, which may be expressed as $2 + \frac{3}{5}$, and *is* expressed thus, $2\frac{3}{5}$.

The converse method can be similarly demonstrated.

II. *By diagram.*—Draw a rectangle and divide it into 13 equal parts.



AC = 1 unit = 5 parts
CD = 1 unit = 5 parts
∴ AD = 2 units = 10 parts
And DE = $\frac{3}{5}$ units
∴ AE = $2\frac{3}{5}$ units
But AE = 13 parts, or $\frac{13}{5}$ units
∴ $\frac{13}{5}$ = $2\frac{3}{5}$

Addition and Subtraction of Fractions.

Add $\frac{2}{3}$ and $\frac{3}{4}$. Subtract $\frac{2}{3}$ from $\frac{3}{4}$.

I. *By cubes.*—Take 12 cubes of equal size and build these up into a unit. Divide these into 3 *equal parts*, as A, B, and C. Then

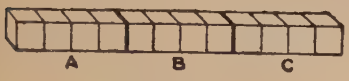


Fig. 1.

Fig. 2.

separate 2 of these parts as in Fig. 2. There are now 2 parts out of 3 or 8 parts out of 12; *i.e.* there are $\frac{2}{3}$ or $\frac{8}{12}$.

Now take 12 similar cubes and divide them into 4 *equal parts*, as D, E, F, and G. Now separate 3 of these parts, as in Fig. 4. Three

Fig. 3.

Fig. 4.

parts out of the 4 have now been removed, or 9 parts out of 12;

i.e. there are $\frac{3}{4}$ or $\frac{9}{12}$

Addition.—Count the cubes. There are 8 in one group and 9 in the other; or—

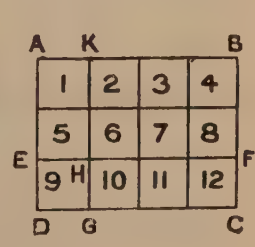
$$8 + 9 = 17$$

$$\textit{i.e.}\ \tfrac{2}{3} + \tfrac{3}{4} = \tfrac{8}{12} + \tfrac{9}{12} = \tfrac{17}{12} = \tfrac{12}{12} + \tfrac{5}{12} = 1 + \tfrac{5}{12} = 1\tfrac{5}{12}$$

Subtraction.—There are 9 cubes in one and 8 cubes in the other group.

Then $\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$

II. *By diagram.*—Draw a line AB, and divide it into four equal parts. Draw AD at right angles to AB, and divide it into three equal parts of the same length as the parts in AB. Complete the parallelogram.



Then AEFB = $\frac{2}{3}$ or $\frac{8}{12}$
and KGCB = $\frac{3}{4}$ or $\frac{9}{12}$

Adding, we have $\frac{2}{3} + \frac{3}{4}$ or $\frac{8}{12} + \frac{9}{12} = \frac{17}{12} = 1\frac{5}{12}$.
Subtracting, we have $\frac{3}{4} - \frac{2}{3}$,

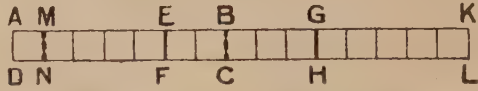
$$\textit{i.e.}\ \text{KGCB} - \text{AEFB} = \tfrac{3}{4} - \tfrac{2}{3} = \tfrac{9}{12} - \tfrac{8}{12} = \tfrac{1}{12}$$

Multiplication of fractions.

To multiply a vulgar fraction by an integer.

By diagram. **Example.**—$\frac{5}{7} \times 3$.

As the numerator will contain 15 units, 15 divisions will be required.



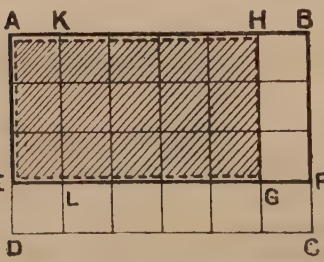
Let Fig. ABCD = unity, *i.e.* it will contain 7 divisions.

Then AMND = $\frac{1}{7}$ of a unit
and AEFD = $5 \times$ AMND = $\frac{5}{7}$ of a unit
also $3 \times$ AEFD = $3 \times \frac{5}{7}$ = AKLD = $\frac{15}{7}$.

To multiply a Vulgar Fraction by a Vulgar Fraction.—Take $\frac{5}{6}$ of $\frac{3}{4}$ as an example.

By diagram.—In constructing the figure, it should be noticed that the two denominators must be multiplied together to determine the number of little squares necessary, *i.e.* they represent a rectangle 6 by 4.

Draw AB and divide it into 6 equal parts. Draw AD at right angles to AB and equal to 4 of these equal parts. Complete the rectangle ABCD. Through each of the points of division draw lines parallel to AB and AD. Then let ABCD = unity.



Then AEFB = $\frac{3}{4}$ of ABCD
and AELK = $\frac{1}{6}$ of AEFB
= $\frac{1}{6}$ of $\frac{3}{4}$
and AEGH = $\frac{5}{6}$ of AEFB
= $\frac{5}{6}$ of $\frac{3}{4}$

Then AEGH is the figure required.

But AEGH contains 15 squares, and the total number of squares is 24, $\therefore$ AEGH is $\frac{15}{24}$ of the whole, *i.e.* $\frac{5}{6} \times \frac{3}{4} = \frac{15}{24}$.

To prove that the multiplier and the multiplicand can be interchanged without altering the product.—To show that multiplicand $\times$ multiplier = multiplier $\times$ multiplicand.

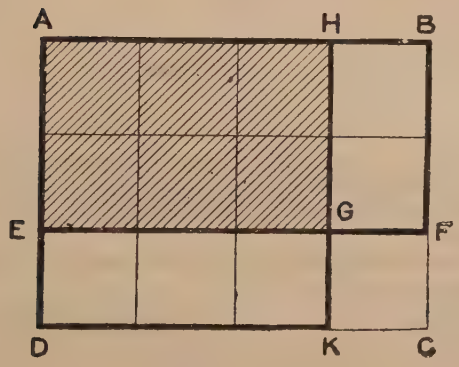
$$i.e.\ \frac{2}{3} \times \frac{3}{4} = \frac{3}{4} \times \frac{2}{3}$$

The diagram (p. 71) is useful to show how the product of two or more fractions may be *smaller* than either of the factors.

This is otherwise a hard matter to grasp at first, as it is apparently contrary to all previous experience.

Let ABCD = the unit.

Then AEFB = $\frac{2}{3}$ of ABCD
and AEGH = $\frac{3}{4}$ of AEFB
= $\frac{3}{4}$ of $\frac{2}{3}$ of ABCD . . . . . (i.)
= $\frac{6}{12}$.



Again, let ABCD be the unit.

Then ADKH = $\frac{3}{4}$ of ABCD
and AEGH = $\frac{2}{3}$ of ADKH
= $\frac{2}{3}$ of $\frac{3}{4}$ of ABCD . . . . . (ii.)
= $\frac{6}{12}$.

Hence, comparing (i.) and (ii.), we see that $\frac{3}{4} \times \frac{2}{3} = \frac{2}{3} \times \frac{3}{4}$, for both equal $\frac{6}{12}$; *i.e.* the multiplier and multiplicand can both be interchanged without altering the product.

Harder Examples.—Show that $\frac{1}{2}(\frac{1}{3} - \frac{1}{4}) = \frac{1}{24}$ by a diagram. (K.S.)

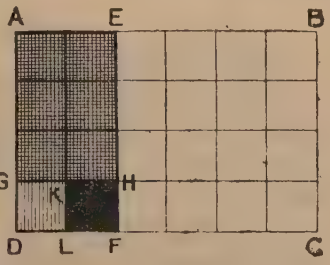
Note.—As the denominator of the product is 24, that number of squares will be required.

Let ABCD = unity = $\frac{24}{24}$.

Then each square = $\frac{1}{24}$

Now AEFD = 8 squares = $\frac{8}{24} = \frac{1}{3}$

and AGHE = 6 squares = $\frac{6}{24} = \frac{1}{4}$



Then AEFD − AEHG = GDFH = 2 squares, and the $\frac{1}{2}$ of 2 squares = 1 square (KHFL), ∴ $\frac{1}{2}$(AEFD − AEHG) = KHFL = $\frac{1}{24}$; *i.e.* $\frac{1}{2}(\frac{1}{3} - \frac{1}{4}) = \frac{1}{24}$.

Division of fractions.

1. *The case in which the divisor is less than the dividend.*

Example.—$2\frac{1}{4} \div \frac{1}{4}$.

Dividend $= 2\frac{1}{4} = \frac{9}{4}$ *Divisor* $= \frac{1}{4}$

The dividend is represented by 9 squares, the divisor by 1, and a glance at the above figures makes it evident that the divisor must be repeated 9 times to make up the dividend. Hence the required quotient is 9.

2. *The case in which the divisor is greater than the dividend.*

Example.—$\frac{1}{4} \div 2\frac{1}{4}$.

Dividend $= \frac{1}{4}$ *Divisor* $= 2\frac{1}{4} = \frac{9}{4}$

The question is: How much of the divisor must be taken in order to equal the dividend? The answer is evidently $\frac{1}{9}$. Hence the quotient $= \frac{1}{9}$.

THEORY.

1. Define a fraction.

2. Name the different kinds of vulgar fractions with which you are acquainted, and give an example of each kind. (C.P., C.W., and P.T.)

3. Distinguish between *proper*, *improper*, *compound*, *complex*, and *decimal fractions*. (C.P., P.T., and O.S.L.)

4. Define factor, numerator, denominator. Explain as clearly as you can why the value of a fraction is increased if its denominator is diminished. (C.W.S. and C.S.)

5. How would you show that the numerator and denominator of a fraction can both be multiplied by the same number without altering the value of the fraction? (C.W., C.M., S.K., and C.PR.)

6. Assuming that if we multiply the numerator and denominator of a fraction by the same number we do not alter the value of the fraction, show how you would apply this assumption to finding the sum of $\frac{2}{3}$ and $\frac{1}{5}$. (P.T. and C.S.)

7. In what ways may the numerator and denominator of a fraction be simultaneously altered without any change in the value of the fraction? (C.S.)

8. If equal amounts be added to the numerator and the denominator of a fraction, show whether the fraction is increased or diminished by the equal additions. (S.A.)

9. Prove that if the numerator and denominator of a fraction be divided by the same number, the value of the fraction is not altered. (C.W.)

10. State and explain the rule for adding fractions together whose denominators are different. (C.PR.)

11. A boy, in answer to a question, says that $\frac{1}{3} + \frac{1}{4} = \frac{1}{12}$. Explain clearly how you would try to convince him of his mistake. (C.W. and K.S.)

12. Explain clearly why the addition of fractions with different denominators is effected by reducing each of the fractions to their least common denominator. (C.M. and C.P.)

13. Prove the rule for the addition of fractions: *e.g.* $\frac{5}{8} + \frac{1}{6}$. (C.W.)

14. Explain, by an example, the reason of the rule for multiplying one arithmetical fraction by another. (L.M. and C.W.)

15. Explain the enlarged meanings of the symbols $\times$ and $\div$ in $\frac{2}{3} \times \frac{9}{7}$ and $\frac{2}{3} \div \frac{9}{7}$. (C.S., C.W., and C.M.)

16. Explain why $\frac{3}{5}$ of $\frac{3}{4}$ equal $\frac{1}{2}$, giving reasons for the steps taken in the simplification. (S.K.)

17. What is meant by multiplication when the multiplier is (i.) 1, (ii.) 0, (iii.) a vulgar fraction? Can you give any meaning to multiplication when the multiplier is a concrete quantity? If so, what is it? (S.A.)

18. Prove that, in order to multiply any multiplicand by a vulgar fraction, we may either first multiply by the numerator and then divide by the denominator, or else first divide by the denominator and then multiply by the numerator. Illustrate your answer by performing the multiplication of $\frac{33}{33}$ (i.) by $\frac{4}{11}$, (ii.) by $\frac{5}{13}$, (iii.) by $\frac{7}{11}$. (S.A.)

19. State and prove the rule for the division of one fraction by another. (C.H.L., C.M., C.W., and K.S.)

20. Show that division by $\frac{2}{3}$ is equivalent to multiplication by $\frac{3}{2}$, and division by $\frac{5}{4}$ is equivalent to multiplication by $\frac{4}{5}$.

21. Write out a rule for the division of one compound fraction by another. (C.M.)

22. What is meant by dividing $\frac{2}{3}$ by $\frac{1}{3}$? (P.T.)

23. Divide $\frac{3}{4}$ by $\frac{1}{2}$, and explain the difference, if any, between division by a fraction and division by an integer. (C.P.)

24. Explain fully why $\frac{3}{4}$ multiplied by $\frac{1}{2}$ gives the same result as $\frac{3}{4}$ divided by 2. (C.W.S.)

25. In dividing a fraction by a whole number, show that we multiply the denominator or divide the numerator by that whole number. (C.W.)

26. Write out the rule for division of fractions, and explain why, in the division of $\frac{5}{8}$ by $\frac{1}{3}$, the quotient is greater than the dividend. (P.T. and C.M.)

27. When you have to divide a fraction by an integer, the quotient may be obtained either by dividing the numerator or by multiplying the denominator by the given divisor. Give an example of this. (P.E.A.)

28. Point out the error in the following answer: *Question.*—Take $\frac{1}{12}$ from the sum of $\frac{2}{3}$ and $\frac{1}{4}$. *Answer.*—$\frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12} - \frac{1}{12} = \frac{10}{12} = \frac{5}{6}$. (P.T.)

29. Write down two fractions, each equal to $\frac{6}{7}$, with complex numerators, and with 11 and 54 as denominators respectively. Show, taking $\frac{6}{7}$, $\frac{9}{11}$, $\frac{13}{54}$ as examples, that, if the sum of the numerators of *any* fractions be taken for a new numerator, and the sum of the denominators for a new denominator, the fraction so formed is intermediate in value between the greatest and least of the fractions chosen. (I.I.)

30. Point out the unnecessary steps in (*a*) and (*b*) and the fault in (*c*), giving in each case a corrected edition—

(*a*) $1\frac{2}{13} \times 7 = \frac{15}{13} \times \frac{7}{1} = \frac{84}{13} = 6\frac{6}{13}$.

(*b*) $1\frac{2}{13} \times 13 = \frac{15}{13} \times \frac{13}{1} = \frac{156}{13} = 12$.

(*c*) $\frac{4}{5} + \frac{5}{6} = \frac{24}{30} + \frac{25}{30} = \frac{49}{30} - 1 = \frac{49}{30} - \frac{30}{30} = \frac{19}{30}$.

Write the question to which the result given in (*c*) is an answer. (P.T.)

31. How do the following pieces of work show a want of intelligence?

(1) $\frac{7}{8} \times 8 = \frac{7}{8} \times \frac{8}{1} = \frac{56}{8} = \frac{7}{1} = 7$.

(2) $\frac{\frac{1}{18} + \frac{1}{10}}{\frac{1}{20} + \frac{1}{36}} = \frac{\frac{28}{180}}{\frac{56}{720}} = \frac{28}{180} \times \frac{720}{56}$, etc.

(3) $2\frac{1}{17} + 1\frac{1}{9} + 3\frac{1}{34} + 3\frac{8}{9}$

$= \frac{35}{17} + \frac{10}{9} + \frac{103}{34} + \frac{35}{9}$ (L.C.M. = 306)

$= \frac{35 \times 18 + 10 \times 34 + 103 \times 9 + 35 \times 34}{306}$, etc.

Work the second and third of these sums properly. (C.M.)

32. Explain the meaning of $\frac{3}{4}$ and $\frac{6}{8}$, and show that they are equal to one another. (C.J.)

33. Explain how the fraction $\frac{9}{12}$ is unaltered in value by cancelling. (C.W.S.)

34. Show by means of a diagram or otherwise that the value of a fraction is not altered when its numerator and denominator are both multiplied by the same whole number. (P.E.A.)

35. Show by a diagram that if the numerator and denominator of a fraction be both multiplied by the same number, the value of the fraction will not be altered. (P.T.)

36. Show that $\frac{3}{4} \div 6 = \frac{1}{8}$. (P.T.)

37. If the numerator and denominator of a fraction be *multiplied* or *divided* by the same number, the value of the fraction is not altered; thus $\frac{1}{3} = \frac{4}{12}$. Prove this by a diagram, taking a straight line as the unit of length. (O.P.)

38. Make a diagram to show that $\frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$. (K.S.)

39. Show by a diagram that $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. (C.W.)

40. Illustrate by a straight line that $\frac{3}{8}$ of $1 = \frac{1}{8}$ of 5, and apply the principle to prove the processes in multiplication and division of ordinary fractions, working the following: $\frac{9}{11} \times \frac{5}{7}$ and $1\frac{13}{15} \div \frac{7}{8}$. (R.U.I.)

41. How would you prove by a diagram that $\frac{3}{7} = \frac{21}{49}$ and $\frac{5}{6} + \frac{3}{4} = 1\frac{7}{12}$? (P.T.)

42. Show by a diagram that $\frac{1}{5}$ of $\frac{1}{7} = \frac{1}{21}$. Define what is meant by the fractions $\frac{3}{7}$ and $\frac{15}{35}$, and show by a diagram that $\frac{3}{7} = \frac{15}{35}$. (K.S.)

43. How are fractions compared? Show by a diagram that if the numerator and denominator of the fraction $\frac{3}{4}$ be each multiplied by 2, its value is not altered. (P.T.)

EXERCISE VIII.

Preliminary Exercises in Vulgar Fractions.

1. Reduce to its lowest terms $\frac{2108}{5813}$. (P.E.A.)

2. Reduce to its lowest terms $\frac{729}{1917}$. (C.PR.)

3. Reduce the following fractions to whole or mixed numbers: $\frac{2516}{13}$, $\frac{427}{11}$, $\frac{849}{13}$. (C.P.)

4. Explain two methods of finding which is the greatest of three given fractions, and use one of the methods in the case of the three fractions, $\frac{3}{62}$, $\frac{7}{114}$, and $\frac{4}{83}$. (C.P.)

5. Reduce to a common denominator, and arrange in order of magnitude, the following fractions: $\frac{5}{27}$, $\frac{11}{21}$, $\frac{5}{6}$, $\frac{4}{15}$, and $\frac{3}{5}$. (P.P.)

6. Find the greatest and the least of the following fractions: $\frac{11}{12}$, $\frac{29}{30}$, $\frac{7}{16}$, $\frac{17}{18}$, $\frac{47}{48}$. (C.W.S.)

7. Arrange the following fractions into three groups: (1) Those which are greater than $\frac{1}{2}$; (2) those which are between $\frac{1}{2}$ and $\frac{1}{4}$; (3) those which are less than $\frac{1}{4}$. Do not find any L.C.M., if you can manage without it.

$$\frac{2\frac{1}{2}}{4},\ \frac{12}{25},\ \frac{2}{3},\ \frac{1\frac{1}{3}}{8},\ \frac{7}{16},\ \frac{7}{15},\ \frac{7}{12},\ \frac{7}{29} \qquad \text{(P.T.)}$$

8. Determine which is the greatest, and which is the least, of the three quantities—

$$\frac{2\frac{1}{2}}{88\frac{1}{4}},\ \frac{2\frac{3}{4}}{97\frac{5}{8}},\ \text{and}\ \frac{3}{105\frac{1}{2}}$$

Addition and Subtraction of Vulgar Fractions.

9. Add together $\frac{2}{3}$, $\frac{8}{10}$, $\frac{7}{12}$, $\frac{2}{13}$, and $\frac{17}{30}$. (C.PR.)

10. Add together $\frac{1}{3}+\frac{2}{7}+\frac{3}{9}+\frac{4}{103}$. (O.B.)

11. Reduce $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}-\frac{13}{6}-\frac{1}{20}$ to a fraction in its lowest terms. (C.PR.)

12. Show how to perform the following subtractions (i.) (ii.), without reducing any mixed number to an improper fraction: (i.) $44\frac{1}{19}-43\frac{17}{1}$; (ii.) $45\frac{17}{47}-43\frac{11}{19}$. (P.T.)

13. Reduce the fractions $\frac{50}{23331}$, $\frac{28}{169989}$, $\frac{39}{153873}$ to a common denominator; then add them together, and bring the result to its lowest terms. (C.H.L.)

14. Which is greater—the difference between $5\frac{23}{32}$ and $2\frac{3}{7}$, or the sum of $\frac{7}{8}+\frac{5}{14}+\frac{1}{6}$, and by how much? (P.T.)

15. What fraction subtracted from $\frac{3}{5}$ of $2\frac{3}{4}$ will make $\frac{5}{8}$ of $1\frac{3}{5}$. (P.T.)

16. Arrange in order of magnitude the fractions $\frac{19}{37}$, $\frac{17}{60}$, $\frac{107}{380}$, and find by how much the sum of the greatest and least exceeds or falls short of twice the other fraction. (C.H.L.)

17. The difference between two numbers is $2\frac{62}{91}$, and the smaller of them is $\frac{54\frac{2}{7}}{\frac{1}{5}\text{ of }8\frac{1}{3}}$: what is the larger number? (P.T.)

18. The sum of three fractions is 2: the first exceeds the second by $\frac{1}{2}$; and the second exceeds the third by $\frac{1}{4}$: find them. (L.M.)

19. Find the average of all proper fractions having integral numerators and 20 for the denominator (0 as a numerator being excluded). (C.W.)

Multiplication and Division of Fractions.

20. Convert $2\frac{2}{15}$ of $\frac{3}{5}$ of $\frac{7}{8}$ into a single fraction.

21. Multiply $3\frac{9}{11}$ of $\frac{3}{20}$ of $6\frac{1}{2}$ by $2\frac{2}{5}$ of $\frac{22}{36}$. (R.M.A.)

22. Find the value of $\frac{3}{14}$ of $\frac{7}{23}$ of $\frac{13}{18}$; and multiply it by $(\frac{17}{42} - \frac{1}{13} + \frac{19}{117})$. (C.P.)

23. Subtract $13\frac{3}{4}$ from $17\frac{3}{4}$. Multiply the same two fractions, and add the two results together. (C.P.)

24. Divide $3\frac{5}{12}$ by $4\frac{11}{28}$. (R.M.A.)

25. Divide $2\frac{2}{15} + 2\frac{27}{35} - 3\frac{4}{21}$ by $2\frac{5}{21} + 3\frac{1}{8} - 4\frac{6}{7}$. (C.PR.)

26. Divide $1456\frac{19}{9234}$ by $\dfrac{5203\frac{9}{136}}{17589\frac{3}{19}}$. (C.P.)

27. Subtract $2\frac{13}{13}$ from $9\frac{34}{35}$, and divide the remainder by $11\frac{1}{13}$. (O.S.L.)

28. Divide the product of $4\frac{1}{11}$ and $2\frac{13}{16}$ by their difference. (O.S.L.)

29. Divide $2\frac{1}{3} + \frac{2}{7}$ of $8\frac{3}{4}$ by $2\frac{3}{4}$ of $2\frac{8}{9} + \frac{3}{10}$. (O.S.L.)

30. Find out which is the greater of $\frac{56}{143}$ and $\frac{72}{187}$, and divide $\frac{16}{11}$ by their difference. (S.K.)

31. Add together $3\frac{1}{3}$, $\frac{4}{5}$, 7; divide what you get by $\frac{2}{3}$; and then see how much you must add to the quotient to make 100. (P.P.)

32. Obtain (i.) the sum, (ii.) the difference, (iii.) the product of the mixed fractions $17\frac{2}{3}$ and $27\frac{3}{5}$; and express all your results in the form of mixed fractions. (C.P.)

33. What number divided by the product of $\frac{10}{11}$, $\frac{1}{3}$, and $\frac{7}{2}$ will give the difference between $3\frac{1}{3}$ and $2\frac{5}{8}$? (C.W.S.)

34. What fraction multiplied by $3\frac{1}{2}$ of $\frac{4}{5}$ produces $2\frac{2}{3}$ of $\frac{4}{13}$? (R.M.A.)

35. What number multiplied by $\frac{3}{13}$ of 3 will equal $\frac{12}{16}$ of $8 \div \frac{26}{57}$? (C S.)

36. How often can you take $\frac{5}{11}$ from $13\frac{4}{9}$? What will be left? (K.S.)

37. How many times is the difference between $\dfrac{\frac{2}{3} \text{ of } \frac{3}{4}}{\frac{1}{2}}$ and $\dfrac{\frac{2}{5} \text{ of } \frac{5}{8}}{\frac{2}{5} \text{ of } 4\frac{1}{2}}$ contained in their sum? (K.S.)

38. By what must $5\frac{2}{3}$ be increased to get $9\frac{1}{3}$? By what must it be multiplied to get the same result? (C.P.)

39. Find (by fractions) a number such that, if one twenty-fourth part of it be taken away, the remainder is 253. (P.E.A.)

40. The product of three numbers is $3057\frac{183}{256}$; the greatest of them is $17\frac{3}{4}$; and of the others the greater is not more than $3\frac{1}{16}$ of the other, nor less than $2\frac{1}{4}$ of it: find the limits within which the least must lie. (C.S.)

CHAPTER VII.

SIMPLIFICATION OF VULGAR FRACTIONS.

Meaning and Force of the Addition, Subtraction, Multiplication, and Division Signs.

In simplifying fractions it is necessary (among other things) to bear in mind—

1. The *meanings* of the signs employed.
2. The *order* in which the operations indicated by those signs are to be performed.

The word "of."

1. *Its meaning.*
   (*a*) Multiplication.
   (*b*) *Very* close connection between the fraction following it and the one preceding it.

2. *Its order.*—The multiplication which it indicates *must* be performed before addition, subtraction, or division is attempted. Thus "of" is said to take precedence of all signs.

The symbol ×

1. *Its meaning.*
   (*a*) Multiplication.
   (*b*) Connection between the fractions preceding and following it *less* close than when "of" is used.

2. *Its order.*—The multiplication which it indicates must be performed before addition or subtraction is attempted.

The symbol ÷

1. *Its meaning.*—Division.
2. *Its order.*—The division which it indicates must be performed before addition or subtraction is attempted.

The symbol — $\left(\text{as in } \frac{3\frac{1}{2}}{1\frac{3}{4}}\right)$

1. *Its meaning.*—Division. Indicates that the *whole* of the numerator is to be divided by the *whole* of the denominator.

2. *Its order.*—The division which it indicates must not be performed until the numerator and denominator have been reduced to simple or improper fractions. (See Example at foot of page.)

If the compound fraction consists wholly of fractions preceded by × or ÷, the × or ÷ signs are taken to refer only to those fractions which *immediately* follow them.

The symbol +

1. *Its meaning.*—Addition.
2. *Its order.*—The operation which it indicates should be performed after all multiplication and division signs have been eliminated.

The symbol −

1. *Its meaning.*—Subtraction.
2. *Its order.*—Same as the addition sign.

To sum up: of the four principal signs +, −, ×, ÷, the multiplication and division signs must be "cleared" or dealt with before the operations indicated by the addition and subtraction signs are attempted.

The + sign connects *terms:* the × sign merely connects *factors.* Viewed in this light, the sign + is a more important sign than the sign × ; hence some call + and − Strong Signs, and × and ÷ Weak Signs.

The following examples will illustrate some of the preceding points :—

Example.—

$\frac{7}{8}$ of $\frac{6}{7} \div \frac{4}{5}$ of $\frac{8}{9}$

$$= \frac{\cancel{7}}{8} \times \frac{\overset{3}{\cancel{6}}}{\cancel{7}} \times \frac{5}{4} \times \frac{9}{\underset{4}{\cancel{8}}}$$

$$= \frac{135}{128} = 1\frac{7}{128} \quad \textit{Ans.}$$

Note (1).—That "of" connects $\frac{8}{9}$ so closely to $\frac{4}{5}$ that it is inverted with it. The effect of the "of" is the same as though $\frac{4}{5}$ of $\frac{8}{9}$ were enclosed in brackets.

Example.—

$\frac{7}{8}$ of $\frac{6}{7} \div \frac{4}{5} \times \frac{8}{9}$

$$= \frac{\cancel{7}}{\cancel{8}} \times \frac{\overset{2}{\cancel{6}}}{\cancel{7}} \times \frac{5}{\underset{2}{\cancel{4}}} \times \frac{\cancel{8}}{\underset{3}{\cancel{9}}} = \frac{5}{6} \quad \textit{Ans.}$$

Note (1).—That $\frac{8}{9}$ is not connected so closely to $\frac{4}{5}$ by the multiplication sign as it was connected by the "of" sign. Hence each fraction is governed only by that sign which *immediately* precedes it.

(2) In each case the fraction is dealt with *as a whole,* i.e. the whole fraction is accounted for at each step.

Example.—

$$\frac{\frac{1}{2} + \frac{1}{4}}{\frac{1}{2} - \frac{1}{4}}$$

$$= (\tfrac{1}{2} + \tfrac{1}{4}) \div (\tfrac{1}{2} - \tfrac{1}{4})$$

$$= 3 \quad \textit{Ans.}$$

Example.—

$$\tfrac{1}{2} + \tfrac{1}{4} \div \tfrac{1}{2} - \tfrac{1}{4}$$

$$= \tfrac{1}{2} + (\tfrac{1}{4} \div \tfrac{1}{2}) - \tfrac{1}{4}$$

$$= \tfrac{3}{4} \quad \textit{Ans.}$$

Note that the bar has the force both of division and of brackets.

Example.—

$$\frac{3\frac{2}{3} - 5\frac{1}{4} \div 3\frac{1}{2} + 4\frac{5}{6}}{3\frac{2}{3} \times 5\frac{1}{4} - 3\frac{1}{2} \times 4\frac{5}{6}} \text{ (O.S.L.)}$$

$$= \frac{3\frac{11}{3} - 5\frac{21}{4} \div \frac{7}{2} + 4\frac{29}{6}}{3\frac{11}{3} \times 5\frac{21}{4} - \frac{7}{2} \times 4\frac{29}{6}}$$

It is advisable to reduce to improper fractions here, because all the fractions will be speedily concerned in either multiplication or division. As a rule, however, such reduction should be avoided.

$$= \frac{\frac{11}{3} - \frac{21}{4} \times \frac{2}{7} + \frac{29}{6}}{\frac{11}{3} \times \frac{21}{4} - \frac{7}{2} \times \frac{29}{6}}$$

Invert the fraction that comes after the division sign.

$$= \frac{\frac{11}{3} - \frac{3}{2} + \frac{29}{6}}{\frac{77}{4} - \frac{203}{12}}$$

Deal with × signs before operating with + or − signs.

$$= \frac{44 - 18 + 58}{231 - 203}$$

Multiply by 12 to clear the denominators.

$$= \frac{84}{28} = 3 \quad Ans.$$

The Use of Brackets.

Let us examine the expression—

$$2\tfrac{1}{2} + 1\tfrac{3}{4} \times 1\tfrac{1}{3}$$

We see that it consists of two *terms*, the first, $2\frac{1}{2}$, and the second expressed by the factors $1\frac{3}{4} \times 1\frac{1}{3}$. To show this more clearly, we *may* place the terms in brackets, thus —

$$(2\tfrac{1}{2}) + (1\tfrac{3}{4} \times 1\tfrac{1}{3})$$

But suppose we desire the expression to mean that the sum of $2\frac{1}{2}$ and $1\frac{3}{4}$ is to be multiplied by $1\frac{1}{3}$, we *must* obviously employ some means of *binding* the $2\frac{1}{2}$ and $1\frac{3}{4}$ closely together, thus—

$$(2\tfrac{1}{2} + 1\tfrac{3}{4}) \times 1\tfrac{1}{3}$$

Note the difference—

between

$$(2\tfrac{1}{2} + 1\tfrac{3}{4}) \times 1\tfrac{1}{3}$$
$$= 4\tfrac{1}{4} \times 1\tfrac{1}{3}$$
$$= \frac{17}{4} \times \frac{4}{3}$$
$$= \frac{17}{3} = 5\tfrac{2}{3} \quad Ans.$$

Note from the above that brackets should be "cleared" (*i.e.* their contents reduced to simple fractions) before other operations are attempted.

and

$$2\tfrac{1}{2} + 1\tfrac{3}{4} \times 1\tfrac{1}{3}$$
$$= 2\tfrac{1}{2} + \frac{7}{4} \times \frac{4}{3}$$
$$= 2\tfrac{1}{2} + \frac{7}{3}$$
$$= 2\tfrac{1}{2} + 2\tfrac{1}{3}$$
$$= 4\tfrac{5}{6} \quad Ans.$$

Note from the above that "weak" signs are cleared before the operations indicated by the "strong" signs are attempted.

If we wish to multiply the sum of $2\frac{4}{5}$ and $3\frac{3}{10}$ by the sum of $1\frac{1}{4}$ and $2\frac{1}{2}$, it would not be correct to write the problem as follows :—

$$2\tfrac{4}{5} + 3\tfrac{1}{10} \times 1\tfrac{1}{4} + 2\tfrac{1}{2}$$

for this would mean that *three* terms $2\frac{4}{5}$, $3\frac{1}{10} \times 1\frac{1}{4}$, and $2\frac{1}{2}$ were to be *added* together, and our problem evidently states that *two* terms are to be *multiplied* together. Brackets are necessary to show clearly what is meant, thus—

$$(2\tfrac{4}{5} + 3\tfrac{1}{10}) \times (1\tfrac{1}{4} + 2\tfrac{1}{2})$$

or it is permissible to omit the multiplication sign between the two brackets and write thus—

$$(2\tfrac{4}{5} + 3\tfrac{1}{10})\,(1\tfrac{1}{4} + 2\tfrac{1}{2})$$

Similarly, if we wished to divide $2\frac{4}{5} + 3\frac{1}{10}$ by $1\frac{1}{4} + 2\frac{1}{2}$, we could write—

$$\frac{2\frac{4}{5} + 3\frac{1}{10}}{1\frac{1}{4} + 2\frac{1}{2}} \text{ or } (2\tfrac{4}{5} + 3\tfrac{1}{10}) \div (1\tfrac{1}{4} + 2\tfrac{1}{2})$$

but the division sign between the brackets must not be omitted, for its omission would mean that the two quantities were to be multiplied together.

Occasionally brackets of the forms [ ], { } are employed, and more rarely still, a bar or vinculum —— is used in place of a bracket.

Example.

$$\begin{aligned} 2\{\tfrac{1}{2} \div (\tfrac{1}{3} \times \overline{\tfrac{1}{4} + \tfrac{1}{5}})\} &= 2\{\tfrac{1}{2} \div (\tfrac{1}{3} \times \tfrac{9}{20})\} \\ &= 2\{\tfrac{1}{2} \times \tfrac{3}{1} \times \tfrac{20}{9}\} \\ &= \tfrac{2}{1} \times \tfrac{1}{\cancel{2}} \times \tfrac{\cancel{3}}{1} \times \frac{20}{\cancel{9}_{3}} \\ &= \tfrac{20}{3} = 6\tfrac{2}{3} \quad \textit{Ans.} \end{aligned}$$

Note (1) That the innermost bracket is first "cleared."

(2) That the division sign before the bracket ( ) causes all the fractions within that bracket to be inverted, but only after the addition (or subtraction) signs have been eliminated.

(3) That the 2 before the bracket { } multiplies all the terms within that bracket.

"Plus" or "Minus" Signs before Brackets.

To express the sum of $2\frac{4}{5}$ and $2\frac{1}{4} - 1\frac{1}{4}$, we *may* write thus—

$$(2\tfrac{4}{5}) + (2\tfrac{1}{4} - 1\tfrac{1}{4})$$

or simply—

$$2\tfrac{4}{5} + 2\tfrac{1}{4} - 1\tfrac{1}{4}$$

Note that the second bracket is preceded by a plus sign, and that the sign within the bracket is unchanged on the removal of the bracket.

Now, suppose we wish to express the *difference* between $4\frac{4}{5}$ and $2\frac{1}{4} + 1\frac{1}{4}$. If we write our expression thus—

$$4\frac{4}{5} - 2\frac{1}{4} + 1\frac{1}{4}$$

we are taking $2\frac{1}{4}$ from the sum of $4\frac{4}{5}$ and $1\frac{1}{4}$, which is not the operation we have to perform. Brackets will again help us to express our meaning, thus—

$$4\frac{4}{5} - (2\frac{1}{4} + 1\frac{1}{4})$$

or, recollecting that both $2\frac{1}{4}$ and $1\frac{1}{4}$ are to be taken from $4\frac{4}{5}$, we may re-write thus—

$$4\frac{4}{5} - 2\frac{1}{4} - 1\frac{1}{4}$$

$$\text{hence } 4\frac{4}{5} - (2\frac{1}{4} + 1\frac{1}{4}) = 4\frac{4}{5} - 2\frac{1}{4} - 1\frac{1}{4}$$

Note that when the bracket is preceded by a minus sign, the sign within the bracket is changed on the removal of the bracket.

Similarly—

$$4\frac{4}{5} - (2\frac{1}{4} - 1\frac{1}{4}) = 4\frac{4}{5} - 2\frac{1}{4} + 1\frac{1}{4}$$

Thus we learn that if there is a *plus* sign before a bracket, the signs in the bracket are unaltered on the removal of the bracket; but that if there is a *minus* sign before the bracket, then the plus signs within the bracket must be changed to minus signs, and the minus signs must be changed to plus signs, on the removal of the bracket.

Example.—Simplify—

$$\frac{\{(5\frac{1}{18} - 2\frac{5}{6}) \text{ of } 2\frac{3}{5}\} \times \frac{3}{4}}{1\frac{3}{4} + 2\frac{7}{12}}$$

In the case of two or more brackets, clear the inner one first.

$$= \frac{\left\{\left(5 - 2 + \frac{1 - 15}{18}\right) \text{ of } 1\frac{3}{5}\right\} \times \frac{3}{4}}{1 + 2 + \frac{9 + 7}{12}}$$

Avoid reducing to improper fractions except to multiply or divide.

$$= \frac{\left\{\left(4 - 2 + \frac{19 - 15}{18}\right) \text{ of } 1\frac{3}{5}\right\} \times \frac{3}{4}}{3\frac{16}{12}}$$

1 or $\frac{18}{18}$ is taken from 5 and added to $\frac{1}{18}$, making $\frac{19}{18}$, or the expression might be written—

$$5 - 2 - 1\frac{4}{18}$$

$$= \frac{\{2\frac{2}{9} \times 1\frac{3}{5}\} \times \frac{3}{4}}{1\frac{3}{3}}$$

The answer is not 0, but 1, for in reality 1 should be written above each quantity that cancels out.

$$= \frac{20}{9} \times 1\frac{3}{5} \times \frac{3}{4} \times 1\frac{3}{15} = 1 \quad \textit{Ans.}$$

Example.—

$$2\tfrac{4}{11} - 1\tfrac{7}{33} \text{ of } 2\tfrac{2}{7} - \frac{1}{2 + \frac{1}{1 + \frac{3}{5}}}$$

Here there are two "quantities," or terms, separated by a minus sign. To make this clearer we *may* enclose them in brackets.

$$= \left(2\tfrac{4}{11} - 1\tfrac{7}{33} \text{ of } 2\tfrac{2}{7}\right) - \frac{1}{2 + \frac{1}{1 + \frac{3}{5}}}$$

$$= \left(\frac{2 - 1 + \frac{12 - 7}{33}}{8 - 4 + \frac{44 - 8}{143}} \text{ of } 2\tfrac{2}{7}\right) - \frac{1}{2 + \frac{1}{1\frac{3}{5}}}$$

There is no need yet to bring any quantity to an improper fraction.

$$= \left(\frac{1\frac{5}{33}}{4\frac{36}{143}} \text{ of } 2\tfrac{2}{7}\right) - \frac{1}{2 + \frac{1}{\frac{8}{5}}}$$

$$= \left(\frac{\frac{38}{33}}{\frac{608}{143}} \times \frac{16}{7}\right) - \frac{1}{2\frac{5}{8}}$$

$$= \left(\frac{\overset{1}{\cancel{38}}}{\underset{3}{\cancel{33}}} \times \frac{\overset{13}{\cancel{143}}}{\underset{16}{\cancel{608}}} \times \frac{\overset{1}{\cancel{16}}}{\underset{1}{\cancel{7}}}\right) - \frac{1}{\frac{21}{8}}$$

Inverting the divisor, we can multiply the three fractions in the first bracket together.

$$= \tfrac{13}{21} - \tfrac{8}{21}$$

$$= \tfrac{5}{21}. \quad \textit{Ans.}$$

Approximations. Continued Fractions.

It is often advantageous to be able to approximate fractions, in order to facilitate working and to test the approximate correctness of the result of the simplification of a compound or complex fraction.

Example.—*Approximate* $\frac{17}{24}$.

Increasing the numerator by 1, we have $\frac{18}{24}$ or $\frac{3}{4}$
Decreasing „ „ 1, „ $\frac{16}{24}$ „ $\frac{2}{3}$

Thus $\frac{2}{3}$ and $\frac{3}{4}$ are both approximate values of $\frac{17}{24}$.

Example—

$$4\tfrac{1}{9} - 1\tfrac{5}{18} + 3\tfrac{5}{9} + 12\tfrac{9}{16} - \tfrac{11}{27} - \tfrac{1}{4}$$

Approximating, we have—

$$4\tfrac{1}{9} - 1\tfrac{5}{18} + 3\tfrac{5}{9} + 12\tfrac{9}{18} - \tfrac{4}{9} - \tfrac{2}{9}$$

$$= 19 - 1 + \frac{2 - 5 + 10 + 9 - 8 - 4}{18}$$

Each denominator is made 9 or 18.

$\frac{11}{27}$ is approximated as $\frac{12}{27}$ or $\frac{4}{9}$.

$= 18\frac{3}{9}$, as an approximate answer.

The correct answer is $18\frac{559}{432}$.

Another mode of approximating a fraction leads to some interesting and important results.

Example.—*Take* $\frac{34}{55}$ *as the fraction to be approximated.*

Dividing the numerator (34) by the numerator (34), and the denominator (55) by the numerator (34), we get—

$$\frac{34}{55} = \frac{1}{1 + \frac{21}{34}}$$

Discarding the fraction ($\frac{21}{34}$), $\frac{34}{55}$ is "approximated" as 1.

Dividing numerator (21) and denominator (34) by numerator (21), we get—

$$= \frac{1}{1 + \frac{1}{1 + \frac{13}{21}}}$$

Discarding the fraction $\frac{13}{21}$ and simplifying the complex fraction, $\frac{34}{55}$ is approximated as $\frac{1}{2}$.

Now treating $\frac{13}{21}$ as we treated $\frac{21}{34}$—

$$= \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{8}{13}}}}$$

Proceeding as before, the approximation is now $\frac{2}{3}$.

Continuing the process—

$$= \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{5}{8}}}}}$$

And now $\frac{3}{5}$.

$$= \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{3}{5}}}}}}$$

And now $\frac{5}{8}$.

$$= \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{2}{3}}}}}}}$$

And now $\frac{8}{13}$.

$$= \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}}}}}$$

And now $\frac{13}{21}$.

And so on.

A fraction having a complex form like the above is called a *continued fraction.* A knowledge of the way in which a continued fraction is built up will be of help to us in its solution.

Example.—*Simplify*—

$$4 + \cfrac{1}{4 + \cfrac{1}{4 + \frac{1}{4}}}$$

$$= 4 + \cfrac{1}{4 + \cfrac{1}{4\frac{1}{4}}}$$

$$= 4 + \cfrac{1}{4 + \cfrac{1}{\frac{17}{4}}}$$

$$= 4 + \frac{1}{4\frac{4}{17}}$$

$$= 4 + \frac{1}{\frac{72}{17}}$$

$$= 4\tfrac{17}{72}. \quad \textit{Ans.}$$

When we resolved our simple fraction into a continued fraction, we arrived at the fraction at the bottom of the right-hand side last.

Hence, in simplifying the continued fraction, we shall *begin* with the bottom right-hand fraction and work back to the simple form.

THEORY

1. State your method of simplifying a complex fraction; and, in accordance therewith, simplify $\frac{5\frac{3}{4}}{2\frac{2}{9}}$. (S.A.)

2. Distinguish between your treatment of the "of" and × signs in the solution of complex fractions.

3. What is understood by *strong* and *weak signs*? What is their order of precedence, and why?

4. What effect has a minus sign before a bracket upon the contents of the bracket?

5. What is a continued fraction? Show how such a fraction is built up?

EXERCISE IX.

Complex and Compound Fractions involving Multiplication and Division only.

1. $\dfrac{8\frac{1}{2} \text{ of } 9\frac{3}{5} \text{ of } 5}{12\frac{1}{7} \text{ of } 1\frac{1}{11} \text{ of } 30\frac{3}{5}}$. (C.P.)

2. $\frac{33}{39} \times 2\frac{1}{7} \times \frac{2\frac{1}{4}}{4\frac{3}{4}} \div \frac{5}{4}$. (C.P.)

3. $\frac{3}{14}$ of $\dfrac{4\frac{6}{7}}{6\frac{1}{8}}$ of $\dfrac{6\frac{8}{11}}{11\frac{2}{7}}$. (C.P.)

4. $\dfrac{\frac{3}{4}}{\frac{1}{7}}$; $\dfrac{2}{\frac{3}{4}}$; $\dfrac{\frac{2}{3}}{\frac{4}{7}}$; $\dfrac{\frac{2}{3}}{\frac{7}{7}}$; $\dfrac{2}{\frac{3}{4}}$. (C.S.)

Complex Fractions involving Addition and Subtraction.

5. $\dfrac{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}{2\frac{1}{4} + 4\frac{1}{8} + 5\frac{1}{4}}$. (C.P.)

6. $\dfrac{2\frac{11}{12} - 2\frac{13}{44} + \frac{7}{8}}{3\frac{1}{4} + 1\frac{13}{24} - 3\frac{5}{6}}$. (P.T.)

7. $\dfrac{87_{12}}{5\frac{11}{18}+5\frac{5}{6}} - \dfrac{11\frac{3}{8}}{16\frac{1}{12}+5\frac{5}{8}}$. (O.S.L.)

8. $\dfrac{\frac{8}{15}+\frac{9}{23}}{\frac{9}{17}+\frac{7}{16}} \times \dfrac{\frac{4}{11}+\frac{21}{17}}{\frac{11}{15}+\frac{12}{19}}$. (P.T.)

9. $\dfrac{4\frac{2}{3}+1\frac{3}{45}}{3\frac{1}{3}+13\frac{1}{15}} \sim \dfrac{18\frac{1}{4}-5\frac{7}{8}}{3\frac{1}{2}\text{ of }5\frac{1}{8}}$. (C.P.)

10. $\dfrac{\frac{1}{7}+\frac{2}{7\times 8}+\frac{3}{7\times 8\times 9}}{\frac{1}{6}+\frac{2}{6\times 7}+\frac{3}{6\times 7\times 8}}$. (C.P.)

11. $\dfrac{5\frac{19}{17}+8\frac{13}{13}}{\frac{2}{3}-\frac{1}{13}} \times \dfrac{4\frac{2}{19}+3\frac{16}{29}}{\frac{11}{23}+\frac{2}{11}}$. (C.S.L.)

12. $\dfrac{4\frac{3}{13}-2\frac{2}{3}}{2\frac{5}{11}+1\frac{4}{7}} \div \dfrac{5\frac{17}{18}-4\frac{7}{8}}{3\frac{3}{4}+1\frac{2}{3}}$. (C.H.L.)

13. $\dfrac{3\frac{3}{4}+1\frac{7}{9}}{18+4\frac{1}{9}} \div \dfrac{15+13\frac{3}{7}}{74-16\frac{5}{7}}$. (L.M.)

14. $\dfrac{14\frac{3}{4}+12\frac{1}{3}}{10\frac{17}{18}-9\frac{7}{8}} \div \dfrac{7\frac{5}{11}+4\frac{3}{7}}{8\frac{3}{13}-6\frac{2}{3}}$. (C.S.L.)

15. $\dfrac{8\frac{7}{9}-7\frac{2}{9}+5\frac{5}{6}-4\frac{1}{3}}{9\frac{9}{10}-8\frac{13}{15}+7\frac{7}{8}-6\frac{6}{7}} \div \dfrac{\frac{1}{2}\text{ of }\frac{2}{3}\text{ of }\frac{3}{5}}{\frac{1}{7}\text{ of }\frac{14}{15}\text{ of }\frac{1}{2}}$. (C.P.)

16. $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}}$ of $\dfrac{\frac{1}{4}-\frac{1}{5}}{\frac{1}{5}+\frac{1}{6}}$ of $\dfrac{\frac{1}{6}-\frac{1}{7}}{\frac{1}{7}+\frac{1}{8}}$ of 1155. (C.PR.)

Strong and Weak Signs.

17. $\frac{4}{7}$ of $\frac{5}{8}$ of $\frac{17}{20} - \frac{1}{4}$ of $\frac{5}{14}$ of 2. (C.P.)

18. $\frac{11}{17}$ of $21\frac{9}{7} - \frac{11}{16}$ of $10\frac{5}{9} + \frac{11}{21} \div \frac{33}{61}$. (C.S.L.)

19. $2\frac{1}{3}$ of $3\frac{1}{3} - 3\frac{1}{7} \div 7\frac{1}{14} - 5\frac{1}{3}$. (C.S.L.)

20. $2\frac{3}{4} \div 1\frac{4}{7} \times 5\frac{1}{3} - \frac{3}{4}$ of $1\frac{1}{4} + \frac{1}{2}$. (C.H.L.)

21. $3\frac{7}{13}$ of $\frac{429}{1127} \div 71 + 2\frac{3}{70}$ of $\frac{3}{65}$ of $\frac{35}{41} - \frac{7}{180}$ of $1\frac{11}{81}$. (C.W.)

22. $\dfrac{2\frac{1}{4}-\frac{2}{5}\text{ of }1\frac{5}{6}}{\frac{1}{5}\text{ of }3\frac{1}{3}+\frac{7}{8}}$. (C.PR.)

23. $\dfrac{\frac{3}{4}\text{ of }\frac{7}{9}+\frac{7}{12}\text{ of }\frac{9}{16}}{\frac{1}{3}-\frac{7}{9}\text{ of }\frac{3}{14}+\frac{15}{32}}$. (C.P.)

24. $\dfrac{45\frac{5}{8}\text{ of }\frac{1}{13}\frac{0}{0}+\frac{17}{68}\text{ of }2\frac{5}{68}}{1\frac{2}{7}\text{ of }2\frac{8}{9}-\frac{1}{2}\times\frac{3}{7}\times 8\frac{5}{9}}$. (C.W.)

25. $\dfrac{17}{151}$ of $\dfrac{3\frac{5}{8}-2\frac{3}{4}+1\frac{1}{2}}{2\frac{1}{3}\text{ of }1\frac{1}{7}-1\frac{1}{2}\text{ of }\frac{3}{7}}$. (L.M.)

EXERCISE X.

Preliminary Exercises in the Use of Brackets.

1. (*a*) $(4\frac{1}{8}+\frac{5}{4})$ of $(1\frac{1}{2}-\frac{7}{10})$.
(*b*) $4\frac{1}{8}+\frac{5}{4}$ of $1\frac{1}{2}-\frac{7}{10}$.

2. (*a*) $(5\frac{1}{3}-2\frac{1}{2})\times(2\frac{1}{4}+\frac{3}{5})$.
(*b*) $5\frac{1}{3}-2\frac{1}{2}\times 2\frac{1}{4}+\frac{3}{5}$.

3. (*a*) $(\frac{4}{5}-\frac{3}{11})\times(2\frac{3}{4}+3\frac{3}{8})$.
(*b*) $\frac{4}{5}-\frac{3}{11}\times 2\frac{3}{4}+3\frac{3}{8}$.

4. (*a*) $3\frac{1}{3}+1\frac{1}{6}\times\frac{3}{10}-6\frac{1}{4}\div 25$.
(*b*) $(3\frac{1}{3}+1\frac{1}{6})\times\frac{3}{10}-(6\frac{1}{4}\div 25)$.
(*c*) $(3\frac{1}{3}+1\frac{1}{6})\times(\frac{3}{10}-6\frac{1}{4}\div 25)$.
(*d*) $(3\frac{1}{3})+(1\frac{1}{6}\times\frac{3}{10})-(6\frac{1}{4}\div 25)$.

5. (*a*) $\frac{3}{8}+\frac{1}{4}$ of $\frac{3}{7}-\frac{5}{36}$.
(*b*) $(\frac{3}{8}+\frac{1}{4})(\frac{3}{7}-\frac{5}{36})$.

6. (*a*) $\frac{3}{4}$ of $\frac{2}{3}+5\frac{1}{2}$ of $6\frac{7}{8}+\frac{1}{4}$ of $\frac{8}{10}$.
(*b*) $(\frac{3}{4})$ of $(\frac{2}{3}+5\frac{1}{2})$ of $(6\frac{7}{8}+\frac{1}{4})$ of $(\frac{8}{10})$.
(*c*) $(\frac{3}{4}$ of $\frac{2}{3}+5\frac{1}{2})$ of $(6\frac{7}{8}+\frac{1}{4}$ of $\frac{8}{10})$.

7. (*a*) $4\frac{1}{8}+\frac{5}{4}$ of $1\frac{1}{2}-\frac{7}{16}$.
(*b*) $(4\frac{1}{8}+\frac{5}{4})$ of $(1\frac{1}{2}-\frac{7}{16})$.

8. (*a*) $\frac{7}{8}\times 9-2\frac{1}{2}$.
(*b*) $\frac{7}{8}\times(9-2\frac{1}{2})$.

9. (*a*) $\frac{1}{9}$ of $5\frac{2}{3}-\frac{1}{4}$ of $\frac{5}{11}$ of $6\frac{3}{10}$.
(*b*) $(\frac{1}{9})$ of $(5\frac{2}{3}-\frac{1}{4})$ of $(\frac{5}{11}$ of $6\frac{3}{10})$.
(*c*) $\{(\frac{1}{9})$ of $5\frac{2}{3}-(\frac{1}{4}$ of $\frac{5}{11})\}$ of $6\frac{3}{10}$.

10. (*a*) $26-14-\frac{3}{4}+3\frac{1}{3}$.
(*b*) $26-14-(\frac{3}{4}+3\frac{1}{3})$.
(*c*) $26-(14-\frac{3}{4}+3\frac{1}{3})$.
(*d*) $26-\{14-(\frac{3}{4}+3\frac{1}{3})\}$.

Fractions involving Brackets.

11. $(\frac{1}{2}+\frac{3}{4}+\frac{5}{6}) \div (\frac{2}{3}+\frac{1}{4}+\frac{7}{6})$. (C.P.)

12. $(\frac{3}{13}+\frac{5}{17}+\frac{7}{19}) \times (\frac{2}{11}+\frac{16}{31}+\frac{51}{121})$. (C.S.L.)

13. $(13\frac{5}{8}$ of $5\frac{1}{4}$ of $2\frac{3}{5}) - (14\frac{3}{10}$ of $3\frac{5}{7}$ of $3\frac{7}{2})$. (C.P.)

14. $11\frac{1}{4} - 1\frac{3}{5}$ of $1\frac{1}{16} + 1\frac{3}{8}$ of $(5\frac{1}{4} - 2\frac{3}{4})$. (O.S.L.)

15. $(\frac{3}{8}+\frac{3}{5})\frac{5}{6} - \frac{7}{8}(3\frac{1}{2} - 2\frac{3}{5}) + 16\frac{4}{13}$. (C.P.)

16. $(1\frac{23}{56} + 4\frac{1}{8} \times 1\frac{7}{11}) \div (5\frac{1}{5} - 7\frac{7}{8} \div 9\frac{3}{25})$. (O.S.L.)

17. $(7\frac{1}{8} + 5\frac{6}{7} - 6\frac{3}{11}) \div \{(8\frac{2}{3}$ of $\frac{1}{7})$ of $(7\frac{4}{9} \div 8\frac{3}{8})\}$. (C.P.)

18. $\frac{5}{16} \times \left(\frac{2}{5}\text{ of}\frac{22\frac{1}{2} - 11\frac{1}{2}}{15\frac{1}{2} - 12} + \frac{1}{3}\right)$. (C.P.)

19. $\dfrac{(9\frac{1}{3}$ of $4\frac{5}{7}) \div 8\frac{1}{3}}{2\frac{7}{9} - \frac{5}{12} + 1\frac{7}{36}}$. (S.K.)

20. $\dfrac{2 \times 1\frac{6}{11} \times (6\frac{2}{3} - 1\frac{1}{6})}{1\frac{34}{35} - \frac{1}{12}} \div \dfrac{3\frac{3}{4} - 1\frac{13}{20}}{3\frac{5}{12} + 1\frac{29}{60}}$. (C.H.L.)

21. $\left(\dfrac{1\frac{1}{6}$ of $3\frac{4}{7}}{1\frac{1}{5}$ of $6\frac{1}{3}} - \dfrac{4\frac{1}{3} - 1\frac{4}{9}}{4\frac{1}{3} + 2\frac{1}{6}}\right) \div \dfrac{4\frac{1}{2} - 1\frac{11}{16}}{6\frac{3}{4} + 3\frac{3}{8}}$. (C.P.)

22. $\left(\dfrac{7\frac{2}{3} + 4\frac{5}{6} - 8\frac{3}{5}}{18\frac{3}{7} - 7\frac{1}{6} + 3\frac{7}{15}}\right) \div \left(\dfrac{3\frac{2}{7} - 2\frac{1}{5} + 17\frac{3}{8}}{5\frac{1}{5} - 4\frac{2}{7} + 3\frac{1}{5}}\right) \times (30\frac{37}{149}$ of $14\frac{84}{117})$. (C.P.)

23. $\dfrac{\frac{3}{4}$ of $\frac{7}{9}$ of $1\frac{5}{33} - 2\frac{1}{4}$ of $3\frac{2}{3}$ of $\frac{1}{72}}{4\frac{1}{2} - (3\frac{1}{3} + 4\frac{2}{7}) + 3\frac{7}{8} + \frac{3}{56}}$. (C.P.)

24. $\dfrac{4\frac{2}{7}$ of $1\frac{1}{19} - 16\frac{2}{3}$ of $\frac{52}{171}}{16\frac{2}{3}$ of $(\frac{6}{19} - \frac{2}{7})$ of $\frac{39}{95}}$. (K.S.)

25. $\dfrac{44\frac{3}{5} - 3\frac{3}{11}$ of $10\frac{1}{12}}{3\frac{1}{2} + \frac{7}{65}$ of $13\frac{13}{14}} + \dfrac{(5\frac{3}{4} - 3\frac{2}{3})$ of $2\frac{5}{9}}{1\frac{1}{4}$ of $(2\frac{1}{2} + 1\frac{1}{3})}$. (S.K.)

26. $\dfrac{5 + 2\frac{7}{9}$ of $\frac{27}{50}}{30 - 6\frac{6}{11}$ of $2\frac{1}{5}} + \dfrac{6\frac{5}{6}}{23\frac{3}{7}}$ of $(6\frac{1}{21} - 5\frac{9}{14})$. (S.K.)

27. (*a*) $\dfrac{5\frac{3}{4} + 2\frac{5}{9}$ of $(11\frac{1}{2} - 4\frac{3}{5})}{(5\frac{3}{4} - 2\frac{5}{9})$ of $11\frac{1}{2} - 4\frac{3}{5}}$. (S.A.)

(*b*) $\dfrac{(5\frac{3}{4} + 2\frac{5}{9})$ of $11\frac{1}{2} - 4\frac{3}{5}}{(5\frac{3}{4} - 2\frac{5}{9})$ of $(11\frac{1}{2} + 4\frac{3}{5})}$. (S.A.)

28. $\dfrac{(\frac{1}{3} - \frac{1}{9})$ of $(\frac{1}{3} + \frac{1}{5})}{\frac{1}{2} - \frac{1}{4}$ of $\frac{1}{3} + \frac{1}{6}} \div \dfrac{(\frac{1}{3} + \frac{1}{4})$ of $(\frac{1}{3} - \frac{1}{6})}{\frac{1}{9}$ of $\frac{1}{10} + \frac{2}{7}$ of $1\frac{2}{5}}$. (C.PR.)

29. $\dfrac{(\frac{1}{3} + \frac{2}{5} + \frac{3}{14}) - (\frac{1}{4} + \frac{1}{11} + \frac{14}{44})}{3 - (\frac{1}{3}$ of $\frac{5}{11}) - (\frac{5}{11}$ of $\frac{8}{19}) - (\frac{8}{19}$ of $\frac{1}{3})}$. (C.P.)

30. $\dfrac{4(\frac{1}{3} - \frac{1}{4}) - 9(\frac{1}{2} - \frac{1}{4}) + 16(\frac{1}{2} - \frac{1}{3})}{2(\frac{1}{3} - \frac{1}{4}) - 3(\frac{1}{2} - \frac{1}{4}) + 4(\frac{1}{2} - \frac{1}{3})}$. (C.S.)

31. $\dfrac{9(3\frac{1}{3}$ of $\frac{7}{25})}{5\left(\frac{6}{4}$ of $\frac{2}{5}\right)} \div \frac{4}{7}\left(\dfrac{1\frac{1}{2}$ of $3\frac{1}{3}}{\frac{5}{12}$ of $4\frac{4}{5}}\right)$. (C.S.)

32. $\dfrac{\{(\frac{1}{10}-\frac{1}{11})1\frac{1}{4}-(\frac{1}{91}-\frac{1}{1\,9})(1\frac{1}{3}+\frac{1}{4})\}}{(1\frac{1}{2}-\frac{1}{8})\frac{1}{7}-\frac{23}{364}}$. (L.M.)

33. $\dfrac{(\frac{2}{3}-\frac{2}{11}+\frac{16}{21})\div\left(2+\dfrac{1}{2+\frac{1}{2}}\right)}{(\frac{3}{77}+\frac{3}{253}-\frac{1}{21})\times\left(3-\dfrac{1}{3-\frac{1}{3}}\right)}$. (L.M.)

34. $\dfrac{2\frac{2}{7}\times\frac{4\frac{2}{3}}{6\frac{6}{7}}\times(4-2\frac{2}{7})\div(\frac{1}{10}+1\frac{1}{4})}{1-\dfrac{1}{3+\frac{1}{3}}}$ (O. and C.S.C.)

35. $\dfrac{2\frac{6}{17}+(3\frac{3}{19}\text{ of }2\frac{4}{17})}{4\frac{4}{19}\times\{(1\frac{1}{4}\text{ of }3\frac{2}{17})-1\frac{10}{17}\}}$. (C.W.)

36. $\dfrac{\frac{4}{5}\text{ of }1\frac{2}{3}\text{ of }2\frac{2}{3}\text{ of }2}{\frac{7}{6}+\frac{5}{12}-\frac{3}{4}}-\left\{3\frac{1}{3}\div\left(\dfrac{1-\frac{15}{16}}{\frac{1}{3}-\frac{1}{8}}\right)\right\}$. (C.P.)

37. $\frac{660}{1409}\times\left\{6\frac{3}{24}+\dfrac{70}{20}-2\frac{4}{23}\times\left(\dfrac{1\frac{5}{6}+3\frac{1}{20}}{28\frac{7}{9}-25\frac{2}{3}}\right)\div1\frac{179}{700}\right\}$. (C.M.)

38. $\left\{\frac{1}{8}\text{ of }(\frac{1}{10}-\frac{1}{11})\div\dfrac{\frac{1}{7}-\frac{1}{9}\div(\frac{4}{9}+\frac{4}{11})}{(\frac{1}{7}+\frac{1}{9})\div(\frac{4}{9}-\frac{4}{11})}\right\}\times\dfrac{\frac{1}{3}+\frac{1}{2}\div(\frac{1}{7}-\frac{1}{9})}{(\frac{1}{3}+\frac{1}{7})\div\frac{1}{7}-\frac{1}{9}}$. (C.S.L.)

39. $\dfrac{55\frac{7}{11}\text{ of }23\frac{9}{13}\text{ of }13\frac{13}{17}-431\frac{1}{2}\text{ of }17\frac{1}{7}\text{ of }15\frac{2}{9}}{5\frac{2}{15}\text{ of }\left(1\frac{1}{2}-\dfrac{1}{2\frac{1}{4}}\right)+5\frac{1}{3}\text{ of }\left(1-\dfrac{1\frac{1}{4}}{2\frac{1}{12}-\frac{3}{4}}\right)+5\frac{7}{9}\text{ of }\dfrac{3\frac{1}{4}\text{ of }2\frac{1}{2}-1}{5-\frac{2}{3}\text{ of }2\frac{1}{2}}}$. (C.S.)

40. $\dfrac{4\frac{2}{5}\div13\frac{1}{7}+2\frac{2}{5}}{4\frac{2}{3}+\frac{1}{3}\text{ of }4\frac{1}{4}+(1\frac{13}{15}+2\frac{1}{5})\times\frac{2}{3}}+\dfrac{4\frac{2}{5}+3\frac{4}{5}-5\frac{3}{4}}{2\frac{1}{5}+1\frac{1}{5}\times7\frac{1}{2}-2\frac{3}{4}}$. (S.A.)

EXERCISE XI.

Continued Fractions.

1. $\dfrac{3}{1-\dfrac{3}{4+\frac{5}{8}}}$. (P.T.)

2. $\dfrac{1}{1+\dfrac{1}{2-\frac{2}{15}}}$. (C.P.)

3. $\dfrac{2}{3+\dfrac{4}{5+\frac{6}{7}}}$. (K.S.)

4. $1+\dfrac{1}{2+\dfrac{1}{3+\frac{1}{4}}}$. (P.T.)

5. $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{1+\frac{1}{4}}}}$. (P.E.A.)

6. $\dfrac{4}{2+\dfrac{3}{5+\dfrac{5}{7+\frac{7}{9}}}}$. (C.P.)

7. $\dfrac{1}{1+\dfrac{3}{2+\dfrac{1}{2-\frac{3}{5}}}}$. (C.W.)

8. $1-\dfrac{2}{8-\dfrac{7}{1+\dfrac{1}{2+\frac{1}{3}}}}$. (C.PR.)

9. $\dfrac{1}{2+\dfrac{2}{3+\dfrac{3}{4+\frac{5}{6}}}}$. (O.S.L.)

10. $\dfrac{1}{2+\dfrac{2}{3+\dfrac{3}{4+\frac{4}{5}}}}$. (C.J.)

12. $\dfrac{1}{1-\dfrac{1}{1\dfrac{1}{1\frac{1}{3}}}}$

11. $\dfrac{2\frac{1}{2}}{\frac{8}{11}-\dfrac{2}{3\frac{1}{3}+\dfrac{1}{2\frac{1}{2}-1\frac{1}{3}}}}$. (C.H.L.)

EXERCISE XII.

Complex and Compound Fractions giving Negative or Zero Results.

1. $\frac{1}{5}$ of $2\frac{1}{2}-3$.

2. $\frac{1}{5}+\frac{2}{3}$ of $\frac{1}{4}-\frac{1}{9}\div\frac{2}{5}-1$.

3. $(\frac{3}{5}-\frac{1}{7})\div(1\frac{3}{4}+\frac{5}{8})-2$.

4. $(\frac{3}{4}$ of $\frac{1}{13}$ of $3\frac{1}{4})-(7\frac{1}{2}-6\frac{5}{14})$ (C.P.)

5. $1-\dfrac{\dfrac{1}{5+\dfrac{3}{1-\dfrac{2}{4\frac{1}{3}}}}}{12+\left(\dfrac{\frac{3}{4}}{\frac{1}{5}}\div\dfrac{\frac{3}{4}}{5}\right)}\times\dfrac{9}{14}$ of $\dfrac{\left(\dfrac{3}{3\frac{1}{2}}\text{ of }1\frac{1}{6}\right)\frac{1}{6}}{7(1\frac{1}{2}\text{ of }\frac{3}{11})}-3$

6. $1+\dfrac{2\dfrac{1}{4\dfrac{3}{5\dfrac{4}{6\frac{2}{3}}}}}{3\dfrac{1}{5\dfrac{3}{6\dfrac{4}{33}}}}-2$.

7. $\dfrac{\frac{1}{3}+\frac{1}{5}+\frac{1}{7}}{\dfrac{1}{2\frac{1}{2}}+\dfrac{1}{3\frac{1}{2}}+\dfrac{1}{4\frac{1}{4}}}\times 1\frac{3}{11}$ of $7\frac{1}{3}-1$.

8. $1+\dfrac{9(3\frac{1}{4}\text{ of }\frac{7}{15})}{5\left(\frac{6}{\frac{4}{5}}\text{ of }\frac{2}{3}\right)}\div\dfrac{4}{7}\,1\frac{1}{7}\left(\dfrac{1\frac{1}{3}\text{ of }3\frac{1}{3}}{\frac{5}{12}\text{ of }4\frac{4}{5}}\right)-2$.

9. $\left(7\frac{3}{5}+\frac{5}{9}\text{ of }\dfrac{8\frac{2}{3}}{12\frac{1}{2}}\right)\times\dfrac{5}{8}$ of $\dfrac{1}{10\frac{5}{12}-1\frac{1}{3}}-\dfrac{50}{99}$.

10. $\left(3\frac{1}{9}-\frac{2}{3}\text{ of }\dfrac{8\frac{1}{3}}{7}+\dfrac{2\frac{2}{5}}{5\frac{1}{4}}\right)\div\dfrac{27}{6\frac{1}{8}-4\frac{3}{4}}-\dfrac{3}{8}$.

EXERCISE XIII.

Miscellaneous.

1. $\dfrac{6\frac{1}{5}\times 5\frac{1}{6}}{6\frac{1}{5}+5\frac{1}{6}}\div(2\frac{1}{11}+1\frac{69}{88})$. (K.S.)

2. $\dfrac{3\frac{2}{3}+2\frac{1}{4}\text{ of }7\frac{1}{5}}{3\frac{2}{3}\text{ of }2\frac{1}{4}+7\frac{1}{5}}\div\dfrac{\frac{2}{3}}{\frac{1}{2}}+2$. (C.P.)

3. $\dfrac{4\frac{1}{2}+3\frac{3}{4}-6\frac{1}{8}}{7\frac{1}{4}-5\frac{2}{3}+2\frac{1}{2}}\div\dfrac{371\frac{3}{4}}{577\frac{1}{2}}$. (C.P.)

4. $5\frac{1}{3}$ of $\dfrac{1}{1\frac{1}{3}+\dfrac{1}{2\frac{1}{4}}}\div\dfrac{4\frac{1}{5}+5\frac{1}{4}}{4\frac{1}{4}+3\frac{2}{3}}$. (C.PR.)

5. $\left(\dfrac{2\frac{1}{4}-\frac{3}{8}\times 1\frac{5}{6}}{\frac{1}{5}\times 3\frac{1}{3}+\frac{13}{36}}-\dfrac{1}{2\frac{1}{2}}\right)\div\dfrac{3}{5}$. (O.R.)

6. $\dfrac{1\frac{1}{2}\times 4\frac{2}{7}\times 2\frac{1}{3}}{1\frac{1}{3}\times 1\frac{6}{7}\times\left(\dfrac{1}{4\frac{1}{2}}+\dfrac{1}{6\frac{1}{2}}\right)}$. (C.S.L.)

7. $\dfrac{10\frac{1}{3} - 2\frac{3}{7} \text{ of } 2\frac{3}{8}}{(10\frac{1}{3} - 2\frac{3}{4}) \text{ of } 2\frac{5}{8}} \div \dfrac{1}{3 + \dfrac{1}{1\frac{1}{3}}}$. (O.S.L.)

8. $\dfrac{3\frac{3}{7}}{\frac{1}{5}\left(\dfrac{3}{3\frac{1}{2}} \text{ of } 7\right)} + \dfrac{5\frac{5}{8}}{4\frac{1}{6}} \text{ of } \dfrac{4 - (3\frac{1}{3} - 1\frac{1}{5})}{3\frac{1}{15} - 2\frac{1}{60} + \frac{11}{20}}$. (S.K.)

9. $\dfrac{\dfrac{1}{1\frac{1}{2}} + \dfrac{1}{1\frac{1}{3}} + \dfrac{1}{1\frac{1}{4}}}{3 - \left\{\dfrac{1}{1 - \frac{1}{16}} - \dfrac{1}{4 - \frac{8}{17}}\right\}}$. (C.J.)

10. $\dfrac{\frac{1}{9} \text{ of } 3\frac{6}{13} + \dfrac{2}{3\frac{1}{4}} - \dfrac{1}{6\frac{1}{2}}}{5 + \dfrac{1}{2\frac{1}{6}} + \frac{3}{13} \text{ of } 2\frac{1}{3}}$. (S.K.)

11. $\dfrac{6\frac{1}{4} - 3\frac{1}{16}}{3\frac{2}{5} \text{ of } 1\frac{1}{4}} \div \dfrac{12\frac{1}{15} - 11\frac{2}{3}}{3\frac{1}{3} \text{ of } \dfrac{1}{4\frac{1}{6}}}$. (S.K.)

12. $\dfrac{7\frac{3}{14} - 5\frac{2}{21}}{8\frac{6}{7} - 7\frac{5}{8}} \times \dfrac{1}{2 + \dfrac{1}{14 + \frac{1}{3}}}$. (O.S.L.)

13. $\dfrac{2\frac{2}{3} \text{ of } (\frac{4}{7} \div \frac{3}{5}) - \left(1\frac{1}{4} + \dfrac{2 - \frac{1}{5}}{5}\right)}{4\frac{1}{2} - 3\frac{1}{4} + 2\frac{3}{8} \text{ of } \frac{5}{16}}$. (C.P.)

14. $\dfrac{3\frac{1}{7}}{5\frac{1}{3} + \dfrac{1}{7\frac{1}{3} + \dfrac{1}{9\frac{1}{3}}}}$. (C.P.)

15. $1\frac{5}{6} \text{ of } \dfrac{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}}{2\frac{1}{2} - 3\frac{1}{3} + 4\frac{1}{4}} \times \dfrac{2\frac{1}{2} + \dfrac{\frac{3}{16}}{1\frac{3}{8}}}{\dfrac{3}{4\frac{1}{2}} + \dfrac{4\frac{1}{2}}{3}}$. (L.M.)

16. $\dfrac{1 + \frac{2}{3} - \frac{3}{4}}{1 + \frac{3}{2} - \frac{4}{3}} \times \dfrac{1}{1 + \dfrac{1}{2 + \frac{1}{3}}} \times \left(1 + \dfrac{1}{1 + \frac{1}{10}}\right)$. (L.M.)

17. $1\frac{5}{6} \text{ of } \dfrac{\frac{1}{2} + 1\frac{1}{2} + \frac{1}{4}}{2\frac{1}{2} - 3\frac{1}{3} + 4\frac{1}{4}} \times \dfrac{2\frac{1}{2} + \dfrac{\frac{3}{16}}{1\frac{3}{8}}}{\dfrac{3}{4\frac{1}{2}} + \dfrac{4\frac{1}{2}}{3}}$. (C.S.)

18. Multiply $1 - \dfrac{\frac{3}{10}}{(\frac{1}{2} + \frac{1}{5})^2}$ by $1 - \dfrac{\frac{1}{10}}{\frac{1}{4} - \frac{1}{10} + \frac{1}{25}}$. (S.K.)

19. Show that $2 - \dfrac{4}{5 - \dfrac{9}{7 - \frac{16}{9}}} = \dfrac{1}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}}$. (C.S.L.)

20. Subtract $1\dfrac{1}{1\frac{1}{5}} \text{ of } \dfrac{2\frac{3}{7}}{4\frac{1}{9} - \dfrac{2}{2\frac{1}{3}}}$ from $\left\{3 + \dfrac{2\frac{1}{3} - 1\frac{5}{6}}{1\frac{1}{40}\left(2\frac{1}{4} + \dfrac{3\frac{1}{2}}{6}\right)}\right\}$. (R.M.A.)

EXERCISE XIV. / 4

Problems in Vulgar Fractions.

1. The distance between two wickets was marked out for 22 yards, but the yard measure made use of was $\frac{7}{12}$ of an inch too short. Find the true distance which was marked out. (P.P.)

2. One-third of one-third of one-third of a ship is sold for £2456 8*s*. What is the value of a five-sixteenths share? (C.P.)

3. A father leaves an estate of 10106 ac. 2 rds. 32 po. to his five sons, the eldest to take one-fourth of the whole, and the remainder to be divided equally among the others. What was the share of each? (C.P.)

4. Out of a certain sum of money a boy spends $\frac{3}{5}$, and then $\frac{1}{4}$ of the remainder. He has 4*s*. 6*d*. left. What had he at first? (P.T.)

5. Of the leaves of a tree, one-fourth part fall in the first week of October, one-third part of the remainder in the next week, and two-fifths of those still remaining in the next. There are then 1350 leaves left. How many were there at first? (C.P.)

6. I read $\frac{3}{8}$ of a book on one day, and $\frac{4}{5}$ of the remainder on another day. If there were now 30 pages unread, how many pages did the book contain? (S.L.C.)

7. For one-third of a mile a submarine cable was laid overland, $\frac{3}{13}$ of it was suspended in the water, and $\frac{7}{8}$ rested on the bed of the sea. What was the length of the cable? (C.P.)

8. An undergraduate on a certain afternoon spent at his bookseller's two-sevenths of the money in his purse, and then at his grocer's two-fifths of the remainder. He afterwards went for a ride, paying for horse-hire two-thirds of what he had over, and then returned to his rooms, possessing 5*s*. How much did he spend each way? (C.PR.)

9. It is required to divide 36 into three such parts, that one-half of the first, one-third of the second, and one-fourth of the third may be equal to each other. (L.M.)

10. A train consists of engine, tender, and eight carriages. The weight of the engine and tender equals $\frac{5}{24}$ of the total weight of the train, and is 72 tons less than the weight of the carriages. Find the weight of a carriage. (P.T.)

11. A and B respectively received $\frac{3}{19}$ and $\frac{4}{17}$ of a certain sum of money, and C the remainder. A received £107 14*s*. 9*d*. (i.) What sum did B get, and (ii.) what fractional part was A's share of C's share? (C.P.)

12. Divide £4 2*s*. 6*d*. among three persons, A, B, and C, so that A's share may be $\frac{2}{3}$ of B's, and B's share $1\frac{1}{2}$ of C's. (S.K.)

13. Divide £48 0*s*. $2\frac{1}{2}$*d*. between C and D, so that $\frac{2}{3}$ of C's share may be $\frac{4}{5}$ of D's share. (C.P.)

14. If $\frac{1}{18}$ of a certain sum of money be added to $\frac{1}{3}$ of the same sum, the total is found to be less than $\frac{1}{2}$ of the sum by $5\frac{1}{2}d.$ What is the sum? (K.S.)

15. If the difference in value between $\frac{2}{3}$ and $\frac{7}{11}$ of an estate is £321 6*s.* 8*d.*, what is the value of the whole? (C.S.L.)

16. The highest score in an innings was $\frac{2}{3}$ of the total, and the next highest was $\frac{2}{3}$ of the remainder. These scores differ by 8 runs. What was the total? (C.W.)

17. From a cask of wine, containing 64 gals., 8 gals. are drawn out, and the cask is filled up with water. If the same process is repeated a second and then a third time, what will be the number of gallons of wine left in the cask? (O.S.L.)

18. Out of a cask of wine, of which a fifth part had leaked away, 10 gallons were drawn, and then the cask was two-thirds full. How much did it hold? (P.T.)

19. A man loses $\frac{3}{8}$ of his capital, and afterwards loses $\frac{2}{5}$ of the remainder. What fraction of it has he left? (L.M.)

20. If $\frac{1}{4}$ of a property is left to A, $\frac{2}{3}$ of the remainder to B, and $\frac{1}{3}$ of the rest to C, and £100 each to five other persons, what was the original value of the property? (C.P.)

21. A man spends $\frac{2}{7}$ of his money, and then $\frac{4}{15}$ of the remainder, then $\frac{4}{21}$ of the original sum, and finds he has 6*s.* 8*d.* left. How much had he at first? (C.P.)

22. A coal merchant sells to one customer $\frac{1}{8}$ of his stock, to a second customer $\frac{1}{3}$ of what remains, and to a third $\frac{1}{10}$ of what still remains. He has now 41 tons 6 cwt. 98 lbs. left. What had he to begin with? (O.S.L.)

23. Find the expense of a tour in which the hotel expenses amounted to $\frac{2}{3}$ of the whole, the cost of travelling to $\frac{2}{7}$ of the remainder, and all other expenses together to £5 12*s.* 6*d.* (C.H.L.)

24. A had at first £1 8*s.*, B paid him $2\frac{3\frac{1}{7}}{1\frac{2}{5}}$ of £1 11*s.* 6*d.*, and found that he had remaining $\frac{1}{13}$ of what A then had. How much had B at first? (R.M.A.)

25. For the ordinary parts of the Previous Examination 572 men put down their names; $\frac{8}{11}$ of the number entered for Part I., $\frac{9}{13}$ for Part II. How many men entered for both parts? (C.PR.)

26. Half my money I leave at home, $\frac{1}{3}$ I lend a friend, and spend $\frac{1}{5}$ of the rest. My friend can return me only $\frac{2}{3}$ of what I lent him; $\frac{3}{4}$ of what I left at home is stolen. What fraction of my money have I now? If I spend 2*s.* 3*d.* less than was stolen, what had I at first? (C.M.)

27. An equal sum is paid to each of 12 men; $\frac{1}{3}$ of them are paid in half-crowns, $\frac{1}{3}$ in shillings, and $\frac{1}{3}$ in sixpences. If 136 coins are paid away, find the amount. (C.W.)

28. A's income is $\frac{2}{3}$ of B's income; A's expenditure is $\frac{1}{2}$ of B's income; B's expenditure is equal to the whole of A's income. Show that each has the same sum left. (C.S.)

29. An exhibition has 17525 visitors. Of these a fifth hold season tickets, each costing a certain number of half-sovereigns. Of the remainder, a fifth purchase day tickets costing the like number of half-crowns, and the rest pay the same number of shillings for their evening tickets. The proceeds of the exhibition are £7991 8*s*. What were the three entrance charges? (C.P.)

30. If a third-class railway fare is $\frac{2}{3}$ of the second class, and the second $\frac{3}{4}$ of the first, how far can a third-class passenger travel for 4*s*. 2½*d*. when a first-class passenger pays £1 1*s*. 0½*d*. for 101 miles? (C.W.S.)

31. A man walking 18 miles finds that in 1 hr. 40 min. the distance he has walked is $\frac{2}{7}$ of the remaining distance. Find his rate of walking. (C.PR.)

32. A journey of 624 miles was made by coach, steamer, and rail. The distance by coach was $\frac{1}{15}$ of that by rail, and the distance by sea was $\frac{2}{3}$ that by rail. The fare per mile by sea was $\frac{1}{4}$ of that by rail, and the fare per mile by coach was half as much again as that by rail. What was the expense of the whole journey, coach fare being 2½*d*. per mile? (C.M.)

33. Find the rent of an estate whereof $\frac{5}{12}$ are let at £2 3*s*. an acre, and the remainder, consisting of 20 ac. 3 rds. 12 po., at £1 16*s*. 8*d*. an acre. (S.K.)

34. A man left half of his property to his wife, $\frac{1}{3}$ of the remainder to his eldest son, and the rest to be divided equally among five younger children, each of whom got £600. How much did the wife get? (C.J.)

35. A paid £12 13*s*. 2¾*d*. to B, B paid $\frac{1}{4}$ of that sum to C, and C paid £5 16*s*. 10½*d*. to **A**; then each one of them had £25 17*s*. 6*d*. How much money respectively had they before the payments were made? (C.P.)

36. Divide £3 18*s*. 4*d*. among three persons, A, B, and C, so that B may receive $\frac{7}{8}$ as much as A, and C 5*s*. more than B. (S.K.)

37. Divide £4 7*s*. 6*d*. between A and B as follows: first, $\frac{1}{3}$ to A, and $\frac{1}{7}$ to B; after this division, $\frac{4}{11}$ of the remainder to A, and the rest to B. (C.P.)

38. Divide £90 6*s*. among A, B, and C, so that A may receive $\frac{3}{4}$ as much as B receives, and C $\frac{1}{5}$ as much as A and B together. (S.K.)

39. The elder of two brothers inherited from his father $\frac{2}{7}$ more money than the other did. Since their father's death the elder brother has increased his capital to the extent of $\frac{3}{7}$ of its original value, and the younger brother has lost £707. He finds that he has now only £43 for every £100 that his elder brother has. How much had each of them at first? (C.S.L.)

40. A legacy is divided between three brothers, so that the eldest has $\frac{2}{5}$ of it, the second $\frac{3}{8}$, and the youngest the remainder, which is £35 less than the share of the eldest. Find the value of the legacy. (C.J.)

41. A man directs that his property shall at his death be divided amongst his four sons, A, B, C, and D, so that A shall receive half as much again as B, B half as much again as C, and C half as much again as D. Find what fraction of the whole estate each son will receive; and, if the share of A exceeds that of D by £2869, find how much each son receives. (C.S.L.)

42. Divide £10 between 4 men, 5 women, 6 boys, and 7 girls, so that each woman's share is $\frac{7}{8}$ of each man's, each boy has half the difference between a woman's and a girl's share, each girl has the difference between a man's and a woman's share. (C.W.)

43. After driving some distance on the way from one place to another, a gentleman observed that $\frac{1}{3}$ of the distance he had driven was $\frac{3}{8}$ of the distance he then had to drive; also that, when he had driven $4\frac{1}{2}$ miles farther, $\frac{1}{2}$ of his journey would be accomplished. How many miles had he to drive when he made the observation? (C.P.)

44. A boy was given the value of $\frac{6}{7}$ of a certain sum, and was asked to find what the sum was. He multiplied the value given by $\frac{6}{7}$, instead of dividing by that fraction, and so got an answer too little by £32 10*s*. What was the certain sum? (C.P.)

45. Of the voters in a town, $\frac{16}{33}$ promised to vote for X, and $\frac{15}{33}$ for Y. All these vote, and no others. But $\frac{1}{22}$ of the whole number promised both X and Y. On the election day 9 out of every 10 of these double promises vote for Y, who heads the poll by 40 votes. Find the total number on the register. (C.S.)

46. In an election A receives promises from half the constituency, B from $\frac{19}{40}$; 400 voters, however, had promised both candidates, of whom 300 vote eventually for B, and 100 for A. The electors who had not promised either do not vote. B wins by 80 votes. What were the numbers for each? (C.S.L.)

47. A girl is told that 2 lbs. 2 ozs. 15 drs. of sand is to be equally divided into 7 heaps, and that each heap is to be equally divided among 7 boxes, and she is asked to find how much sand there will be in each heap and how much in each box. She divides accordingly, but neglects in each division the fraction of a dram. Of how much sand does she deprive each box? (P.T.)

48. The products obtained by multiplying together each pair of three given numbers are 3, $4\frac{2}{3}$, $8\frac{2}{3}$ respectively. What are the three numbers? (C.S.)

49 A father left $\frac{8}{15}$ of his estate to his elder and the remainder to his younger son. The elder spent $\frac{4}{25}$ of his share, and divided the remainder equally among his three sons; the younger spent $\frac{24}{175}$ of his share, and divided the remainder equally among his four daughters. One of the sons of the elder married one of the daughters of the younger brother. What fraction of the estate had they between them? (C.S.)

50. A man with a fixed yearly income and a fixed daily expenditure saves in ordinary years $\frac{1}{27}$ of his income. In leap years he saves £4 4*s*. 9*d*. What is his income? (C.M.)

CHAPTER VIII.

DECIMALS.

Let us examine the number 888. The digit 8 in the *third* place (counting from the right) stands for 8 *hundreds*, the digit 8 in the *second* place stands for 8 *tens*, and the digit 8 in the *first* place stands for 8 *units*. As before explained (p. 3), each digit has a *local* value, *i.e.* a value dependent on its position. If the digit 8 be moved from the third to the second place, it represents 8 *tens* instead of 8 *hundreds*, i.e. *its present value is only one-tenth of its former value.*

Similarly, when the digit 8 is moved from the second to the first place, it becomes only one-tenth of its former value.

The question now arises: Can we move the digit 8 still further to the right? Let us try, and see what the result is.

Hundreds.	Tens.	Units.	
8	8	8	8

The digit 8 moved one place further to the right than the units' place, is, according to the rule enunciated above, only *one-tenth* of its former or unit value, *i.e.* our 8 *units* have become 8 *tenths*.

Similarly, by a move to the second place on the right (counting from the units' place) our 8 *tenths* become 8 *hundredths*.

Hence the following:—

Thousands.	Hundreds.	Tens.	Units.	Tenths.	Hundredths.	Thousandths.
8	8	8	8	8	8	8

The Decimal Point.—But the first four digits represent *whole numbers*, whereas the last three digits represent quantities *less than unity*. It is advisable to have some mark separating the whole numbers from the fractions; hence what is known as the *decimal point* is used. Thus 8888·888.

Notes.—(1) The decimal point is written *above* the line thus, ·, in contradistinction to the sign ., which is written *on* the line, and which signifies multiplication.

(2) It would be better if it were customary to place the dot above the digit in the units' place, for then the symmetrical arrangement of the digits to the right and left of the units' place would be more apparent. This, however, would necessitate the use of another convention for repeating decimals.

∞ Hundreds.	∞ Tens.	8	8	∞ Tenths.	∞ Hundredths.
			•		

Definitions of a Decimal Fraction.

(1) *A Decimal Fraction is a fraction whose denominator is some power of* 10.

According to this definition, both $\frac{7}{10}$ and ·7 are decimal fractions, but with this difference, that in $\frac{7}{10}$ the denominator is expressed, whereas in ·7 it is not. In decimal fractions, as commonly understood, the denominator is not expressed. Hence the following (and preferable) definition :—

(2) *A Decimal Fraction is a quantity less than unity expressed in the ordinary system of notation.*

Notes.—(1) The 'ordinary system of notation' is a decimal system, *i.e.* each significant digit represents a multiple of some power of 10.

(2) The development of the Arabic (or decimal) system for the expression of quantities less than unity is mainly due to English mathematicians at the beginning of the seventeenth century.

Reading a Decimal.

The following are correct methods of reading a decimal fraction, *e.g.* ·123 :—

(1) Point (or decimal) one, two, three.

(2) One hundred and twenty-three thousandths.

(3) One tenth, two hundredths, three thousandths.

The first is the *usual* method ; the others have the advantage of showing the value of the digits.

The following is an incorrect (but, unfortunately, too common) method of reading a decimal fraction : —

Point (or decimal) one hundred and twenty-three.

This means either 123 units, or perhaps 123 tenths. It certainly does not mean 123 thousandths.

Effects of Ciphers on Decimals.

Compare the value of ·34 and ·340.

$$\cdot 34 = \tfrac{3}{10} + \tfrac{4}{100} = \tfrac{34}{100}$$

$$\cdot 340 = \tfrac{3}{10} + \tfrac{4}{100} + \tfrac{0}{1000} = \tfrac{34}{100} : \text{ the same as } \cdot 34 \text{ above.}$$

Hence, ciphers placed to the *right* of a decimal do not affect its value unless significant digits follow.

Again, *compare the value of* ·34 and ·034.

$$\cdot 34 = \tfrac{3}{10} + \tfrac{4}{100} = \tfrac{34}{100}$$
$$\cdot 034 = \tfrac{0}{10} + \tfrac{3}{100} + \tfrac{4}{1000} = \tfrac{34}{1000}$$

which is *one-tenth* of $\frac{34}{100}$ (or ·34) above.

Hence, a cipher placed to the *left* of the digits forming a decimal fraction decreases the value of the decimal to one-tenth of its former value. Similarly, two ciphers placed to the *left* give a decimal *one-hundredth* of the first decimal.

Effect of moving the Decimal Point.

In the expression 25·34, if the decimal point be moved *one* place to the *right*, the expression becomes 253·4, *i.e.* it is now *ten* times its original value. If it be moved *two* places to the right, the expression becomes *one hundred* times its original value, and so on.

Conversely, if the decimal point be moved *one* place to the left, the resulting expression is *one-tenth* of its original value, if moved *two* places, *one-hundredth* of its original value, and so on.

Addition and Subtraction of Decimals are performed in exactly the same way as addition and subtraction of whole numbers.

Just as in addition of whole numbers we place units under units, tens under tens, etc., so, in addition of decimal fractions, we place *tenths* under *tenths*, *hundredths* under *hundredths*, etc., because only quantities of *like* denomination can be added together. When an addition sum is put down in accordance with the above directions, it will be observed that the decimal points fall under one another.

Hence the following working rule: *Arrange the numbers so that the decimal points come in the same vertical line.*

Multiplication of Decimals.

1. *Multiplication by some power of* 10.

5 × 10 = 50, *i.e.* 5 *units* become 5 *tens*
55 × 10 = 550, *i.e.* 5 *tens* and 5 *units* become 5 *hundreds* and 5 *tens*

i.e. in multiplying a whole number by 10, we move the digit or digits multiplied *one place to the left.* Applying this principle to decimals—

·5 × 10 = 5, *i.e.* 5 *tenths* become 5 *units*
·55 × 10 = 5·5, *i.e.* 5 *tenths* and 5 *hundredths* become 5 *units* and 5 *tenths*

Hence, in multiplying a decimal fraction by 10, we move the digit or digits one place to the left, or *we move the decimal point one place to the right*, which amounts to the same thing.

To multiply by 100, we move the decimal point *two* places to the right; and, generally speaking—

To multiply by the *first* power of 10 (*i.e.* 10), move the decimal point *one* place to the right.

To multiply by the *second* power of 10 (*i.e.* 100), move the decimal point *two* places to the right.

To multiply by the nth power of 10 (*i.e.* 10^n), move the decimal point n places to the right.

2. *Multiplication by numbers other than some power of* 10.

The cases in which the multiplier is a whole number or a decimal fraction or a mixed number are all reducible to the same rule.

Rule.—*Ignore the decimal points and treat the digits as though they were integers. Mark off from the right-hand digit of the product as many places as there are decimal places in the multiplier and multiplicand together, and prefix the point. If there are not sufficient digits in the product, prefix ciphers until the required number of digits is obtained.*

Examples.—

(*a*) 7·23 × 2·3.

723	
23	
2169	
1446	
16629	

There is a total of three decimal places in the multiplier and multiplicand; so, marking off three places from the right of the product we get 16·629. *Ans.*

(*b*) ·0723 × ·023.

723	
23	
2169	
1446	
16629	

Seven decimal places have to be marked off. There are only five digits. Hence two noughts are prefixed, and we get ·0016629. *Ans.*

Proof of Example (*a*).—By regarding 7·23 and 2·3 as whole numbers, we have multiplied together numbers that are respectively 100 and 10 times greater than the numbers we were required to multiply. The product is therefore 100 × 10 times or 1000 times greater than it should have been. Hence to obtain the correct product we must divide by 1000, and we do this by moving the decimal point three places to the left (see p. 96).

Division of Decimals.

(*a*) *The case in which the divisor is a whole number.*

Example.—1·42 ÷ 5.

$$5 \mid \underline{1\cdot42} \quad \cdot284$$

We proceed as in ordinary division. 5 is not contained an integral number of times in one unit. We change the one unit into ten tenths.

The fifth part of 14 tenths is 2 tenths, with a remainder of 4 tenths. We prefix the decimal point, and write down 2 tenths in the quotient. The remainder 4 tenths, together with 2 hundredths, make 42 hundredths, the fifth part of which is 8 hundredths (which we place in the quotient), and a remainder of 2 hundredths or 20 thousandths. The fifth part of 20 thousandths is 4 thousandths, which we place in the quotient.

(*b*) *The case in which the divisor is a decimal.*

Example.—16·629 ÷ 7·23.

Place in fractional form thus: $\frac{16\cdot629}{7\cdot23}$.

If both the numerator and the denominator be multiplied by 1000, we shall get rid of the decimals.

$$\frac{16\cdot629}{7\cdot23} = \frac{16\cdot629 \times 1000}{7\cdot23 \times 1000} = \frac{16629}{7230}$$

$$\begin{array}{r} 7230)16629(2\cdot3 \\ \underline{14460} \\ 21690 \\ \underline{21690} \end{array} \quad \textit{Ans.}\ 2\cdot3.$$

Example.—·00000011 ÷ 1·1.

$$\frac{\cdot00000011}{1\cdot1} = \frac{\cdot00000011 \times 100000000}{1\cdot1 \times 100000000} = \frac{11}{110000000}$$

$$\begin{array}{r} 110000000)110000000(\cdot0000001 \\ \underline{110000000} \end{array} \quad \textit{Ans.}\ \cdot0000001.$$

The divisor is not contained in the dividend any whole number of times. Put the decimal point. Add a nought to the dividend. The divisor is not contained in the dividend any *tenths* of times. Place a *nought* in the quotient. Place another nought in the dividend. It is not contained any *hundredths* of times. Place a second nought in the quotient, and so on.

Rule.—*Multiply both numerator and denominator by such a power of* 10 *as will rid both of all decimals, and proceed with the division.*

An objection to the use of this method in the above example is that many useless noughts are employed. This may be overcome by the following consideration. If we divide the divisor by 10, and the quotient also by 10, we make no difference in the result.

$$\begin{array}{r} 110000000)11 \\ \underline{11} \end{array} \qquad (\cdot0000001$$

It is evident that the quotient will contain the digit 1. Place it in the quotient, temporarily as 1 whole number. Divide the divisor by 10000000; similarly divide the quotient by 10000000, and we obtain ·0000001 in the quotient.

The Remainder in Division of Decimals.—It should be carefully noted that if the divisor and dividend have been multiplied by the same number, the remainder obtained must be divided by the *same* number in order to obtain the *true* remainder.

Example.—*How many pieces, each* 3·05 *in. long, can be cut from a rod* 21 *ft.* $2\frac{3}{4}$ *in. long; and what will be the length of the remaining piece?* (C.S.L.)

$$\text{Number of pieces} = \frac{21 \text{ ft. } 2\frac{3}{4} \text{ in.}}{3{\cdot}05 \text{ in.}} = \frac{254{\cdot}75 \text{ in.}}{3{\cdot}05 \text{ in.}} = \frac{254{\cdot}75 \times 100}{3{\cdot}05 \times 100} = \frac{25475}{305}$$

305)25475(83
2440
1075
915
160

The number of pieces is 83.

Now to deal with the remainder.

The 160 as it stands is 160 in. But because numerator and denominator were 100 times their true value, the remainder is 100 times its true value. The true remainder is therefore $\frac{160 \text{ in.}}{100}$, or 1·6 in.

Ans. Number of pieces, 83.
Length of remaining piece, 1·6 in.

This matter is difficult to some students, so we shall proceed to verify our result.

Length cut off = 3·05 in. × 83 = 253·15 in.
Hence the remaining piece = 254·75 in. − 253·15 in. = 1·6 in. *Ans.*

An explanation of the reason why, in the division of one decimal by another (each fraction being less than unity), an integral quotient may be obtained, is seen in the fact that the quotient represents the number of times the divisor is contained in the dividend.

For an explanation of the fact that in a division of decimals the quotient may be larger than the dividend, see p. 67.

THEORY.

1. What do you mean by a *decimal?* (C.P. and P.T.)

2. State exactly what is meant by ·032 of any whole number. (S.K.)

3. Write down the series of fractions (their denominators being powers of 10) which is equivalent to the decimal ·12345. (C.P.)

4. What effect is produced by placing a cipher—(*a*) To the right of a whole number; (*b*) to the left of a whole number; (*c*) to the right of a decimal; (*d*) to the left of a decimal, but after the decimal point? (I.I.)

5. If we add ciphers to the end of a decimal fraction the value of the fraction is unaltered, but if we add ciphers at the end of a whole number the value of the number is changed. Explain this. (K.S.)

6. Define *decimal fractions.* Explain how it is (i.) that the value of a decimal fraction is *not affected* by ciphers affixed to the right of it;

and (ii.) that the value is *decreased* by prefixing ciphers to the left of it. Take as an example ·4, ·400, ·40000, ·004, ·00004. (C.P.)

7. Explain, by reference to the corresponding vulgar fractions, why ·307 is equal to ·30700. In what other way can you explain this equality? Why is it wrong to read ·307 as *decimal three hundred and seven*? (P.T.)

8. Prove the rule for placing the decimal point in multiplication of decimals. (C.W.)

9. Give the reason of the ordinary rule for fixing the position of the decimal point in a product. (L.M.)

10. Why is the number of decimal places in the product of two decimals equal to the sum of the decimal places in the multiplier and multiplicand? Illustrate by 6·375 and ·048. (P.T.)

11. What is the rule for placing the decimal point in the answer in the multiplication of one decimal by another? Show by an example that the rule is correct in that particular case. (K.S.)

12. What method do you employ for the division of decimals? How would you explain it? (K.S.)

13. Give a rule for placing the point in the division of one decimal by another, and show that the rule applies in the case of $1{\cdot}0001 \div {\cdot}000010001$. (P.T.)

14. Give a rule for division of decimals which shall include all cases of division, and explain why the division of one decimal by another, each fraction being less than unity, may give an integral quotient. (P.T.)

15. Explain the reason for the rule for fixing the position of the *point* in the quotient of two decimals. (L.M. and C.H.L.)

16. Prove the rule for the division of decimals. (C.P. and C.PR.)

17. Explain the method by which you can most simply multiply or divide a decimal by 1000. (C.J.)

18. How would you explain the fact that in division of decimals the quotient may be larger than the dividend? Give an example. (K.S.)

19. Give short notes of a first lesson in decimal fractions. (C.M.)

20. Prove the rule for the conversion of a vulgar fraction to a decimal fraction, *e.g.* $\frac{3}{4}d.$ to a decimal fraction of $1d.$ (C.W.)

EXERCISE XV.

Finite Decimals. The Four Rules.

Addition and Subtraction.

1. Find the sum of 16·94, ·00037, 1·79375, 2·185, and ·090087. (C.J.)

2. Add together 71206·00123, 42·123, ·006, and ·100069. (C.S.)

3. Find the value of $1{\cdot}2345 + 23{\cdot}4567 - 345{\cdot}6789 + 4567{\cdot}8901$. (C.P.)

4. Add together 896·51, ·00867, 1·92, and ·806. (C.S.)

5. Find in decimals the difference between ·8695234 and $\frac{108}{125}$. (P.T.)

6 Add together 3·40765, 537·063, and ·84379. Subtract the result from 601·0473. (R.M.A.)

Multiplication.

7. Multiply 47·00658 by ·05062. (O.S.L.)

8. Multiply the sum of ·0083 and ·8003 by 8·003. (P.T.)

9. Multiply 16·384 by ·375625. (O.S.L.)

10. Multiply the sum of 42·35 and ·0047 by the difference between 6·259 and 17·25.

11. Find the continued product of ·01, ·001, and ·0001.

12. What number divided by 2·11 will give 2·95 as a quotient?

13. A train is travelling at the rate of 42·12 miles per hour. How far will it go in ·16 of an hour?

14. Multiply 613·24 by 45, and by ·000045. (O.R.)

15. Multiply ·00124 by ·0325. (O.J.)

16. Square 11·111. (C.P.)

17. Multiply 240·1803 by 2·401803, and, from your result, write down the square of 2401·803. (C.P.)

18. If x stand for ·278, write down the values of $x \times 10$, $x \times 1$, $x \times \cdot 1$, $x \times \cdot 01$, $x \times \cdot 001$. Show from the results that the value of $\frac{\cdot 0156}{\cdot 278}$ is greater than ·01, but less than ·1. (L.M.)

Division.

19. (*a*) Divide 555·0592 by 93·76 (O.R.); (*b*) 429 by ·026 (O.J.); (*c*) 429·55 by ·01375 (C.S.L.); (*d*) 105·5906 by ·00703 (C.H.L.); (*e*) 3217 by ·00625 (K.S.); (*f*) ·9651 by 187·5. (K.S.)

20. Multiply 4·23506 by ·00423; and divide ·004 by ·04, by 4000, and by 23·67 respectively. (C.P.)

21. Multiply 3·5 by ·008; and divide 60·2 by ·002 and ·0602 by 2000. (O.R.)

22. Divide 4·953 by ·1524, and from this result deduce the quotient when 495·3 is divided by ·0001524. (C.H.L.)

23. Divide ·00015573 by the square of ·23 increased by one-tenth of the cube of ·2. (C.P.)

24. Divide 5·75 by ·0184. From your result write down the quotients when ·0575 is divided by 1·84, and 5750 is divided by 18·4. (K.S.)

25. If, in a division sum, the dividend is 483·758 and the quotient is 99·95, what is the divisor? (C.P.)

26. The product of two numbers is 5·65504, and their quotient is 6·4: find them. (C.M.)

27. Find the length of the remaining piece when portions each ·1234 in. long are cut off as often as possible from a ribbon 89·576 in. long. (S.A.)

28. Two lines are 41·06328 in. and ·0438 in. long respectively. How many lines as long as the latter can be cut off from the former? What will be the length of the remaining line? (C.S.)

29. The product of two numbers is 191·1. One of the numbers is 546: find the other.

30. $\dfrac{\cdot 304 \times \cdot 002 \times 1 \cdot 8}{\cdot 0009 \times \cdot 038}$.

Miscellaneous Examples.

31. Divide the difference between the thousandth part of 375 and the hundredth part of 16 by the ten-thousandth part of 43. (P.T.)

32. Add the following numbers: (i.) ·144578, (ii.) ·8923495, (iii.) 17·34, (iv.) 2·8912, and (v.) ·00734801. Multiply (i.) and (iii.) together, and

(ii.) and (iv.) together. Subtract one result from the other, and add ·0270216456. (C.P.)

33. Divide the difference between one-tenth of three thousandths and one-hundredth of nine ten-thousandths by the sum of one-millionth of three hundred and ninety-five and one-hundredth of nine-thousandths. (P.T.)

34. Write down the product of ·001 by 100, and of ·02 by ·002: also the quotient of ·001 divided by ·0001, and of 20 divided by 2000. Give the reason (not the rule) for *one* of the multiplications and *one* of the divisions. (C.P.)

35. Subtract ·992748 from 1; divide the result by ·0074; and prove the truth of your division by vulgar fractions.

36. Divide the sum of $\frac{7}{22}$ and $\frac{1604}{5011}$ by their difference, and express one-millionth part of the quotient as a decimal. (C.P.)

37. Subtract ·004789 from 24·07484; multiply the remainder by ·0005; and divide the latter result by 4000. (C.P.)

38. Divide the difference between 5·5225 and the square of ·075 by 126·1. (C.P.)

39. Subtract seven-tenths and three-thousandths from one and five-hundredths and four ten-thousandths. Multiply 364·78 by 8·5005. In each case verify your result by vulgar fractions. (C.P.)

40. Divide the sum of $\frac{\cdot0391}{2\cdot3}$ and $\frac{9\cdot49}{13}$ by ·83.

41. Subtract 0·07063 from 1·003032. State the rule for multiplying one decimal by another. Find the product of 0·0256 and 1·0071, and divide the product by 2·7975. (S.K.)

42. 2·001 × ·013 ÷ ·0023. (C.P.)

43. (·0011 × ·091 ÷ ·0035) + (·015 × ·507 ÷ ·39). (S.K.)

44. (1·61 × ·0209 ÷ ·00253) − (2·03 × ·336 ÷ 32·48). (S.K.)

45. (7·35 × ·0143 ÷ 15·015) − (·152 × ·033 ÷ 2·09). (S.K.)

46. $\dfrac{25 - \cdot0025 + 3\cdot82 \div \cdot25}{8055\cdot5 \times \cdot05}$. (O.R.)

47. Evaluate $\dfrac{110\cdot6 \times \cdot018 \div \cdot08}{1\cdot5 \times \cdot08 + \cdot09}$. (C.PR.)

48. Reduce to its lowest terms the following fractional expression, and find its value to four decimal places: $\dfrac{\frac{1}{2} + \frac{2}{3} + \frac{3}{7}}{\frac{5}{2} + \frac{1}{3} + \frac{1}{7}}$. (S.K.)

49. $\dfrac{6\cdot501 - 3\cdot07 + 2\cdot124}{5\cdot5}$. (O.R.)

50. $\dfrac{(2\cdot31)^3 - (1\cdot69)^3}{(2\cdot31)^2 - (1\cdot69)^2}$. (C.S.L.)

CHAPTER IX.

DECIMALS: THE FOUR RULES—CONTRACTED METHODS.

Addition.

Example.—*Add together* 14·57361, ·80137, 32·4739, 1·0029, ·11387, 5·43689 *correct to three places of decimals.*

14·5736	
·8014	
32·4739	We re-write the addends correct to four places in order to obtain a "carrying" figure to be added to the addends of the third place. If a decimal in the fifth place consists of a digit 5 or over 5 in value, one is added to the digit in the fourth place, *e.g.*—
1·0029	
·1139	
5·4369	·80137, the second addend, becomes ·8014.
54·4026	
54·403. *Ans.*	We then re-write the sum correct to three places.

Subtraction.—The contracted method much resembles that employed in addition, except that, if both subtrahend and minuend are shortened both in excess or both in defect, it is unnecessary to retain more digits in the working than are required in the answer.

Example.—*Take* ·987659321 *from* 2·037626115 *correct to five places of decimals.*

2·03763	Here both minuend and subtrahend are shortened in excess, but if the subtrahend be shortened in excess and the minuend in defect, or *vice versâ*, the difference will not be correct unless both minuend and subtrahend are extended one digit more than the required number.
·98766	
1·04997	

Multiplication.

Example.—*Multiply* 4·316 *by* 4·316 *correct to three places of decimals.*

Ordinary Method.		*Contracted Method.**
4·316	4·316	4·316
4·316	4·316	6134
25896	17264	17264
4316	1294 8	1295
12948	43 16	43
17264	25 896	26
18·627856	18·627 856	18·628

* Where possible, all decimal workings should be properly contracted.

These methods are placed side by side for the student's observation. The first is the ordinary or conventional method. The second is not the ordinary method, but it yields exactly the same result as the first. In the second method, we begin to multiply by the *left*-hand digit, and place the right-hand digit of each partial product one place to the *right* instead of to the left, as in the ordinary method.

Now, suppose we have to multiply 4·316 by 4·316 correct to three places of decimals, we can reverse the multiplier as shown in the contracted method, which is equivalent to multiplying as in the second method.

The first partial product 17264 is found in the usual way. To obtain the second partial product, the last digit (6) of the multiplicand is disregarded, save that we mentally multiply 3 by 6, and since the product 18 is nearer 20 than 10, we carry 2 to the next product, and then proceed as in ordinary multiplication.

To obtain the third partial product, we disregard (1) the next digit of the multiplicand, save that we note it for a carrying figure. As, however, the product is only 2, we disregard it altogether. The fourth partial product is obtained in a similar manner.

Rule—

1. Mark off in the multiplicand as many places plus one * as the number of decimal places required to be found.

2. Place the *units*' digit of the multiplier under the last of the decimal places of the multiplicand. Write the digits of the multiplier in the reverse order, and if any digit of the multiplier has not a digit above it in the multiplicand, place a cipher there. *See example below.*

3. Find the first partial product as in ordinary multiplication.

4. Find the second partial product as in ordinary multiplication, but ignore the right-hand digit of the multiplicand, except for the purpose of ascertaining the carrying figure.

5. Find the third partial product, but ignore the *second* right-hand digit, except for ascertaining the carrying figure. Take care to put the first digit of all these partial products in the same vertical line.

6. Add up the partial products, and mark off in the sum the required number of decimal places.

Example.—Multiply 52·687640812 by 18·703216231 correct to six places of decimals.

```
          ×
052·687640812
132 61230781
-------------
  5268764081
  4215011265
   368813486
     1580629
      105375
        5269
        3161
         105
          16
-------------
985·4283387   Ans. 985·428339.
```

* This is to ensure accuracy.

Division.

The following rule for contracted division will explain itself in the light of the contracted multiplication exercise given on p. 104 and the contracted division exercise below.

1. If there are more decimal places in the divisor than there are in the dividend, retain one more divisor than the number required and strike out the rest.

2. Proceed in the ordinary way (p. 98) until all the digits in the dividend have been brought down.

3. When the number of digits in the dividend is less than the number of digits in the divisor, instead of affixing ciphers as in ordinary division of decimals, strike off a digit from the divisor and proceed one step with this curtailed divisor.

4. Repeat this process, striking off the digits of the divisor in succession, until the required number of decimal places in the dividend has been obtained.

Note.—In multiplication it is desirable to add in the nearest multiple of 10 from the multiple of the digit just struck off.

Example.—Divide 985·4283387 by 18·703216231 correct to six decimal places.

Note the connection between this example and the preceding one.

Retaining only *seven* decimal places (*i.e.* one more than the number required in the quotient) in the divisor, we have—

$$\frac{985\cdot4283387}{18\cdot7032162} = \frac{9854283387}{187032162}$$

```
187032162)9854283387(52·687640
          935160810
          ---------
           502675287
           374064324
           ---------
           128610963
           112219297
           ---------
            16391666
            14962573
            --------
             1429093
             1309225
             -------
              119868
              112219
              ------
                7649
                7481
                ----
                 168
```

Or, using the Italian method, we have—

```
187,0,3,2,1,6,2)9854283387(52·687640
                ----------
                 502675287
                 ---------
                 128610963
                 ---------
                  16391666
                  --------
                   1429093
                   -------
                    119868
                    ------
                      7649
                      ----
                       168
```

Example.—*Find correct to five places of decimals the value of the infinite series—*

$$1+\frac{1}{2}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|5}}+\ldots \quad \text{(L.M. and C.S.)}$$

The notation $\underline{|3}$, $\underline{|4}$, $\underline{|5}$, is used to express the continued product of all the integers from 1 to 3, 1 to 4, 1 to 5. They are read as "factorial three," "factorial four," etc.

Then we might re-write thus—

$$1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\frac{1}{1.2.3.4.5}+$$

Each fraction is expressed as a decimal: each fraction is obtained from its predecessor by division, and in order to ensure accuracy, each is carried out to seven places of decimals.

$$\begin{aligned}
&1 && = 1\cdot\\
&\frac{1}{1\times 2} && = \cdot 5\\
&\frac{1}{1\times 2\times 3} && = \cdot 1666667\ldots\\
&\frac{1}{1\times 2\times 3\times 4} && = \cdot 0416667\ldots\\
&\frac{1}{1\times 2\times 3\times 4\times 5} && = \cdot 0083333\ldots\\
&\frac{1}{1\times 2\times 3\times 4\times 5\times 6} && = \cdot 0013889\ldots\\
&\frac{1}{1\times 2\times 3\times 4\times 5\times 6\times 7} && = \cdot 0001984\ldots\\
&\frac{1}{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8} && = \cdot 0000248\ldots\\
&\frac{1}{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9} && = \cdot 0000028\ldots\\
&\frac{1}{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9} && = \cdot 0000003\ldots\\
& && \overline{1\cdot 7182819\ldots}
\end{aligned}$$

There is no need to find the value of the next or any subsequent term, as their values cannot be expressed by any significant digits in seven places of decimals, and hence will not affect the result.

Ans. 1·71828.

Example.—*Find correct to five decimal places the sum of—*

$$\frac{1}{8}+\frac{1}{8^2}+\frac{1}{8^3}+\frac{1}{8^4}+\frac{1}{8^5}+\ldots \quad \text{(C.S.)}$$

This may be solved by a similar process.

$$\frac{1}{8} = \cdot 1250000$$

$$\frac{1}{8^2} = \cdot 0156250, \text{ etc., etc.} \quad \textit{Ans.} \cdot 14285\ldots$$

NOTE.—*After decimals have been mastered, all remainders, as a rule, should be given as decimals, and not as fractions.*

EXERCISE XVI. 16

Finite Decimals. Contracted Methods.

1. 8·097 + 9761·03 + 598 + ·07912 + 52052 + 359·07316 correct to 3 places.

2. 1·1 + 20·02 + 13 + 2·845 + 1·0001 correct to 3 places.

3. 82·537 + 2000 + 1·354 + ·006 + 13 correct to 2 places.

4. 347·859 + ·010101 + 639 + 2·573 + 11·01115 + ·32784 correct to 5 places.

5. 7600 + 3·1009 + 473·842691 + ·07 + ·00001 + 1·1 correct to 4 places.

6. 777·88 + 99·555 + 441·765 + 79·0456773 correct to 6 places.

7. 70·304 − 29·30465 correct to 3 places.

8. ·37124 − ·1234567 correct to 6 places.

9. From 2000 take the sum of 99·876, 375·3, and 894·43245 correct to 3 places.

10. (210·346 + 57·36 + 44·56 + 79) − 83·450932 correct to 5 places.

11. Find to within one-thousandth the sum of 16·4184, 62·6187, 22·438, and 15·0376.

12. 5·0175 × ·225 correct to 5 places.

13. 292·5 × ·00067 correct to 5 places.

14. 11·007 × ·00678 correct to 7 places.

15. ·123 × ·0203 × ·02 correct to 7 places.

16. ·07 × 3·972 × 975 correct to 2 places.

17. 9·71 × 5·836 × ·0001 correct to 5 places.

18. 3·705 ÷ ·0705 correct to 3 places.

19. ·237191 ÷ ·0232 correct to 4 places.

20. Multiply 768·4087 by 35·567, retaining integers only.

21. Divide 626·34276 by 143·567 to within one hundred thousandth of the whole.

22. Square 7·14389673 correct to 3 decimal places.

23. Divide 1 by 1·2599210 correct to 5 significant decimals.

24. $\frac{2{\cdot}44948974}{1{\cdot}41421356237}$ correct to 4 decimal places.

25. Find to a millionth the reciprocal of 3·203458612.

26. Find to 5 decimal places the difference between the square of 8·50163 and the cube of 3·712. (C.W.)

27. Multiply 5·7295 by 1·61423 correct to 5 places of decimals. (I.I.)

28. Find the value of 3·14159 × ·123456 to 3 decimal places with the least possible amount of work. (C.W.)

29. Multiply 3·14159 by ·69897 correctly to 7 places of decimals. (C.M.)

30. Find the values, correct within one ten-thousandth of the whole, of 5327·643 × 781·24395 and 7384·291 ÷ 4·382672, using methods of approximate calculation. (C.M.L.)

31. Divide 24·946742 by 7 to 4 significant figures. Assuming that a metre = 39·37079 in., express a square metre in square feet and decimals of a square foot correct to 3 significant figures. (K.S.)

32. Divide ·736 by 2·85, and 2·85 by ·7036, in each case to 4 places of decimals, and find the product formed by multiplying the quotients together. (S.K.)

33. Divide 5, correctly to 3 places of decimals, by ·01234. (C.P.)

34. Divide ·3 by ·17 to 4 places of decimals, and 71 by ·0013 to the end of the integral part of the answer. (C.P.)

35. Divide 73438·6445 by 4·12345, and ·004321 by ·00043212, correct to 5 places of decimals. (C.P.)

36. Find the first 5 decimal places of the quotient of ·1373 divided by ·021. (S.K.)

37. Find to 8 decimal places the value of—

$$\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots$$

38. Find to 4 decimal places the value of—

$$\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\ldots$$

39. Find correct to 7 places of decimals—

$$1+\frac{1}{1}+\frac{1}{2}+\frac{1}{\underline{3}}+\frac{1}{\underline{4}}+\frac{1}{\underline{5}}+\ldots$$

40. Find the value of $1+\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\ldots$ correct to 5 places of decimals. (O.S.L.)

EXERCISE XVII. 17

Problems involving Finite Decimals.

1. Divide 100 into two parts such that one may exceed the other by 8·75. (P.T.)

2. How many pieces each 6·1 in. long can be cut from a rod 21 ft. $2\frac{3}{4}$ in. long; and what will be the length of the remaining piece? (C.S.L.)

3. The product of two numbers is ·15252. If we increase one of them by ·02 the product becomes ·1525446. Find the numbers. (P.T.)

4. What decimal fraction, when diminished by $\frac{1}{10}$ of itself, becomes ·45? (P.T.)

5. How much does a creditor to the amount of £9518 receive when his debtor pays £·625 in the pound? Prove this sum by vulgar fractions. (C.W.)

6. If the circumference of a circle be taken as $3\frac{1}{7}$ times the diameter, how many turns in a minute will a wheel, 4·55 ft. across, make at the rate of 10·4 mi. in 1·25 hr.? (C.P.)

7. A bicycle wheel has a circumference of 7·145 ft., and makes 6500 revolutions in an hour. Find the distance in miles travelled in $5\frac{1}{4}$ hrs., as nearly correct as you can give it to two places of decimals only. (C.S.L.)

8. A small object weighs ·009 of an ounce. How many such objects would be required to turn the scale against a weight of 1 cwt. 1 qr. 1 lb., and by what weight would the mass of small objects exceed this weight? (P.T.)

9. How many times can a cup containing ·013 of a pint be completely

filled from a jug containing a quart; and how much (expressed as a decimal of a pint) will remain in the jug after the last complete filling of the cup? (P.T.)

10. A cup contains ·7735 litre. How often must it be filled to obtain 1575 litres of liquid at least? (P.E.A.)

11. How many doses, each weighing ·00176 gr., can be obtained from a powder weighing ·096 gr., and what is the weight of the remaining portion of the powder? (C.P.)

12. Prove that $3 + \frac{1}{7 + \frac{1}{16}} = 3{\cdot}14159$ nearly, and find the square of 3·14159 correct to 3 places of decimals. (C.W.)

13. How often is ·35 of an inch completely contained in 12·52 in., and what remains? (P.T.)

14. If A owns ·58 of a ship, and B owns the rest; and if the difference in the money value of their shares is £37·46, what is the value of the ship? (P.T.)

15. Into how many equal parts must ·08475 of a ton be divided in order that each one of the parts may be ·791 lb.? (C.P.)

16. How many revolutions in a minute will be made by a carriage-wheel 4·55 ft. in diameter when the carriage is going at the rate of 5·2 mi. in 1·25 hr.? (Take $\frac{22}{7}$ as the ratio of the circumference to the diameter.) (C.M.)

17. A weight of 1 lb. is placed in one scale of a balance, and pins, each weighing ·0013 of an ounce, are put one by one in the other scale until the balance turns. Then one pin is taken out of the scale. How many pins are left in the scale, and by how much (in ounces) does the 1 lb. weight outweigh them? (P.T.)

18. The water of a certain company contains ·002 of organic impurity, and ·003 of metallic impurity, and ·02 of chalk. Find what vulgar fraction of the whole is pure water. (C.J.)

19. I post 3 cards bought at 1*d.* each, 3 bought at 2 for $1\frac{1}{2}d.$, 7 bought at 10 for 6*d.*, and 7 bought at 10 for $5\frac{1}{2}d.$ How much more have I paid for the posted cards than I should have paid if I had bought them at $\frac{1}{2}d.$ each? Give (1) the exact result decimally, and (2) the result to the nearest halfpenny, saying how you know it to be the nearest. (P.T.)

20. A book, before binding, consists of 324 leaves (each $9\frac{1}{2}$ in. by $6\frac{1}{4}$ in.), and weighs 1 lb. 3 oz. Express decimally, in ounces, the weight of as much paper of the book as would cover a square yard. (P.T.)

21. An imperial gallon contains 277·274 cub. in. Find (correct to three decimal places) the number of gallons in a cubic foot. (P.E.A.)

22. Given that a gallon of water weighs 10 lbs. avoirdupois, that 1 lb. avoirdupois contains 7000 grs., and that a cubic inch of water weighs 252·458 grs., calculate the number of cubic inches in a gallon to three places of decimals. (C.P.)

23. A cubic metre of water contains 1000 litres, and a cubic foot of water contains $6\frac{1}{4}$ gals. If a linear metre is 3·28 ft., express correctly to four decimal places, a litre as a decimal of a gallon. (P.T.)

24. The Snowdon Tramway has a continuous rise of 3140 ft. in $4\frac{3}{8}$ miles. The average gradient is said to be 1 in 7·83. Compare these statements, and determine the true average gradient from the first statement to three places of decimals. (C.P.)

25. If 127,196 sovereigns weigh one ton, find the weight of a sovereign in grains, correct within $\frac{1}{10}$ of a grain, assuming that 1 lb. avoirdupois is equal to 7000 grs. (C.H.L.)

26. Standard gold consists of eleven parts by weight of pure gold to one part of alloy, and 40 lbs. Troy of standard gold is coined into 1869 sovereigns. Find the weight of a sovereign in grains correct to three places of decimals. (O.S.L.)

27. A train goes 2046 mi. in 78·05 hrs.; its average pace is 45 mi. per hour: how long was it delayed by stoppages? (S.A.)

28. John added together 3·1285, ·0129, 12·63, and 3·7. He read them all as decimals of £1, but one was really a decimal of a florin. The answer should have been £8·1044. Which line did he misread? (P.T.)

29. Find, to within 10,000 sq. mi., the area of the earth's surface, having given that its radius is 3959 mi., and that the surface of a sphere is 3·1416 times the square on the diameter. (L.M.)

30. Assuming that the circumference of the earth is 24000 mi., and that diameter is to circumference as 1 : 3·1416, express the diameter of the earth (*a*) in miles, and (*b*) as a decimal of its circumference (decimal to four places only). (C.W.)

31. Two straight rods, initially 23 ft. and 12 yds. long respectively, were heated. The former expanded ·000412 of its initial length, and the sum of the expansions of the two rods was $\frac{1}{4}$ in. What decimal of its initial length, correct to six decimal places, did the latter rod expand? (C.P.)

32. A parish map is drawn on the scale of a foot to a mile. Express, as the decimal of a square inch, the size on this map of a field which contains an acre. (C.S.L.)

33. The area of a circle may be obtained by multiplying the square of its radius by 3·1416. Find radius of a circle containing 3848·46 sq. in. (P.E.A.)

34. Three measurements are taken of the diameter of three different circles known to be 6, 5, and 4 in. in circumference; these give the diameters as 1·9, 1·58, and 1·27 in. respectively. Find the ratio of the circumference to the diameter as determined by the mean of these three measurements to *five* places of decimals. (C.P.)

35. If the area of a circle be 76·54321874 sq. in. and the ratio of the circumference to the diameter (π) be 3·14159265, find the length of the diameter *correct to two places of decimals.* [Area of a circle is π times the square of the half-diameter.] (C.S.)

CHAPTER X.

CONVERSION OF VULGAR FRACTIONS INTO DECIMALS. PURE AND MIXED CIRCULATING DECIMALS.

The Conversion of Decimal into Vulgar Fractions, and of Vulgar Fractions into Decimals.

We have dealt with such fractional expressions as $\frac{1}{5}$, $\frac{47}{48}$ (commonly called vulgar fractions), and also with such expressions as ·2, ·35, commonly called decimal fractions, or sometimes decimals, and we have now to consider how vulgar fractions may be converted into decimals, and *vice versâ*.

To express a Decimal as a Vulgar Fraction.

Such an expression as ·47 may be read as forty-seven hundredths, (p. 95), and we can write 47 hundredths as $\frac{47}{100}$.

Similarly, ·019 = 19 thousandths = $\frac{19}{1000}$.

and ·000007 = 7 millionths = $\frac{7}{1000000}$.

It will be observed that the numerator consists of the digits of the decimal, and that the denominator contains as many ciphers as there are decimal places in the original decimal.

Hence the following *rule for the conversion of a decimal into a vulgar fraction* :—

(1) *For the numerator.* Write the digits of the decimal exclusive of any ciphers that may precede the digits.

(2) *For the denominator.* Write unity, and after it put as many ciphers as there were decimal places in the original decimal.

Example.—*Convert* ·053 *into a vulgar fraction.*

$$\cdot 053 = \frac{53}{1000}. \quad \textit{Ans.}$$

To express a Vulgar Fraction as a Decimal.

(*a*) *The case in which the denominator is a power of* 10.

Such a vulgar fraction as $\frac{59}{100}$ may be read as 59 hundredths, and this may be written as ·59. The conversion here resolves itself into a consideration of decimal notation.

(b) The case in which the denominator is not a power of 10.

1. A vulgar fraction may be converted into a decimal fraction (and thereby into a decimal) by multiplying its terms by such a number as will make the denominator a power of 10.

$$\frac{2}{5} = \frac{2 \times 2}{5 \times 2} = \frac{4}{10} = \cdot 4$$

$$\frac{7}{8} = \frac{7 \times 125}{8 \times 125} = \frac{875}{1000} = \cdot 875$$

2. Such a decimal as $\frac{15}{32}$ has to be dealt with in another way. A vulgar fraction may be regarded as representing a quotient resulting from the division of its numerator by its denominator (p. 60). Applying this knowledge, we proceed as follows :—

```
32)15·0(·46875
   128
   ---
   220
   192
   ---
    280
    256
    ---
     240
     224
     ---
      160
      160
      ---
```

By Contracted Method.

```
32)15·0(·46875
     220
      280
       240
        160
```

The decimal point in the quotient is obtained in the same manner as in division of decimals (p. 98).

But if we take such a fraction as $\frac{1}{3}$, and divide it as above described, we get 3 constantly recurring in the quotient, and a little consideration will show us that the process of division can never terminate, for there is no digit by which we can multiply 3 to obtain a product ending in 0.

Repeating Decimals.

Thus $\frac{1}{3}$ may be represented as ·3333 . . .

This quality of perpetual or indefinite repetition is represented thus : $\dot{3}$.

Note (1) That . *on* the line represents multiplication, 3 . 3 = 9.
(2) That · *above* the line represents the dividing mark between whole numbers and decimal fractions, 3·3.
(3) That · above a digit in a decimal represents indefinite repetitions of that digit, $\dot{3}$.
(4) That repeating decimals are also known as *recurring*, *circulating*, and *periodic* decimals.

If a group of decimals repeats, the dot is placed only over the first and last of the repeating digits, *e.g.* $\cdot\dot{1}42857\dot{1} = \cdot 142857142857142857 \ldots$

The group of digits which recurs constitutes *the period*. When all the digits repeat, the decimal is called a *pure circulating decimal*, *e.g.* $\cdot\dot{2}\dot{4}$. When some non-repeating decimals are followed by a repeating period, the decimal is called a *mixed circulating decimal*, *e.g.* $\cdot 24\dot{2}\dot{7}$.

Limiting Value.—A *limit* is a fixed quantity which a varying quantity tends to approach in value, but which it can never exceed or even equal.

Take the case of a polygon enclosed in a circle. The number of the sides of the polygon may be increased indefinitely, but the circle will always be the "limit" of the polygon, *i.e.* the length of the circumference of the circle will always be the limit of the total lengths of the sides of the polygon. The greater the number of sides, the greater the total lengths of the sides, but that total will never equal the length of the circumference of the circle.

When we state that $\frac{1}{3} = \cdot3333333333\ldots$, $\frac{1}{3}$ is the "fixed quantity," $\cdot3333333333\ldots$ is the "varying quantity," which tends to approach the "fixed quantity" in value. Hence $\frac{1}{3}$ is sometimes spoken of as the *limiting value* of $\cdot3$, for it is the value which $\cdot3$ can never equal, but to which it is constantly getting closer, and to which it may get as closely as we please.

How closely $\cdot3$ may get to $\frac{1}{3}$ may be seen from the following consideration. The difference between $\frac{1}{3}$ and $\cdot3333333333\ldots$ is $\frac{1}{3\,000\,000\,000}$, which fraction of a century amounts to only about $\frac{1}{10}$ of a second, so that, although $\cdot3333333333\ldots$ is never (and can never be) equal to $\frac{1}{3}$, it is for all practical purposes indistinguishable from it.

When we write that $\frac{1}{3} = \cdot3333\ldots$, the sign $=$ is used in a restricted sense. The two quantities are not actually equal; they "tend to equality."

Since the decimal $\cdot333333\ldots$ seems to "grow out" of $\frac{1}{3}$, the latter is sometimes called the *generating fraction* of the former.

The meaning of $\cdot\dot{9}$.

$$\cdot\dot{9} = \tfrac{9}{9} = 1$$

But $\cdot\dot{9}$ is a decimal, *i.e.* a quantity less than unity (p. 95), and we have made it equal to unity. How is this apparent contradiction to be explained?

Now, $\cdot\dot{9}$ denotes $\cdot99999999\ldots$ to infinity; while—

$\cdot9 = \frac{9}{10}$, which is less than unity by $\frac{1}{10}$
$\cdot99 = \frac{99}{100}$, ,, ,, ,, $\frac{1}{100}$
$\cdot999 = \frac{999}{1000}$, ,, ,, ,, $\frac{1}{1000}$, etc.

i.e. for every additional 9 the difference between the decimal and unity becomes $\frac{1}{10}$ of what it was before, which means that $\cdot\dot{9}$ is constantly approximating to unity, but is never unity itself.

Unity is the *limiting value* of $\cdot\dot{9}$; it is the value which $\cdot\dot{9}$ can never exceed (and, indeed, can never equal), but to which it is constantly approaching, and to which it may approach as closely as we please.

$\dot{9}$ differs from 1 by a quantity which is less than any assignable quantity however small, and therefore it may be considered as equal to 1.

Note.—When ·9 occurs at the end of a decimal, it should be omitted, and the preceding decimal increased by unity. Thus—

$$\cdot 0\dot{9} = \cdot 1$$
$$\cdot 07\dot{9} = \cdot 08, \text{ etc.}$$

To convert a Pure Recurring Decimal into a Vulgar Fraction.

Example.—*Convert* $\cdot\dot{3}$ *into a vulgar fraction.*

The decimal = ·3333...
From ten times the decimal, which = 3·333...
take the decimal itself, which = ·333...
Then the remainder, or nine times the decimal = 3·

$\therefore$ the decimal = $\frac{3}{9} = \frac{1}{3}$ *Ans.*

Example.—*Find the generating fraction of* $\cdot 6\dot{2}1\dot{}$.

The decimal = ·621621621...
From 1000 times the decimal, or 621·621621...
take the decimal itself, or ·621621...
Then the remainder, or 999 times the decimal = 621·

$\therefore$ the decimal = $\frac{621}{999} = \frac{23}{37}$
Thus the generating fraction = $\frac{23}{37}$ *Ans.*

Note that we multiply by such a power of ten as will bring all the digits of the period to whole numbers.

Rule for the conversion of a Pure Recurring Decimal into a Vulgar Fraction.

For the numerator, take one of the periods.

For the denominator, put down as many nines as there are decimal places in the period.

The following alternative verification will be of interest to the more advanced student.

Take a pure circulating decimal, such as $\cdot\dot{3}\dot{7}$.

$$\cdot\dot{3}\dot{7} = \cdot 37373737\ldots$$
$$= \frac{37}{100} + \frac{37}{100^2} + \frac{37}{100^3} + \frac{37}{100^4} \cdots$$

which is a geometrical progression decreasing for an infinite number of terms, with $\frac{1}{100}$ as a common ratio, and $\frac{37}{100}$ as the first term.

$$\text{Then the sum} = \frac{37}{100} + \frac{37}{100^2} + \frac{37}{100^3} + \frac{37}{100^4} \cdots$$

$$\text{then } \tfrac{1}{100} \text{ of the sum} = \frac{37}{100^2} + \frac{37}{100^3} + \frac{37}{100^4} \cdots$$

$$\text{By subtraction, } \tfrac{99}{100} \text{ of the sum} = \tfrac{37}{100}$$

$$\text{and the sum itself} = \frac{37}{\cancel{100}} \times \frac{\cancel{100}}{99} = \frac{37}{99}$$

To convert a Mixed Recurring Decimal into a Vulgar Fraction.

Example.—*Convert* $\cdot 7\dot{3}$ *into a vulgar fraction.*

the decimal = $\cdot 733333\ldots$
from 100 times the decimal, or $73{\cdot}3333\ldots$
take 10 „ „ or $7{\cdot}3333\ldots$
then the difference, or 90 times the decimal = 66·
$\therefore$ the decimal = $\frac{66}{90} = \frac{11}{15}$ *Ans.*

Example.—*Convert* $\cdot 0\dot{5}6\dot{}$ *into a vulgar fraction.*

The decimal = $\cdot 056565656\ldots$
from 1000 times the decimal, or $56{\cdot}565656\ldots$
take 10 „ „ or $\cdot 565656\ldots$
then the difference, or 990 times the decimal = 56·
$\therefore$ the decimal = $\frac{56}{990} = \frac{28}{495}$ *Ans.*

Note that we first multiply by such a power of ten as will bring all the digits of the decimal to whole numbers, and then by such a power of ten as will bring the non-repeating digits to whole numbers.

Rule for the conversion of a Mixed Recurring Decimal into a Vulgar Fraction.

(In the rule given below, all digits are to be considered as whole numbers.)

(1) *For the numerator*, subtract the non-repeating digits from the whole.
(2) *For the denominator*, put a *nine* for every repeater, and after the nines put a *nought* for every non-repeater.

Example.—*Find the generating fraction of* $5{\cdot}16\dot{1}88\dot{1}$.

$$5{\cdot}16\dot{1}88\dot{1} = 5\frac{161881 - 16}{999900} = 5\frac{161865}{999900} = 5\frac{327}{2020} \quad \textit{Ans.}$$

The Kind of Vulgar Fractions that will produce Terminating Decimals.

We have seen that a vulgar fraction may be converted into a terminating decimal if a number can be found by which both numerator and denominator may be multiplied so that a power of 10 may result as the new denominator (p. 111). So, if the denominator of a given fraction is such a number that when multiplied by some other number the product is a power of 10, then that given fraction is convertible into a terminating decimal.

What we have to discover is: What are the digits which when multiplied by some other digit produce a power of 10?

Now, 10 is a power of 10 (p. 61), so we may restate our problem

more simply thus: What are the digits which when multiplied by some other digit will give 10, or, more simply still, *what are the prime factors of* 10?

Excluding itself and unity, the only prime factors of 10 are 2 and 5.

So *if the denominator of a fraction in its lowest terms contains no other prime factors than* 2 or 5, *or* 2 *and* 5, *then that fraction will produce a terminating decimal.*

Thus the following will produce terminating decimals:—	*And the following will produce recurring decimals:—*
$\frac{7}{8} = \frac{7}{2 \times 2 \times 2}$	$\frac{5}{6} = \frac{5}{3 \times 2}$
$\frac{9}{40} = \frac{9}{2 \times 2 \times 2 \times 5}$	$\frac{7}{15} = \frac{7}{5 \times 3}$
$\frac{13}{125} = \frac{13}{5 \times 5 \times 5}$	$\frac{37}{750} = \frac{37}{3 \times 5 \times 5 \times 5 \times 2}$

Note that the fraction must be in its *lowest terms*. At first sight we might think that such a fraction as $\frac{38}{95}$ would produce a recurring decimal because its denominator evidently has 19 as a factor. But $\frac{38}{95} = \frac{19 \times 2}{19 \times 5}$. Thus 19 disappears on reduction to lowest terms.

To determine the number of digits in the finite decimal obtainable from an equivalent vulgar fraction.

Consider the following cases:—

(*a*) $\frac{3}{8} = \frac{3}{2 \times 2 \times 2}$

$= \frac{3 \times 5 \times 5 \times 5}{2 \times 2 \times 2 \times 5 \times 5 \times 5}$

$= \frac{375}{1000}$

$= 0{\cdot}375$

Since 2 occurs 3 times in the denominator, in order to change $2 \times 2 \times 2$ into $10 \times 10 \times 10$, we multiply numerator and denominator by $5 \times 5 \times 5$.

Hence because the factor 2 occurs 3 times in the denominator, there will be 3 digits in the decimal.

(*b*) $\frac{39}{5000} = \frac{39}{2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 5}$

$= \frac{39 \times 2}{2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 5}$

$= \frac{78}{10000}$

$= {\cdot}0078$

Because 5 occurs 4 times in the denominator, in order to change 5^4 into 10^4 we multiply by 2^4; but as 2^3 is already present in the denominator, it is necessary to multiply only by 2.

Hence because the factor 5 occurs 4 times, there will be 4 places in the decimal.

Note that when 2 and 5 both occur, that factor which occurs the oftener will determine the number of places in the decimal.

From such examples as above we deduce the following:—

The number of digits in any finite decimal is the same as the

greatest number of times that either of the prime factors 2 or 5 occurs in the denominator of the equivalent vulgar fraction when expressed in its lowest terms.

Every vulgar fraction in its lowest terms not having twos or fives as prime factors of its denominator must produce a pure circulating decimal.

Every vulgar fraction must produce—

(*a*) A terminating decimal, *or*

(*b*) A pure circulating decimal, *or*

(*c*) A mixed circulating decimal.

Since the fraction does not have 2's or 5's in its denominator, it cannot produce a terminating decimal (p. 115).

If it produced a mixed circulating decimal, such a circulating decimal, when reduced to a vulgar fraction, would have at least one nought following the 9 or 9's in the denominator. The denominator would thus be a multiple of 10. But the denominator cannot be a multiple of 10, because it does not have 2 or 5 as prime factors of its denominator; so it cannot produce a mixed circulating decimal.

Thus it has been seen that the fraction in question cannot produce a terminating or a mixed circulating decimal. Therefore it must produce a pure circulating decimal.

The Number of Digits in the Period of a Pure Circulating Decimal.

Let us consider the number of possible digits in the pure circulating decimal of which $\frac{1}{7}$ is the generating fraction.

In dividing by 7 the only possible remainders are 0, 1, 2, 3, 4, 5, or 6. Such remainders as 7, 8, 9 are impossible, for every remainder must be less than the divisor.

If 0 occurs as a remainder the decimal will terminate, which leaves us with the following possible remainders for the non-terminating decimal: 1, 2, 3, 4, 5, 6, *i.e. six* possible remainders in all.

If the division does not terminate when all the six have occurred, the next remainder must be the same as one of the remainders previously obtained, and the next digit in the quotient will be the same as one of the previous digits in the quotient, *i.e.* the decimal will now begin to repeat. It is evident that a *limited* number of digits must repeat in the quotient, and in this case the limit is 6, *i.e.* one less than the denominator 7.

If D = the denominator, then the possible remainder may be 1, 2, 3, 4 . . . (D − 1), *i.e.* the *maximum* number of different remainders is (D − 1), and therefore the *maximum* number of digits in the decimal before recurrence commences is (D − 1), that is, the maximum possible number of digits in the period is at least one less than the number indicated by the denominator of the vulgar fraction.

Note that the number of repeating digits cannot exceed the limit above defined, and may be less than this limit.

Thus $\frac{1}{3} = \cdot\dot{3}$ (*one* digit, not *two*)
$\frac{1}{6} = \cdot 1\dot{6}$ (*two* digits, not *five*)
$\frac{1}{101} = \cdot\dot{0}0\dot{9}\dot{9}$ (*four* digits, not *one hundred*)

The Complementary Law.—If a vulgar fraction in its lowest terms produces a circulating decimal having an even number of places in its recurring period, then the last half of the digits of the recurring period may be found from the first half by subtracting in regular order the known digits from 9.

Thus, having found that $\frac{1}{17} = \cdot 94117647058\ldots$, we see that with the digits 058, the complement of the first part is appearing, for 941 + 058 = 999. We thus infer that the second seven is the last digit of half the decimal. Subtracting 9, 4, 1 respectively from 9 we re-obtain 058, which have been already found. Subtracting 1, 7, 6, 4, 7 successively from 9, we find the remaining digits, 8, 2, 3, 5, 2.

Hence $\frac{1}{17} = \cdot\dot{9}41176470588235\dot{2}$.

This law, however, holds good only if the denominator be (1) prime, or (2) the product of primes none of which by itself yields half the whole period above referred to, or (3) the product of the primes in (2) with a power of 2 or 5 or both.

The Number of Repeating and Non-repeating Digits in a Mixed Circulating Decimal.

The number of non-repeating decimals will always be equal to the highest power which either of the factors (2 or 5) has in the denominator (p. 116).

The number of repeating decimals will be at most one less than the number indicated by the denominator (p. 117).

Cyclic Order.

The repeating decimal $\cdot\dot{1}42857\dot{}$, of which $\frac{1}{7}$ is the generating fraction, deserves attention.

(1) Arrange these digits inside a circle, as shown.

(2) Note the order of the digits in ascending magnitude, as shown by the Roman numerals outside the circle.

(3) If we begin with 1 (the *first* in order of ascending magnitude) and read off the digits inside the circle, we get $\cdot\dot{1}42857\dot{} = \frac{1}{7}$.

(4) If we begin with 2 (the *second* in order of ascending magnitude) and read off, we get $\cdot\dot{2}85714\dot{} = \frac{2}{7}$.

(5) Similarly—

$$\cdot\dot{4}2857\dot{1} = \frac{3}{7}$$
$$\dot{5}7142\dot{8} = \frac{4}{7}$$
$$\cdot\dot{7}1428\dot{5} = \frac{5}{7}$$
$$\cdot\dot{8}5714\dot{2} = \frac{6}{7}$$

Note.—The student should learn in order the digits 142857. If he has to bring $\frac{4}{7}$ to a decimal, he should say 7 into 40 goes 5. Five then is the first digit of the recurring decimal, and $\frac{4}{7} = \cdot\dot{5}71428\dot{}$.

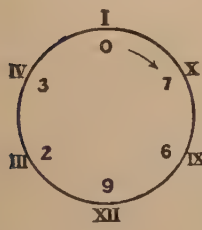
Reason for this cyclic property.

If we show the quotients and remainders, we shall see at a glance the reason for this cyclic property. We can state thus: $1_3, 4_2, 2_6, 8_4, 5_5, 7_1$, where the large digits on the line represent the quotients, and the small digits below the line the remainders.

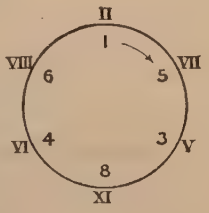
In dividing 1 by 7 (in reducing $\frac{1}{7}$ to a decimal) we get a remainder 3. We have now to divide 3 by 7, and this is just like having to bring $\frac{3}{7}$ to a decimal, and so on with the rest of the quotients and remainders.

Note that the remainders (expressed by the small digits below the line) are in exactly the same order as the Roman numerals on the circumference of the circle.

The repeating decimals of which $\frac{1}{13}$, $\frac{2}{13}$, etc., are the generating fractions, can also be similarly treated.



$\frac{1}{13} = \cdot\dot{0}76923$
$\frac{2}{13} = \cdot\dot{1}53846$
$\frac{3}{13} = \cdot\dot{2}30769$
$\frac{4}{13} = \cdot\dot{3}07692$
$\frac{5}{13} = \cdot\dot{3}84615$
$\frac{6}{13} = \cdot\dot{4}61538$
$\frac{7}{13} = \cdot\dot{5}38461$
$\frac{8}{13} = \cdot\dot{6}15384$
$\frac{9}{13} = \cdot\dot{6}92307$
$\frac{10}{13} = \cdot\dot{7}69230$
$\frac{11}{13} = \cdot\dot{8}46153$
$\frac{12}{13} = \cdot\dot{9}23076$



Note (1) That these digits fall into two classes, as shown above.
(2) That the order of the digits in ascending magnitude is shown by the Roman numerals outside the circle (cf. p. 118).

Suppose we find $\cdot 307692$ in a sum. We recognize it as one of the "thirteenths," and we have next to determine to which group of thirteenths it belongs.

Now, 13 divides into 40 three times, and also into 50 three times, so the fraction might be $\frac{4}{13}$ or $\frac{5}{13}$. To determine *which*, carry the division to the *second* place, when it will be at once seen that $\frac{4}{13}$ is the fraction required.

Examining once more the three circles, we note that the sum of

two opposite digits is in every case equal to 9. Consequently such expressions as $\cdot\dot{4}2857\dot{1}$ and $\cdot 076923$ really resolve themselves into two halves, $\cdot\dot{4}28 \mid 57\dot{1}$, $\cdot 076 \mid 923$, and when we have found the

first half we can find the second by subtracting the digits already found from 9. Thus in $\cdot\dot{4}28571\dot{}$, $9 - 4 = 5$, which is the digit in the fourth place; $9 - 2 = 7$, which is the digit in the fifth place; and so on. (See also p. 118.)

Additional methods for the reduction of mixed circulating decimals to vulgar fractions, and vice versâ.

Example.—

Reduce $\cdot 276\dot{4}28571\dot{}$ *to a vulgar fraction.*

$$\cdot 276\dot{4}28571\dot{} = \cdot 276\tfrac{3}{7} = \frac{276\frac{3}{7}}{1000} = \frac{(7 \times 276) + 3}{7000} = \tfrac{1935}{7000} = \tfrac{387}{1400} \quad \textit{Ans.}$$

Or another and perhaps better method—

$$\begin{aligned} \cdot 276\dot{4}28571\dot{} &= \cdot 276 + \cdot 000\dot{4}28571\dot{} \\ &= \frac{276}{1000} + \frac{\frac{3}{7}}{1000} \\ &= \frac{276 \times 7}{7000} + \frac{3}{7000} \\ &= \frac{1935}{7000} = \frac{387}{1400} \quad \textit{Ans.} \end{aligned}$$

Example.—*Convert* $\frac{1}{17}$ *into a decimal.*

$$\begin{array}{r} 17)1\cdot00(\cdot0588235 \\ 150 \\ 140 \\ 40 \\ 60 \\ 90 \\ 5 \end{array}$$

$$\tfrac{1}{17} = \cdot 0588235\tfrac{5}{17}$$

This $\frac{5}{17}$ represents, not $\frac{5}{17}$ of unity, but $\frac{5}{17}$ of the unit to which the preceding digit 5 refers. In this case it represents $\frac{5}{17}$ of one ten-millionth.

$$\begin{aligned} \text{Now, } \tfrac{5}{17} &= \cdot 0588235\tfrac{5}{17} \times 5 \\ &= \cdot 2941176\tfrac{8}{17} \\ \text{so } \tfrac{1}{17} &= 0\cdot 5882352941176\tfrac{8}{17} \\ \text{Now, } \tfrac{8}{17} &= \cdot 4705882352941\tfrac{12}{17} \ (\textit{i.e. } \tfrac{1}{17} \times 8) \\ \text{so } \tfrac{1}{17} &= \cdot 0588235294117647 \mid 058823529411 \end{aligned}$$

The digits are obviously beginning to repeat. From p. 117 we know that the maximum number of repeaters is 16; so we can write—

$$\tfrac{1}{17} = \cdot 0588235294117647\dot{} \quad \textit{Ans.}$$

THEORY.

1. Prove the rule for reducing a pure recurring decimal to a vulgar fraction. (C.W. and K.S.)

2. What is meant by a "pure," and what by a "mixed" circulating decimal? State and explain, by an example, the rule for converting the latter into a vulgar fraction. (C.W., C.P., O.S.L., C.S., and C.M.)

3. Prove that $\cdot 3\dot{2} = \frac{29}{90}$. (C.W.)

4. Give the rule for converting a recurring decimal into a vulgar fraction. Why is such a process necessary? (K.S.)

5. Describe a method, depending on first principles only, for the conversion into a vulgar fraction of a recurring decimal fraction in which the period has three digits. (C.H.L.)

6. Give, in general terms, the reason why, in reducing a mixed circulating decimal to a fraction, you subtract the finite part? (C.S.)

7. The pure recurring decimal $\cdot\dot{4}28571\dot{4}$ can be written $\cdot\dot{4}28\dot{5}7142\dot{8}$ or $\cdot\dot{4}28571428\dot{5}7\dot{1}$ (amongst many other ways). Investigate the processes for reducing these three equivalent decimals to a vulgar fraction, and give, in a general form, your arguments to show that they must lead to identical results. (C.S.)

8. Write a short essay upon the conversion of a vulgar fraction in its lowest terms into a decimal, the denominator being a prime number greater than 5, stating any theorems which shorten the work of conversion. (C.S.)

9. Without assuming the rule for the conversion of a circulating decimal into a vulgar fraction, determine the fraction which is equivalent to the decimal $\cdot 12\dot{3}4\dot{5}$. (O.S.L.)

10. Show that $\frac{37}{10^2-1} = \frac{3737}{10^4-1} = \frac{373737}{10^6-1}$. How does this suggest a rule for reducing a pure recurring decimal, with two recurring figures, to a vulgar fraction? (C.M.)

11. Explain clearly the reasons for the statement that $\cdot\dot{3}$ is equal to $\frac{1}{3}$. (O.S.L. and O.J.)

12. In what sense can $\cdot\dot{9}$ be said to equal unity? (C.M. and P.T.)

13. For what is $\cdot 3\dot{0}9\dot{0}$ an abbreviation? Supposing this decimal written in full, what fraction of the value of the 3 is the value of the second 9 from the point? (P.T.)

14. Define (i.) a vulgar, and (ii.) a decimal fraction. Explain what vulgar fractions reduced will give (i.) terminable, and (ii.) recurring decimals. (C.P.)

15. If a vulgar fraction has to be converted into a decimal, how can it be determined by inspection of the denominator whether the decimal will terminate or not? (O.J.)

16. If a vulgar fraction is converted into a decimal, show that the decimal must either terminate or recur. (O.S.L.)

17. Show that, if the denominator of a vulgar fraction has a prime factor other than 2 or 5, the equivalent decimal cannot terminate. (L.M.)

18. What fractions produce *terminating* and what *recurring* decimals? Explain how it is that *any* fraction must produce either one or the other of these two kinds.

19. Prove that every vulgar fraction which refuses to be reduced to a terminal decimal can be reduced to a recurring decimal. (C.M.)

20. Prove that the decimal will terminate or recur according to the form of the denominator. What sort of decimal will the fraction $\frac{102}{136}$ produce? (C.S.)

21. Explain why a vulgar fraction can be expressed either as a terminating or else as a recurring decimal fraction. Show how to tell beforehand which it will be. Mention any fraction which cannot be so expressed. (S.A.)

22. Explain why all the fractions $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$, when reduced to recurring decimals, contain the same digits in the same order. (L.M.)

23. Reduce $\cdot\dot{1}42857$ to a vulgar fraction. Why is it that these

figures repeat themselves, in order, when multiplied by any whole number less than 7? (L.M.)

24. If the sum of $\frac{1}{2} + \frac{1}{3} + \frac{1}{5}$ be reduced to a decimal, will it be terminating or recurring? (C.P.)

25. In reducing a vulgar fraction to a decimal, how do you discover that the decimal would be recurring, and hence determine the number of figures in the recurring period? (P.T.)

26. How can you tell, by inspection of a vulgar fraction, the number of recurring and non-recurring digits in the decimal it will give rise to? (C.S.)

27. Reduce $\cdot 2\dot{2}\dot{7}$ to a vulgar fraction, and explain the reason of each step in the process. Show that if the recurring part of a mixed circulating decimal be divisible by 9, the equivalent fraction can be reduced to lower terms.

N.B.—You may take a particular example, but the reasoning must be general. (C.S.)

28. How may it be known that a given vulgar fraction gives rise to a circulating decimal? What is the greatest number of digits that occur in the period? (R.U.I.)

29. A vulgar fraction in its lowest terms has 21 for denominator. Prove that, when changed to a decimal, the result must be a *pure* circulating decimal. (I.I.)

30. In bringing $\frac{1}{17}$ to a decimal, show why the number of figures in the circulating period cannot be greater than 16. (C.W.)

31. Give an example of a vulgar fraction such that the equivalent decimal has an infinite number of figures; and show that the significant figures must always repeat themselves before their number becomes as great as the denominator. (L.M.)

32. Show how it may be known by simple inspection that $\frac{1}{7} = \cdot 142857$ is incorrect. (I.I.)

33. The sum of a number of decimals which are all different from one another is to be less than unity, and each decimal is to consist of two figures. What is the greatest number of decimals which can be selected so as to fulfil these conditions? Find also the greatest number of different decimals, each consisting of *three* figures, if their sum is to be equal to unity. (C.S.)

34. Explain the rule for converting a recurring decimal into a vulgar fraction. Find the limit, when an infinite number of terms is taken, of the following series: $\frac{14}{100} + \frac{28}{10000} + \frac{56}{1000000} +$ etc. (C.S.)

35. Show that a pure circulating decimal of any number of figures is the sum to infinity of a decreasing geometrical progression, and verify thereby the ordinary rule for the reduction of any recurring decimal, pure or mixed, to its equivalent ordinary fraction. (L.M.)

36. Find the difference between any given pure circulating decimal of five figures and the non-circulating decimal consisting of the same figures in the same order. Express this difference as a fraction of the latter decimal, and explain the constancy of your result whatever be the number chosen. (C.S.)

EXERCISE XVIII. 18

Reduction of Vulgar Fractions to Finite Decimals and vice versâ.

1. Express as decimals $\frac{1}{8}$, $\frac{1}{16}$, $\frac{7}{64}$, $5\frac{3}{64}$, $2\frac{9}{16}$, $\frac{9}{125}$.

2. Find decimals equivalent to $\frac{54}{125}$, $\frac{19}{20}$, $1\frac{3}{32}$, $\frac{11}{2500}$, $\frac{3657}{100}$, $\frac{1}{78250}$.

3. Express the following as vulgar fractions in their lowest terms: ·7, ·75, ·4125, 17·28, ·0875, 5·072, ·059, ·03125, ·0064, ·0000725.

4. Simplify $\frac{247}{504} \times 1\frac{25}{437} \times 1\frac{64}{143}$; answer in decimal form. (C.W.)

5. Write down the following fractions in a decimal form: $\frac{27}{10}$, $\frac{27}{100}$, $\frac{27}{10000}$; find the product of 1·005 multiplied by ·097, and the quotient of ·0036 divided by ·0144. (S.K.)

6. Express as a decimal $\dfrac{552}{2^5 \times 5^5} + \dfrac{923}{2^5 \times 5^7}$. (P.T.)

7. Express as a decimal $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625}$. (K.S.)

8. Find the fraction with denominator less than 1000 which approximates most closely to 3·14159. (C.S.)

9. Find the value of $\frac{7}{100} + \frac{3}{10} - \frac{9}{100} + \frac{17}{10000}$, and express the result as a decimal of 120. (P.T.)

10. Express $\frac{22}{7}$ and $\frac{355}{113}$ as decimal fractions to 5 places; and prove that their difference is about $\frac{1}{25}$ per cent. of either of them. (L.M.)

EXERCISE XIX. 19

Reduction of Vulgar Fractions to Repeating Decimals and vice versâ.

1. Express as decimal fractions $\frac{2}{3}$, $\frac{6}{55}$, $\frac{11}{70}$, $\frac{27}{22}$, $2\frac{1}{13}$.

2. Find the decimals which represent $\frac{7}{9}$, $\frac{111}{180}$, $\frac{17}{1375}$, $\frac{89}{60}$, $5\frac{3}{52}$.

3. Convert the following into vulgar fractions: $\dot{3}$, $\dot{1}\dot{4}$, $\cdot4\dot{2}7\dot{5}$, $3\cdot2857\dot{1}\dot{4}$, $\cdot0\dot{4}2\dot{3}$.

4. Find the generating fractions of $\cdot6\dot{7}$, $5\cdot0\dot{9}0\dot{9}$, $7\cdot653\dot{1}$, $\cdot093\dot{1}\dot{8}$, $\cdot783\dot{7}\dot{5}$.

5. Convert $\frac{13}{64}$ and $\frac{64}{13}$ into decimals. (C.S.L.)

6. Express as decimals $5 + \frac{3}{10} + \frac{7}{1000}$; $\frac{5}{150}$; $\frac{5}{660}$; and $\frac{6}{7}$. (P.T.)

7. Reduce $\frac{9111058}{1580700000}$ (i.) to its lowest terms, (ii.) to a decimal. (C.P.)

8. Reduce $\dfrac{1}{10^6}$ and $\dfrac{1}{999}$ to decimal fractions. (L.M.)

9. Express decimally $\frac{1}{13}$ and $\frac{1}{7}$, and *deduce* the decimal expressions of $\frac{12}{13}$ and $\frac{20}{91}$. (C.M.)

10. Find the sum of the fractions $\frac{3}{10}$, $\frac{3}{10000}$, $\frac{3}{10000000}$, $\frac{3}{10000000000}$ . . ., there being no limit to the number of these fractions. (C.S.)

11. Simplify $\dfrac{7\frac{1}{2} + 1\frac{17}{22}}{8\frac{2}{3} + 3\frac{23}{33}} - \dfrac{2\frac{1}{4} - 1\frac{2}{3}}{1\frac{2}{3} \text{ of } 1\frac{3}{7}}$, and express the answer as a decimal. (C.W.)

12. Find the fractions respectively equivalent to the decimals ·0̇9, ·0̇9̇, and ·00̇9̇. (C.P.)

13. Convert ·045$\dot{4}\dot{0}$ into a vulgar fraction, and prove the correctness of the method you adopt.

14. Write down the fractions equivalent to ·$\dot{4}$, ·$\dot{1}\dot{5}$, ·$2\dot{5}\dot{5}$, and 17·$75\dot{6}$ respectively, and express them in their lowest terms. (C.P.)

15. Reduce ·$23\dot{4}\dot{5}$ to a vulgar fraction, giving the reason for each step in the process. (L.M.)

CHAPTER XI.

REPEATING DECIMALS: THE FOUR RULES.

Addition and Subtraction of Recurring Decimals.

I. Pure Recurring Decimals.

Suppose we have to add $\cdot\dot{1}\dot{9}$ and $\cdot\dot{2}\dot{1}\dot{3}$, we could reduce the circulating decimals to their equivalent vulgar fractions, but a little consideration will reveal a method which will be found shorter and more likely to be accurate.

$\cdot\dot{1}\dot{9}$ repeats in periods of *two* digits, $\cdot\dot{2}\dot{1}\dot{3}$ in periods of three digits. Thus—

$$\begin{array}{l|l|l|} \cdot 191919 & 191919 & 191919 \\ \cdot 213213 & 213213 & 213213 \end{array}$$

Note that there is a recurrence of the same group of decimals after every *six* places, and also that *six* is the L.C.M. of 2 and 3.

Try two such decimals as $\cdot\dot{2}\dot{1}\dot{3}$ and $\cdot\dot{4}\dot{1}\dot{7}\dot{8}$, and note that there is a recurrence of the same group of decimals after every 12 places of decimals: also that 12 is the L.C.M. of 3 and 4.

Hence the following Rule for the Addition (or Subtraction) of Pure Circulating Decimals :—

(*a*) Find the L.C.M. of the number of places in each circulating decimal.

(*b*) Write each decimal for that number of places indicated by the L.C.M.

(*c*) Proceed as in ordinary addition (or subtraction), taking care, however, to extend the decimals at least one place (and preferably two places) to the right in order to find the digit to be carried to the first column at which we commence.

(*d*) The decimal will repeat for the same number of digits as are indicated by the L.C.M.

Example.—*Find the sum of* $\dot{3}$, $\cdot\dot{0}\dot{6}$, $\cdot\dot{7}\dot{1}\dot{8}$, $\cdot\dot{2}\dot{1}\dot{3}\dot{4}$.

$$\begin{array}{l} \cdot 333333333333\,3\ldots \\ \cdot 060606060606\,0\ldots \\ \cdot 718718718718\,7\ldots \\ \cdot 213421342134\,2\ldots \\ \hline 1\cdot 326079454792\dot{}\quad \textit{Ans.} \end{array}$$

The number of decimal places is 1, 2, 3, 4 respectively. The L.C.M. of 1, 2, 3, 4 is 12, and there will consequently be 12 digits in the repeating decimal.

Continue the decimal to the 13th place, in order to "carry" the correct digit.

In cases where the sum of the extra column exceeds 15, it is advisable to carry the decimal out to yet another place.

II. Mixed Circulating Decimals.

The following investigations will help us :—

$\left.\begin{array}{l}\cdot\dot{1}4285\dot{7}\\ \cdot1\dot{4}2857\dot{1}\\ \cdot14\dot{2}857\dot{1}4\end{array}\right\}$ on reduction, will each be found equal to $\frac{1}{7}$

Hence, the period of a recurring decimal may be considered as beginning to recur from any one of its digits if we consider the preceding digit or digits as non-recurring.

Now, apply this to the following: Find the sum of $16\cdot\dot{8}4\dot{7}$, $4\cdot3\dot{2}\dot{9}$, $\cdot0\dot{1}\dot{6}$.

First write thus—

$$\begin{array}{r}16\cdot847847\ldots\\ 4\cdot329292\ldots\\ \cdot046666\ldots\end{array}$$

The non-repeating digit 3 interferes with the regularity of the recurrence so far as the first place is concerned. Similarly, the non-recurring digits ·04 interfere with the recurrence in the first and second places. Hence there can be no recurrence, at any rate, till after the second place.

$$\begin{array}{r}16\cdot84\\ 4\cdot32\\ \cdot04\end{array}$$

Draw a vertical line after the second digit in order to mark the beginning of the circulating column.

$$\begin{array}{r}16\cdot84\,|\,784784\,7\\ 4\cdot32\,|\,929292\,9\\ \cdot04\,|\,666666\,6\\ \hline 21\cdot22\dot{3}80744\dot{4}\end{array}$$

The number of circulating digits is 3, 2, and 1 respectively: the L.C.M. is 6; therefore the number of places in the circulating period will be 6.

Repeat to six places and one more. Proceed as in addition.

Ans.

Hence the following Rule for the Addition (or Subtraction) of Mixed Circulating Decimals :—

(*a*) *Write the decimals so that their periods commence at the same place. Draw a vertical line.*

(*b*) *Find the L.C.M. of the numbers indicating the number of decimal places in the recurring period of each decimal.*

(*c*) *Continue each decimal to that number of places indicated by the L.C.M. Draw a second vertical line.*

(*d*) *Add (or subtract), taking care to obtain the "carrying" digit as explained on p.* 124.

(*e*) *The digits between the two vertical lines will constitute the recurring period.*

It sometimes happens that the sum (or difference) of recurring decimals constitutes a terminating decimal.

Multiplication.

I. To multiply a recurring decimal by a single digit. Proceed as in ordinary multiplication.

Example.—$5{\cdot}1\dot{6} \times 7$.

$$\begin{array}{r|l} 5{\cdot}1\dot{6}66 & \\ 7 & \\ \hline 36{\cdot}16 & \textit{Ans.} \end{array}$$

Carry the decimal two places beyond the end of the period, so as to ensure the accuracy of the last digit retained in the answer.

II. To multiply a recurring decimal by more than one digit.

Example.—$5{\cdot}1\dot{6} \times 77$.

$$\begin{array}{r|l} 5{\cdot}1\dot{6}66\ldots & \\ 77\cdot & \\ \hline 36{\cdot}16 & \\ 361{\cdot}6 & \end{array}$$

Proceed as in multiplication of finite decimals, making allowance, however, for the "carrying" digit.

$$\begin{array}{r|l} 5{\cdot}1\dot{6}66 & \\ 77\cdot & \\ \hline 36{\cdot}1666 & \\ 361{\cdot}6666 & \\ \hline 397{\cdot}8\dot{3} & \textit{Ans.} \end{array}$$

Before adding, repeat the decimals in the products in order to ensure the accuracy of the "carrying" digit.

III. To multiply two recurring decimals together.

Example.—$42{\cdot}\dot{1}\dot{8} \times {\cdot}2\dot{8}$.

$$\cdot 2\dot{8} = \frac{28-2}{90} = \tfrac{26}{90} = \tfrac{13}{45}$$

$$\text{then } 42{\cdot}\dot{1}\dot{8} \times {\cdot}2\dot{8} = 42{\cdot}\dot{1}\dot{8} \times \tfrac{13}{45}$$

$$\begin{array}{r} 42{\cdot}181818 \\ \times\quad\quad 13 \\ \hline 45)548{\cdot}363636(12{\cdot}1\dot{8}\dot{5} \\ 45 \\ \hline 98 \\ 90 \\ \hline 83 \\ 45 \\ \hline 386 \\ 360 \\ \hline 263 \\ 225 \\ \hline 38 \end{array}$$

Ans. $12{\cdot}1\dot{8}\dot{5}$.

Division.

I. When the divisor is a whole number or a terminating decimal.

Example.—$8\cdot\dot{7}\dot{2} \div 288$.

$$
\begin{array}{r|l}
288)8\cdot727272(\cdot\dot{0}\dot{3} \quad \textit{Ans.} & \\
\quad 864 & \\
\hline
\quad 872 & \text{Proceed as in division of terminating} \\
\quad 864 & \text{decimals, but bring down the digits of} \\
\hline
\quad\quad 8 & \text{the repeating period instead of ciphers.}
\end{array}
$$

Example.—$51\cdot\dot{6} \div \cdot 03$.

$$\frac{51\cdot\dot{6}}{\cdot 03} = \frac{51\cdot\dot{6} \times 100}{\cdot 03 \times 100} = \frac{5166\cdot\dot{6}}{3} = 1722\cdot\dot{2} \quad \textit{Ans.}$$

II. When the divisor is a repeating decimal, it is as well to reduce to vulgar fractions.

THEORY.

1. Two recurring decimals, with periods of 6 and 7 figures respectively are added together. What is the number of places in the period of their sum? (C.M.)

2. A, B, C are 3 recurring decimals. A has 9 recurring figures and 4 non-recurring; B has 6 recurring figures and 5 non-recurring; C has 4 recurring figures and 2 non-recurring. How many recurring figures are there in the decimal representing the sum of A, B, and C? (C.S.)

3. Show how to multiply a repeating decimal by a whole number without reducing the decimal to a vulgar fraction.

4. When one circulating decimal is divided by another, how is the result obtained? (C.W.)

EXERCISE XX. *20*

REPEATING DECIMALS.

Addition and Subtraction.

1. $1\cdot0\dot{1}\dot{2} + 2\cdot\dot{0}1\dot{2} + 3\cdot0\dot{1}\dot{2}\dot{3}$. (O.S.L.)

2. Find the sum of $8\cdot\dot{4}$, $\cdot\dot{3}\dot{2}$, $\cdot\dot{5}6\dot{7}$, $\cdot0\dot{5}\dot{6}$, $\cdot1\dot{7}$, and $4\dot{3}04122\dot{5}$. (C.W.)

3. Add together $4\cdot\dot{5}$, $5\cdot\dot{6}$, $7\cdot1\dot{6}$, and $8\cdot\dot{2}7\dot{9}$. (C.P.)

4. Add, without reducing to fractions, $\cdot\dot{7}\dot{2}$, $2\cdot8\dot{3}$, and $15\cdot9\dot{9}0$ together, giving the result (i.) in the form of a recurring decimal, (ii.) in that of a fraction. (C.P.)

5. Reduce the following fractions to decimals, and add the resulting decimals together: $\frac{7}{60}$, $\frac{1}{810000}$, $\frac{8}{7}$. (C.PR.)

6. Convert $\frac{5}{1287}$, $\frac{9}{2395}$, and $\frac{7}{3905}$ into recurring decimals, and add together the three results. (C.H.L.)

7. Express $2\cdot3\dot{4}\dot{5}\dot{6} - 1\cdot2\dot{3}\dot{4}\dot{5}\dot{6}$ as a circulating decimal. (O.S.L.)

8. Subtract $\cdot003411775\dot{3}$ from the sum of $\cdot21437$ and $\cdot01426\dot{3}$. (C.P.)

9. Subtract $\cdot095238\dot{3}$ from $\cdot\dot{6}$, and express the result as a vulgar fraction in its lowest terms. (C.P.)

Multiplication.

10. $\dot{0}0\dot{9} \times 90{\cdot}8\dot{1} \times 1{\cdot}\dot{2}$.

11. $0{\cdot}07692\dot{3} \times {\cdot}2857\dot{1}\dot{5} \times 10{\cdot}\dot{1}$.

12. ${\cdot}\dot{1}42857 \times {\cdot}2857\dot{1}\dot{4} \times 5{\cdot}\dot{4}$.

13. $3{\cdot}\dot{3} \times 4{\cdot}\dot{4}\dot{5}$. (C.P.)

14. Multiply $59{\cdot}21\dot{4}2857\dot{1}$ by 14. (I.I.)

15. Find the exact product of $12{\cdot}\dot{3}$ and $8{\cdot}0\dot{4}\dot{5}$. (L.M.)

16. Multiply ${\cdot}00\dot{3}$ by ${\cdot}010203\dot{3}$, giving the result accurately as a decimal. (C.P.)

17. Explain the meaning of ·36 and of ·3$\dot{6}$ respectively. Multiply the two together, and give the result in the form of a vulgar fraction in its lowest terms. (C.P.)

18. Multiply ${\cdot}18\dot{2}4\dot{3}$ by 140008 without changing the decimal to a vulgar fraction. (C.P.)

19. Multiply 3·14159 by ${\cdot}\dot{1}42857\dot{1}$ correctly to 7 places of decimals. (C.M.)

20. Multiply $1{\cdot}38\dot{1}$ by 8·3 without reducing them to vulgar fractions. (C.M.)

21. Find the value of $1{\cdot}0\dot{5}\dot{4} \times 1{\cdot}\dot{1}74603\dot{1} \times 1{\cdot}6153\dot{8}\dot{4}$. (C.S.L.)

22. Explain the contracted method employed to determine approximately the product of two decimals. By this method multiply ·1116 by $43{\cdot}7428\dot{5}\dot{3}$, bringing out the product correct to 8 places of decimals. (C.S.)

23. Multiply ${\cdot}1\dot{0}76923\dot{1}$ by ·4$\dot{3}$, (1) by converting each decimal into a vulgar fraction in its lowest terms, multiplying these together, and reconverting their product into a decimal; and (2) by multiplying the decimals as they stand, without converting either of them into a vulgar fraction. (C.H.L.)

24. Multiply $3{\cdot}2857\dot{1}\dot{4}$ by ${\cdot}38461\dot{5}$, and reduce the product to a decimal. (I.I.)

Division.

25. $10{\cdot}2857\dot{1}\dot{4} \div 288$.

26. $5{\cdot}\dot{3} \div 2{\cdot}\dot{3}$.

27. ${\cdot}\dot{0}0\dot{9} \div {\cdot}\dot{0}6\dot{3}$.

28. $6{\cdot}3 \div 1{\cdot}\dot{4} \times {\cdot}\dot{1}42857\dot{1}$.

29. Divide 2635·49 by ·01235, and 13·06 by ·0326. (C.H.L.)

30. Divide ·0109$\dot{0}$ by 30·25 without converting the decimals into vulgar fractions. (C.S.L.)

31. Divide ·0000273 by ·00$\dot{3}$. (C.H.L.)

32. Divide ·0$\dot{1}$ by ·000$\dot{1}$, and $5{\cdot}4\dot{3}$ by $44{\cdot}4\dot{5}$. (C.P.)

33. Divide 16·0524 by ·001008, and 3882·57 by $31{\cdot}5\dot{6}$. (C.W.)

34. Express $\frac{5}{71}$ as a decimal fraction, and find correct to 4 decimal places $\frac{\cdot01825}{\cdot07\dot{1}}$. (O. and C.S.C.)

35. By what must 1001 be multiplied in order to produce ·02? (P.T.)

36. Divide $4{\cdot}8\dot{3}$ by $6{\cdot}\dot{4}$ without reducing to vulgar fractions, and reduce $\frac{49}{83}$ to a recurring decimal. (C.W.)

37. Simplify $\frac{247}{361} \times 1\frac{35}{437} \times 1\frac{64}{115}$, and divide $1{\cdot}25\dot{3}$ by 94, giving the answer in decimal form. (C.W.)

38. Multiply 7642 by ·12$\dot{1}$, and divide 1761 by ·$\dot{5}28\dot{3}$. (O.R.)

39. Multiply $3{\cdot}\dot{5}\dot{4}$ by $7{\cdot}\dot{1}42857\dot{1}$, and divide the product by $2{\cdot}30\dot{4}\dot{5}$, the result being given correct to four decimal places by the shortest method you can select. (S.A.)

40. Find the sum, and also the difference, of $\cdot 02\dot{2}\dot{7}$ and $\cdot\dot{1}\dot{2}$. Divide the first result by the second, and give each of the three results (1), as a vulgar fraction (2), as a decimal. (C.S.)

41. Add, subtract, multiply, and divide the decimals $\cdot 00041$, $\cdot 0000017$; bringing the last result into the form of a recurring decimal. (C.P.)

42. Find the value of $2\cdot 3\dot{0}40709 + 6\cdot 07\dot{0}50208 - 4\cdot\dot{7}53803$ without converting the decimals into vulgar fractions, and divide the result by $\cdot 33\dot{6}2\dot{1}42857$. (C.H.L.)

43. Find, correct to 5 places of decimals, the sum of $3\cdot 27 \div 1\cdot 7$, $\cdot 00327 \div 17$, $327 \div \cdot 0017$, and $3\cdot 27 \div 1\cdot\dot{7}$. (C.P.)

44. Divide $\cdot 0001\dot{2}$ by the product of $\cdot 0\dot{0}2\dot{1}$ and $\cdot\dot{5}23809$. (L.M.)

45. Convert $\frac{1}{630}$ into a decimal; and find, to 4 places of decimals, the value of $(\cdot 3673 - \cdot 256) \div (4\cdot 6 + 2\cdot 527)$.

Multiply $\cdot 0\dot{0}2105$ by 48·3 without converting into vulgar fractions. (C.S.L.)

46. Simplify $10\cdot 9\dot{0} \times \cdot 9\dot{1}2857\dot{1} \div \cdot\dot{4}76190$. (L.M.)

47. Obtain in decimal form the value of—

$$\tfrac{7}{12} \text{ of } (\cdot 00041625 \div \cdot 037 + \tfrac{8}{13} \text{ of } \cdot 012638).$$ (C.P.)

48. Express $\cdot\dot{7}2\dot{9} \times \cdot 016\dot{4} \div (3\cdot\dot{2}\dot{6} + 1\cdot 2\dot{3}\dot{7})$ as a circulating decimal. (O.S.L.)

49. Simplify $(\cdot 09 + 2\cdot 069) \div (\cdot 905 - \cdot 89103)$. (O.J.)

50. Express $\frac{1}{81}$ as a recurring decimal, and simplify—

$$3\cdot\dot{3}\dot{6} \times \cdot 0\dot{1}4\dot{3} \div \cdot\dot{0}12345679\dot{0}.$$ (C.S.L.)

51. Find to 2 places of decimals $(4\cdot 7\dot{3} + 31\cdot 2\dot{5}8\dot{1}) \div (1\cdot 2\dot{1}6\dot{5} - \cdot 5\dot{1})$. (C.P.)

52. Divide $\cdot 01234$ by $\cdot 2347$ to four decimal places, and express $\dfrac{\cdot 6\dot{1}\dot{5} + \cdot 3128\dot{7}}{1\cdot 2\dot{7}}$ in its simplest form. (C.P.)

53. $\dfrac{\cdot 17 + \cdot 65333\ldots}{\cdot 247 \div 2\cdot 21}$. (S.K.)

54. Express as a decimal $\dfrac{\cdot 5\dot{1} \times \cdot 5\dot{1} \times \cdot 5\dot{1} - \cdot 4\dot{5} \times \cdot 4\dot{5} \times \cdot 4\dot{5}}{\cdot 5\dot{1} - \cdot 4\dot{5}}$. (C.H.L.)

55. Find the value (as a decimal) of—

$$\frac{1}{1} + \frac{1}{1\times 2} + \frac{1}{1\times 2\times 3} + \frac{1}{1\times 2\times 3\times 4} + \frac{1}{1\times 2\times 3\times 4\times 5}.$$ (C.M.)

56. Simplify $\dfrac{2\cdot 4\dot{6} - 2\cdot\dot{3}\dot{0}}{\cdot\dot{3} + \cdot 12\dot{7}} + \dfrac{4\frac{1}{3}}{19}$. (C.S.L.)

57. Find the value of $\dfrac{13}{29}$ of $\dfrac{7\cdot\dot{1} - 3\cdot\dot{6}}{2\cdot 0\dot{5} + 2\cdot 9\dot{4}}$ of $\dfrac{7\cdot 25 \text{ of } 3\cdot 6}{10\cdot 725}$. (L.M.)

58. Express $(\cdot\dot{1})^2$ as a repeating decimal. (R.U.I.)

59. $1\frac{3}{4}$ of $\frac{4{\cdot}75}{12{\cdot}6}$ of $\frac{1{\cdot}\dot{1}}{{\cdot}0078125} \div \frac{17{\cdot}\dot{7}}{16{\cdot}35}$. (L.M.)

60. Reduce to a single decimal ($\frac{5}{9}$ of ${\cdot}\dot{2}9\dot{7} + \frac{2}{3}$ of ${\cdot}00\dot{7}$) $\times \frac{31{\cdot}968}{1{\cdot}2155714\dot{2}\dot{8}}$. (C.S.)

EXERCISE XXI. 21

Problems involving Repeating Decimals.

1. A man owns $\frac{2}{13}$ of a ship; he sells ${\cdot}1\dot{3}5\dot{1}$ of his share. What fraction of the ship does he retain? (C.J.)

2. A owns ${\cdot}\dot{3}9\dot{6}$ of an estate, and B the remainder. What is the value of B's share of it, if the estate be worth £4014 10*s.*? (K.S.)

3. A railway train is moving at the rate of $27{\cdot}3\dot{7}$ yards per second. How many miles an hour is this equivalent to? (C.J.)

4. The value of ·1625 of a field at £17 per acre is £40 1*s.* $1\frac{1}{2}d.$ How much is the field worth? If the area of the field is ${\cdot}0805$ of the area of the farm to which it belongs, what is the area of the farm? (C.P.)

5. A woman has a certain number of eggs; she sells ${\cdot}\dot{3}$ of the number and one more to one person; ${\cdot}\dot{3}$ of the remainder to a second person; ·5 of the remainder to a third person: after these sales she had 15 eggs left. How many had she at first? (P.T.)

6. A and B together are 4·5 of C, and C is ·24 of B. What decimal is A of C? (C.P.)

7. How many pounds of tea can be bought for £5, if ${\cdot}06\dot{3}$ of 5·5 lbs. cost ${\cdot}41\dot{6}$ of ${\cdot}1\dot{6}$ of a guinea? (C.S.L.)

8. Of ·285 and ·286, find which is nearer to the true value of ${\cdot}2857\frac{1}{4}$, and by how much nearer? (K.S.)

9. A boy is told to find ·61 of a sum of money. He finds ${\cdot}6\dot{1}$ instead. Is his result too large or too small? If the boy's error amounts to 6*d.*, what is the sum of money? (C.M.)

10. After purchasing goods to the value of £60, a man has spent

$$\tfrac{1}{3}\left\{1\tfrac{1}{11} - \frac{1-\frac{7}{22}}{2-\frac{1}{3}} + \frac{1\frac{2}{5}}{3\frac{1}{2}} - \frac{5\frac{5}{8}}{6\frac{1}{4}} \text{ of } \left(\tfrac{1}{5} - \frac{\frac{1}{2}-{\cdot}\dot{3}}{4\frac{3}{4}-3\frac{2}{5}}\right)\right\} \text{ of his money.}$$

How much has he left? (C.M.S.)

11. One clerk has 72·625, and another has 41·375 sheets to engross; they call in a third clerk and agree to divide the work equally between the three, and to pay the third clerk at the rate of ${\cdot}00\dot{8}\dot{3}$ per sheet. How much will the third clerk receive from each of the two others? (C.W.)

12. Divide £47 17*s.* $6\frac{1}{4}d.$ by ·3727272 . . ., and express in a decimal form the ratio of £143 to the quotient. (S.E.)

13. If ·12775 of ·$2\dot{1}917808\dot{}$ of 1 cwt. 13 lbs. of coal costs ·39375*d.*, what is the price per ton? (C.S.)

14. A clerk's salary is each year increased by $\cdot\dot{1}42857\dot{}$ of that which he has received for the previous year, and for his fourth year of service amounts to £120 1*s.* Find what it is for his seventh year, and express his first year's salary as a fraction of his seventh year's salary. (C.S.)

15. There are two maps of the same estate, one is 6 in. to the mile, and on the other $\cdot0\dot{1}$ sq. in. represents a rood; one map is 17 sq. in. larger than the other. How many acres are there in the estate? (S.A.)

16. Show that there are two vulgar fractions which, in their lowest terms, have 148 for denominator, each one of which is equal to a decimal consisting of two non-recurring figures followed by $\cdot\dot{5}6\dot{7}$. (C.P.)

17. A fraction when reduced to a decimal consists of two non-repeating figures followed by $\cdot\dot{7}8\dot{3}$, and has 148 for its denominator, and the numerator is $\frac{3}{4}$ of the number expressed by the non-repeating figures. Find the numerator of the fraction. (C.S.)

18. A certain fraction having 148 for denominator produces a mixed circulating decimal with two non-repeating figures followed by $\cdot\dot{7}8\dot{3}$. If the non-repeating part be $\frac{2}{3}$ of the numerator, find the latter. (C.S.)

19. A number of sheets of paper, each $\cdot00456\dot{7}8\dot{9}$ in. thick, are taken so that the aggregate thickness just exceeds $9\cdot876\dot{5}$ in. Using decimal fractions throughout, find the amount of the excess. (S.A.)

20. In the decimal 2421, if it be given that the four figures repeat, or the last three, or the last two, or the last one, what would be the difference between the greatest and least of such circulating decimals?

(C.S.)

CHAPTER XII.

VULGAR FRACTIONS AND CONCRETE QUANTITIES.

To find what fraction one concrete quantity is of another of the same kind.

Example.—*What fraction is* 9*s.* 10*d. of* 13*s.* 5*d.*?

We have to consider which quantity is to form the denominator, which the numerator. From the question, 9*s.* 10*d.* is to be put in terms of 13*s.* 5*d.*, *i.e.* 9*s.* 10*d.* is to be *named as a part* of 13*s.* 5*d.*, *i.e.* 13*s.* 5*d.* is to be the *naming* part of the fraction. Hence 13*s.* 5*d.* is put in the denominator, 9*s.* 10*d.* in the numerator.

$$\frac{9s.\ 10d.}{13s.\ 5d.} = \frac{118d.}{161d.} = \frac{118}{161} \quad Ans.$$

Example.—*What fraction of a guinea is* £5 4*s.* 9*d.*?

The *guinea* is to be the *naming* portion of the fraction: hence it is placed in the denominator.

$$\frac{£5\ 4s.\ 9d.}{£1\ 1s.\ 0d.} = \frac{£5\frac{19}{80}}{£1\frac{4}{80}} = \frac{\frac{419}{80}}{\frac{84}{80}} = \frac{419}{84} = 4\frac{83}{84} \quad Ans.$$

Note (1) The greater quantity does not always form the denominator.
(2) The second quantity is not necessarily in the denominator.
(3) The only difficulty is to determine which quantity is to form the denominator. It is the quantity which is to give its *name* to the fraction, and it generally appears after some such expression as "part *of*," "fraction *of*."

Example.—*What fraction of* 8 *lbs.* $8\frac{8}{15}$ *oz. avoirdupois is* $7\frac{13}{21}$ *of* 10 *oz.* 18 *dwt.* 11 *grs.*? (C.P.)

$$\frac{7\frac{13}{21} \text{ of } 10 \text{ oz. } 18 \text{ dwt. } 11 \text{ grs.}}{8 \text{ lbs. } 8\frac{8}{15} \text{ oz. avoir.}} = \frac{160 \times 5243 \text{ grs.}}{21 \times \left(8\frac{1}{2} \times 7000 + \frac{8 \times 7000}{15 \times 16}\right) \text{grs.}}$$

$$= \frac{160 \times 5243 \text{ grs.}}{21 \times 59733\frac{1}{3} \text{ grs.}}$$

$$= \frac{160 \times 5243 \times 3}{21 \times 179200}$$

$$= \frac{107}{160} \quad Ans.$$

Note.—Do not endeavour to find $7\frac{13}{21}$ of 10 oz., etc. Work as shown above.

Example.—*What decimal fraction of a ton must be added to* $\cdot\dot{7}857142\dot{8}$ *of* 1 *cwt. to make it equal to* 6 *qrs.* 11 *lbs.?*

Take a simpler example, *e.g.* "What fraction of 8 must be added to 10 to get 12?" The *number* to be added is 2, and this is $\frac{1}{4}$ of 8; ∴ the fraction $= \frac{1}{4}$.

Expressing this in fractional form, we have—

$$\frac{12 - 10}{8} = \frac{2}{8} = \frac{1}{4}$$

Comparing the two problems, 8 represents 1 ton, 10 represents $\cdot\dot{7}857142\dot{8}$ of 1 cwt., and 12 represents 6 qrs. 11 lbs.

Substituting, we have—

$$\frac{6 \text{ qrs. } 11 \text{ lbs.} - \cdot\dot{7}857142\dot{8} \text{ cwt.}}{1 \text{ ton}} = \frac{\frac{179}{112} \text{ cwt.} - \frac{11}{14} \text{ cwt.}}{1 \text{ ton}}$$

$$= \frac{\frac{179-88}{112} \text{ cwt.}}{20 \text{ cwt.}} = \frac{\frac{91}{112}}{20}$$

$$= \frac{91}{2240} = \frac{13}{320}$$

$$= \cdot 040625 \quad \textit{Ans.}$$

THEORY.

1. Prove the rule for reducing a fraction of any denomination to a fraction of another denomination (*e.g.* to reduce 3*s.* 7½*d.* to the fraction of 12*s.* 8¼*d.*). Why is not the answer in money? (K.S.)

2. Examine the following: To reduce one quantity to the fraction of another, place the first quantity in the numerator, and the second in the denominator.

EXERCISE XXII. 22

Vulgar Fractions and Concrete Quantities.

1. $4\frac{1}{3}$ of £7 2*s.* 6*d.* − $3\frac{2}{5}$ of £8 2*s.* 4*d.* (C.P.)

2. Find difference between $\frac{3}{14}$ of a half-guinea and $\frac{11}{48}$ of a half-sovereign. (P.E.A.)

3. By how much does $(\frac{3}{10} + \frac{1}{8})$ of £5 exceed $(\frac{3}{10} - \frac{1}{8})$ of £5? (P.T.)

4. Subtract $\frac{7}{17}$ of £119 from $\frac{13}{23}$ of £178 5*s.* (C.P.)

5. Find by what amount the sum of $2\frac{1}{9}$ of £1 11*s.* $3\frac{3}{4}$*d.* and $4\frac{6}{7}$ of £2 6*s.* $9\frac{3}{4}$*d.* exceeds $\frac{11}{13}$ of £3 11*s.* $9\frac{1}{4}$*d.*

6. Find, to the nearest penny, the value of $\frac{1}{3}$ of 9*s.* 7*d.* + $\frac{1}{4}$ of £1 6*s.* 9*d.* − $\frac{1}{5}$ of 14*s.* 6*d.* (C.S.L.)

7. Simplify the expression $\dfrac{3\frac{1}{12} - 1\frac{6}{7} \text{ of } 1\frac{2}{7}}{4\frac{1}{7} - 2\frac{1}{3} \text{ of } 1\frac{1}{6}}$ of £39 17*s.* 4*d.* (C.S.L.)

8. Find the value of $\dfrac{6\frac{3}{4} + (\frac{2}{5} \div 1\frac{11}{25}) - (\frac{3}{7} \text{ of } 15\frac{3}{4})}{(\frac{3}{4} \text{ of } 7\frac{3}{4}) - (5\frac{2}{5} \div 3\frac{1}{3})}$ of 9*s.* 1*d.* (C.W.)

9. $\frac{3}{4}$ of $\frac{7}{8}$ of 15*s.* + ($\frac{2}{3}$ − $\frac{3}{5}$) of ($\frac{7}{8}$ − $\frac{1}{3}$) of £1 15*s.* − ($\frac{3}{4}$ ÷ $\frac{2}{5}$ of $\frac{3}{4}$) of 1*s.* 4$\frac{1}{2}$*d.* − $\frac{6}{10}$ of $\frac{3}{7}$ of a farthing. (C.P.)

10. $\frac{\frac{1}{2} \text{ of } 3\frac{3}{7}}{\frac{2}{3} \text{ of } 3\frac{1}{4}}$ of £3 12*s.* 5*d.* ÷ $\frac{5\frac{9}{14}}{\frac{3}{5}}$ of 16*s.* 4*d.* + $\frac{3}{\frac{\frac{1}{7}}{\frac{3}{11}}}$ of 3*s.* 8*d.* (C.S.)

11. $\frac{1}{11 - \frac{2}{3 - \frac{1}{3}}}$ of $\frac{\frac{4}{9} - \frac{6}{19}}{\frac{2}{9} - \frac{3}{19}}$ of 11 hrs. 5$\frac{5}{11}$ min. (P.T.)

12. Find the value of $1\frac{1}{5} \times 4\frac{1}{11} \times \frac{18}{19} \times \frac{2}{11}$ of 3 ac. 1 rd. 30 po. (O.R.)

13. Find the value of $\frac{2}{3}$ of $20\frac{3}{4}$ yds. + $\frac{3}{5}$ of $18\frac{5}{6}$ yds. + $\frac{3}{8}$ of 20 yds. at 3*s.* 9*d.* a yard. (C.W.)

14. Subtract the $\frac{1}{11}$ of 23 mi. 7 fur. 18 po. from $\frac{5}{27}$ of 9 mi. 4 fur. 11 po. (C.P.)

15. Add $\frac{5}{11}$ of a pound Troy to $\frac{5}{11}$ of a pound avoirdupois, and give the result (*a*) in Troy weight, (*b*) in avoirdupois weight. (C.S.)

EXERCISE XXIII. 2 3

Reduction of One Quantity to the Vulgar Fraction of Another.

1. Reduce £3 14*s.* 8*d.* to the fraction of £7. (O.R.)

2. Express 1 yd. 1 ft. 7 in. as a vulgar fraction of 4 yds. 2 ft. 8 in. (L.M.)

3. Reduce 33 yds. to the fraction of a mile. (O.R.)

4. What fraction is (i.) £3 7*s.* 6*d.* of £1000; (ii.) 3$\frac{1}{2}$ in. of 3$\frac{1}{2}$ rods or poles? (C.P.)

5. What fraction of 15*s.* is $\frac{6}{13}$ of one penny? (C.P.)

6. A pound avoirdupois contains 7000 grains Troy. What fraction is a pound Troy of a pound avoirdupois? (C.PR.)

7. A pound avoirdupois contains 7000 grains Troy. What fraction is an ounce avoirdupois of an ounce Troy? (C.PR.)

8. A man runs 4$\frac{7}{8}$ times round a course 930$\frac{1}{4}$ yds. long. What fraction of 3 miles 301$\frac{1}{2}$ yds. does he run? (C.J.)

9. Express $\frac{2}{7}$ of 5 cwt. 10 lbs. as a fraction of $\frac{3}{9}$ of 2 cwt. 1 qr. 4 lbs. 8 oz. (O.S.L.)

10. What part of £1 4*s.* 10$\frac{1}{2}$*d.* is $\frac{2}{7}$ of the sum of $\frac{2}{7}$ of £1 18*s.* 4$\frac{1}{4}$*d.* and $\frac{1}{9}$ of £6 5*s.* 3*d.*? (P.T.)

11. What fraction of £4 6*s.* 6*d.* is $\frac{3}{7}$ of £1 4*s.* − $\frac{2}{11}$ of 19*s.* 6*d.*? If A is $\frac{2}{3}$ of B, and C is $\frac{3}{5}$ of A, what fraction is C of B? (C.P.)

12. Express $\frac{3}{25}$ of 11*s.* 3*d.* + $\frac{7}{8}$ of $\frac{11}{9}$ of 4*d.* − $\frac{7}{9}$ of $\frac{1}{14}$ of 2*s.* 4*d.* as a fraction of 7*s.* 6*d.* (K.S.)

13. Reduce $\frac{7}{19}$ of 3*s.* 11$\frac{1}{2}$*d.* + $\frac{4}{33}$ of £10 14*s.* 6*d.* − $\frac{8}{17}$ of 18*s.* 5*d.* to the fraction of £5 12*s.* 9*d.* (C.W.)

14. Reduce $\frac{5}{11}$ of £23 16*s.* 8*d.* + $\frac{9}{17}$ of £54 16*s.* 6*d.* − $\frac{2}{21}$ of £2 0*s.* 3*d.* to the fraction of £50. (C.W.)

15. Express the sum of $\frac{75}{90}$ of $\frac{1}{264}$ of a mile + $\frac{125}{960}$ of $1\frac{1}{5}$ of $\frac{1}{22}$ of a furlong as a fraction of 13 yds. (P.T.)

16. If $\frac{7}{15}$ of a guinea be taken from $\frac{3}{13}$ of $\frac{5}{9}$ of £4 11*s.*, what fraction of $\frac{1}{15}$ of £3 17*s.* will remain? (C.W.)

17. What fractional part of three half-crowns is $\frac{1}{8}$ of $\frac{4}{9}$ of £2 5*s.* + $3\frac{4}{17}$ of 2*s.* $1\frac{1}{2}$*d.* − $\frac{5}{8}$ of 4*s.* 8*d.*? (P.T.)

18. Find what weight is the same fractional part of 2 tons 12 cwt. 3 qrs. 21 lbs. that £4 6*s.* $5\frac{3}{4}$*d.* is of £6 15*s.* $10\frac{3}{4}$*d.* (C.P.)

19. What fraction of 4*s.* 7*d.* will exceed $\frac{3}{7}$ of $5\frac{3}{5}$ of 2*s.* 1*d.* by 1*s.* 5*d.*? (K.S.)

20. Find the difference between $\frac{\text{£2 17}s.\ 6d.}{\text{£12 18}s.\ 9d.}$ and $\frac{\text{2 cwt. 2 qrs. 10 lbs.}}{\text{12 cwt. 0 qrs. 19 lbs.}}$. (S.K.)

CHAPTER XIII.

DECIMALS AND CONCRETE QUANTITIES.

Value of the Decimal of a Concrete Quantity.

The method employed is so nearly similar to ordinary reduction that an example will suffice.

Example.—*Find the value of* ·047265625 *of a ton in terms of lower denominations.* (C.P.)

·047265625 ton
20
·945312500 cwt.
4
3·781250000 qrs.
28
21·875000 lbs.
16
14·000 oz.

Ans. 3 qrs. 21 lbs. 14 oz.

Note (1) the similarity to ordinary reduction.
(2) that the successive multiplications do not deal with the digits in thick type, which yield the answer.

Example.—*Find the value of* $3{\cdot}342857\dot{1}$ *of* £$\cdot4\dot{9}$. (I.I.)

Since $\cdot428571 = \frac{3}{7}$, and $\cdot\dot{9} = 1$, the sum becomes—

$$3{\cdot}3\tfrac{3}{7} \text{ of } £{\cdot}5 = 3\tfrac{24}{70} \text{ of } £\tfrac{1}{2}$$
$$= \tfrac{234}{70} \times \tfrac{1}{2} = \tfrac{234}{140}$$
$$= £1\ 13s.\ 5\tfrac{1}{7}d. \quad \textit{Ans.}$$

Note that it is necessary to reduce to vulgar fractions.

To express one concrete quantity as the decimal of another of the same kind.

First Method.—*Reduce the lowest denomination to a decimal of the next higher denomination, and prefix the integral part of this higher denomination. Reduce this to the next higher denomination, and so on.*

Example.—*Reduce* 3 *qrs.* 21 *lbs.* 14 *oz. to the decimal of a ton.*

14 oz. = ·875 lb.

28 | 7 | **21**·875 lbs.
14 | 3·125
4 | **3**·78125 qrs.
20 | ·9453125 cwt.
·047265625 ton. *Ans.*

Contrast this with the last example but one, and note that the 3 qrs. and 21 lbs. which were *discarded* in the *descending* reduction are *inserted* in the example in *ascending* reduction now before us.

Example.—*Reduce* £157 19*s.* 7½*d. to the decimal of* £10000. (K.S.)

12 | 7·5*d.*
20 | **19**·625*s.*
10000 | **157**·98125£
·015798125 *Ans.*

SECOND METHOD.—*Proceed just as in the reduction of one quantity to the vulgar fraction of another, and then convert the vulgar fraction into a decimal.*

This method is often long and tedious, and is not universally recommended. The student might work the two preceding examples by this method, in order that the superiority of the first method may be better appreciated.

Example.—*Reduce* 5 *days* 16 *hours to the decimal of a week.* (C.PR.)

$$\frac{5 \text{ days } 16 \text{ hours}}{1 \text{ week}} = \frac{5\frac{16}{24} \text{ days}}{7 \text{ ,,}} = \frac{5\frac{2}{3}}{7} = \frac{17}{21}$$

21)170(·809523̇
200
110
50
80
17 *Ans.* ·809523̇.

Decimalization of Money.

By decimalization of money is meant the conversion of any sum of money into the decimal of a pound.

I. Decimalization of Shillings.

Since 1*s.* = £$\frac{1}{20}$ = £·05 = £(5 × ·01), any number of shillings may be decimalized at once by multiplying the proposed number of shillings by 5 and marking off two places of decimals.

Example.—*Decimalize* 17*s.*

17 × 5 = 85. Marking off two places of decimals, we get ·85
∴ 17*s.* = £·85

II. Decimalization of Farthings.

Since 1 farthing = $\frac{1}{960}$ of £1 = $\frac{1000}{960}$ of £$\frac{1}{1000}$ = $\frac{1000}{960}$ of £·001 = $\frac{25}{24}$ of £·001 = 1$\frac{1}{24}$ of £·001, any number of farthings may be decimalized

by putting down that number of farthings, marking off three places of decimals, and adding $\frac{1}{24}$ of the number of farthings.

Examples.—*Decimalize* $1\frac{1}{2}d.$, $1\frac{3}{4}d.$, $4\frac{1}{4}d.$

£

$1\frac{1}{2}d. = 6$ farthings $= \cdot 006\frac{6}{24} = \cdot 00625$
$1\frac{3}{4}d. = 7$,, $= \cdot 007\frac{7}{24} = \cdot 0072916\dot{6}$
$4\frac{1}{4}d. = 17$,, $= \cdot 017\frac{17}{24} = \cdot 0177083\dot{3}$

III. Decimalization of any Sum of Money.

Any sum may be decimalized by combining these rules.

Example.—*Reduce* 16*s.* $3\frac{3}{4}d.$ *to the decimal of a pound.*

$16s. = £\cdot 80$ (par. I.).
$3\frac{3}{4}d. = 15$ farthings $= £\cdot 015\frac{15}{24} = £\cdot 015\frac{5}{8} = £\cdot 015625$ (par. II.).

Combining these results—

$16s.\ 3\frac{3}{4}d. = £\cdot 8 + £\cdot 015625 = £\cdot 815625$ *Ans.*

Money should be decimalized mentally, but there is often a difficulty in dividing by 24. This may be surmounted by dividing by 2 and then by 12.

Example.—*Decimalize* £6 8*s.* $11\frac{3}{4}d.$

Multiply 8 by 5; put result in first decimal place. £6 8*s.* = £6·4
$11\frac{3}{4}d. = 47$ farthings; consider the result in the third decimal place. ·047
Divide ·047 mentally by 2, getting ·0235.
Divide ·0235 by 12. ·0019583

£6·4489583

£6 8*s.* $11\frac{3}{4}d.$ = £6·4489583 *Ans.*

Decimalization of Money correct to three places of Decimals.

Since there are 960 farthings in a pound, and since a digit in the third decimal place represents $\frac{1}{1000}$ part of unity, we can decimalize to the third place and obtain an answer correct to the nearest farthing. If the number of farthings is greater than 12 (say 19), the fraction of the third place required by Rule II. (p. 137) is $\frac{19}{24}$, and because $\frac{19}{24}$ is greater than $\frac{1}{2}$, to obtain the *nearest* third place, the decimal in the third place must be increased by 1.

Example.—*Express* 4*s.* $9\frac{1}{2}d.$ *as a decimal of one pound correct to the third place.*

£

$4s. = \cdot 2$
$9\frac{1}{2}d. = 38$ farthings $= \cdot 040$

$4s.\ 9\frac{1}{2}d. = \cdot 240$ *Ans.*

The following should be memorized:—

£		£	
2*s.*	$= \cdot 100$	3*d.*	$= \cdot 0125$
1*s.* 6*d.*	$= \cdot 075$	1*d.*	$= \cdot 00416\dot{6}$
1*s.*	$= \cdot 050$	$\frac{1}{4}d.$	$= \cdot 0010416\dot{6}$
6*d.*	$= \cdot 025$		

To convert a decimal of a pound into pounds, shillings, and pence.

(*a*) Write the decimal correct to four places.

(*b*) Divide the digits in the first two places by 5. The quotient gives the number of shillings.

(*c*) Find the difference between the product of 5 times the quotient found in (*b*) and the whole decimal.

(*d*) Mark off one decimal place in the difference obtained in (*c*).

(*e*) Find $\frac{24}{25}$ of this difference. The nearest integer to the remainder is the number of farthings.

Example.—*Convert* £·57606 *into shillings and pence.*

(*a*) Writing decimal correct to four places, we get ·5761.

(*b*) Dividing the digits in the first two places by 5, we get 10, which equals the number of shillings.

(*c*) Subtracting 5×10 from the whole decimal 5761, we get 761.

(*d*) Marking off one decimal place in the difference obtained, we get 76·1.

(*e*) Subtracting $\frac{1}{25}$ of 76·1 from 76·1 (*i.e.* finding $\frac{24}{25}$ of 76·1), we obtain 73·1.

The nearest integer is 73, which is the number of farthings.

$$73 \text{ farthings} = 18\tfrac{1}{4}d. = 1s.\ 6\tfrac{1}{4}d.$$
$$\therefore\ £\cdot57606 = 10s. + 1s.\ 6\tfrac{1}{4}d.$$
$$= 11s.\ 6\tfrac{1}{4}d.\quad \textit{Ans.}$$

Decimalization of Avoirdupois Weight.

The methods of decimalizing money may be extended to avoirdupois weight.

(*a*) Consider the tons as sovereigns.

(*b*) Consider the hundredweights as shillings.

(*c*) Consider quarters as threepenny-pieces.

(*d*) Consider 7 lbs. as 3 farthings.

A comparison of the money table and avoirdupois weight will at once show the reasons for the above.

Example.—*Reduce* 7 *cwt.* 3 *qrs.* 21 *lbs. to the decimal of a ton.*

$$7 \text{ cwt.} = \cdot35$$
$$3 \text{ qrs.} = 3 \times \cdot0125 = \cdot0375$$
$$21 \text{ lbs.} = 3 \times 3 \times \cdot0010416 = \cdot009375$$
$$7 \text{ cwt. } 3 \text{ qrs. } 21 \text{ lbs.} = \cdot396875\quad \textit{Ans.}$$

Note.—The answer is correct to six places of decimals.

EXERCISE XXIV. 2 4

Finite Decimals and Concrete Quantities.

Money.

1. Find the value of ·0875 of £2 18*s.* 4*d.* (O.S.L.)

2. By the best method, find the value in £ *s.* *d.* of ·2989375 of £100. (C.W.)

3. Find the value of ·35 of 50*s.* + ·0625 of a guinea + ·375 of half a crown. (O. and C.H.C.)

4. Find the difference between 2·1456 of 14*s.*, and 1·2789 of £1 7*s.* 6*d.* (C.P.)

5. Find the value in £ *s.* *d.* of $\frac{1}{375}$ of 2*s.* 3*d.* + ·246875 of £3 − ·875 of 15*s.* 6*d.* (K.S.)

6. Find the value of $(1\frac{2}{3} + \frac{2}{7})$ of £3 4*s.* 9*d.* − 1·35 of £3 2*s.* 1*d.* (S.K.)

7. Add together $\frac{11}{35}$ of 4½ guins., ·615 of a £, and $3\frac{3}{11}$ of $4\frac{2}{7}$ of $1\frac{7}{9}$ of 16*s.* 7*d.* (C.PR.)

8. Find the value, to the nearest farthing, of £401·723 × ·0782. (C.W.)

9. Find the value of 1998 × £·092 + 365 × 19*s.* 11·999*d.*, working as shortly as you can. (P.T.)

10. $\dfrac{1\frac{5}{7} \text{ of £2 } 11s.\ 11d.}{1{\cdot}5 \text{ of £4 } 18s.\ 8d. + {\cdot}3 \text{ of £5}}$. (S.K.)

11. Simplify—

$$\frac{5{\cdot}3 \times \text{£}47}{\left({\cdot}02 + \frac{1}{2}\right) \times \text{£}\left(2 + \frac{1}{{\cdot}02}\right) - \left(3 + \frac{1}{{\cdot}03}\right) \times \text{£}\left({\cdot}03 + \frac{1}{3}\right)}.$$ (C.W.)

Avoirdupois.

12. Find the value of ·047265625 of a ton in terms of lower denominations. (C.P.)

13. Reduce ·03042 of 3 tons 15 cwt. 75 lbs. to ounces and the decimal of an ounce. (O.R.)

14. Find the value of 2·307 of 16 cwt. 3 qrs. $13\frac{1}{2}$ lbs. (S.K.)

15. Find the value of ·76 of 1 cwt. + ·275 of 1 qr. + ·555 of 1 lb. (O. and C.H.C.)

16. Find the value of 1·9514 of a ton + 1·5345 of a cwt. + 1·3125 of a quarter expressed in tons, cwts., etc. (C.S.L.)

17. Obtain the value of 1·2 ton 18·4 cwt. 3·5 qrs. 27·3 lbs., at £2 13*s.* 4*d.* per quarter. (C.P.)

Long Measure.

18. Find the value (in yards, feet, and inches) of ·0109375 of a mile. (K.S.)

19. Express 2·03125 miles − 7·3215 furlongs in feet. (K.S.)

20. Find the number of yards in 0·217 of a mile and a half. (S.K.)

EXERCISE XXV.

Decimalization of Money.

1. Reduce the following to the decimal of £1: (*a*) 7*s.* 6*d.*; (*b*) 17*s.* 9*d.*; (*c*) 3*s.* 8¼*d.*; (*d*) 15*s.* 9¾*d.*; (*e*) 18*s.* 10½*d.*

2. Express as the decimal of £1 correct to the third decimal place: (*a*) 2¼*d.*; (*b*) 14*s.* 7½*d.*; (*c*) £1 7*s.* 4¾*d.*; (*d*) £2 15*s.* 2½*d.*; (*e*) £1 2*s.* 0¾*d.*

3. Read the following decimal fractions of £1 in pounds, shillings, and pence (answer correct to nearest farthing): (*a*) ·7; (*b*) ·75; (*c*) ·375; (*d*) 3·879; (*e*) ·2709.

4. Reduce the following to the decimal of a ton: (*a*) 2 cwt. 2 qrs.; (*b*) 17 cwt. 2 qrs. 7 lbs.; (*c*) 13 cwt. 14 lbs.; (*d*) 6 cwt. 2 qrs. 7 lbs.; (*e*) $\frac{3}{8}$ of 2 qrs. 14 lbs.

5. Find the value of the following decimals of a ton: (*a*) ·975; (*b*) ·78125; (*c*) 3·215; (*d*) ·04375; (*e*) ·3265.

EXERCISE XXVI. 26

Reduction of One Quantity to Decimal of Another (Finite Decimals).

1. Reduce 15*s.* 7½*d.* to the decimal of £5; and find the value of ·001 of £6 5*s.* (O.R.)

2. Express $\frac{3}{5}$ of £9 17*s.* 7½*d.* as the decimal of £8. (C.P.)

3. Reduce 5¼ guineas to the decimal of £20, and find the value of 7·5 of 3·45 of £5 10*s.* (O.R.)

4. Find the value of ·275 of £13 2*s.* 6*d.*, and reduce 4*s.* 9*d.* to the decimal of £1. (O.R.)

5. Reduce ·891 of £5 4*s.* 2*d.* to the fraction of £7 16*s.* 9*d.* (O. and C.H.C.)

6. Add together $\frac{3}{5}$ of a guinea, $\frac{17}{36}$ of a half-crown, $1\frac{7}{36}$ shilling, and $\frac{1}{6}$ of a penny, and reduce the whole to a decimal fraction of a pound. (S.K.)

7. Reduce $\frac{7}{18}$ of £39 + $\frac{11}{13}$ of £58 3*s.* 6*d.* − $\frac{17}{27}$ of 4*s.* 6*d.* to the decimal of £107 1*s.* 8*d.* (C.W.)

8. Reduce ·07 of £1 5*s.* + ·675 of £2 1*s.* 8*d.* + ·1875 of 8*d.* to the decimal of £10. (K.S.)

9. Find the value of $\frac{1}{3}$ of £2 0*s.* 6½*d.* + $\frac{2}{5}$ of 17*s.* 4¾*d.* − $\frac{1}{2}$ of 3*s.* 5½*d.*, and reduce your answer to the decimal of £5. (C.W.)

10. Subtract ·0625 of £113 16*s.* 8*d.* from $\frac{23}{24}$ of £50 5*s.* 6*d.*, and find the number by which the result must be multiplied to produce £985 11*s.* 6*d.* (S.K.)

11. What decimal of 13*s.* 4*d.* must be added to 1·3125 of £1 to make the total equal to ·5125 of £5? (P.T.)

12. Reduce $\dfrac{2\frac{1}{7} \text{ of } (3 - 1\frac{1}{2})}{2 + \dfrac{1}{1 + \dfrac{1}{2 + \frac{1}{3}}}}$ of 3*s.* 6*d.* to the decimal of $\frac{37}{308}\frac{4}{7}$ of £100. (C.S.)

13. Reduce 17 cwt. 2 qrs. 24½ lbs. to the decimal of 2 tons 5 cwt. (I.I.)

14. Express ·0125 of a ton as a decimal of $\frac{3}{7}$ of a cwt. (O.S.L.)

15. Subtract ·046 of 2 cwt. 1 qr. from 1·3 of 9 lbs., and give the answer as a decimal of 5 oz. (C.P.)

16. Which is the greater, $3\frac{1}{2}$ lbs. or ·0015 of a ton, and by how much? Give your answer (1) as a decimal of a ton; (2) as a decimal of a stone. (P.T.)

17. What decimal of a mile is 3149·9424 in.? (O.S.L.)

18. Express the difference between ·063 of a mile and ·27 of a furlong as the decimal of 1300 yards. (O.S.L.)

19. Add together ·4 of a mile, ·424 of a rod, ·4246 of a yard, giving the result (i.) as a decimal of a mile, and (ii.) in feet and the decimal of a foot. (C.P.)

20. Determine, in the shortest way you can, how many times (i.) $\frac{7}{16}$ of £1 is contained in $3\frac{3}{4}$ guins., and (ii.) ·26 of half a crown is contained in 2·21 of 15*s.* (C.P.)

EXERCISE XXVII. 27

Repeating Decimals.

Addition, Subtraction, Multiplication, and Division of Compound Quantities.

1. Find the value of £21·1227 + $3\cdot2\dot{5}$*s.* − $8\frac{8}{15}$*d.* (C.P.)

2. Find the value of $3\cdot\dot{6}$ of $\cdot9\dot{5}\dot{4}$ of $\cdot\dot{4}28571\dot{4}$ of 18*s.* 6*d.* (O.R.)

3. Find the value of ·05 of $\cdot\dot{1}0\dot{1}$ of £37 9*s.* 3*d.* (K.S.)

4. Find the value of the sum of $\frac{3}{7}$ of £1 5*s.* 1*d.*, $\cdot\dot{1}0\dot{8}$ of £7 11*s.* $10\frac{1}{4}$*d.*, and ·00625 of £1. (O. and C.S.C.)

5. $\cdot\dot{4}\dot{5}$ of 16*s.* 6*d.* + $\cdot2\dot{2}\dot{7}$ of 9*s.* 2*d.* (O.S.L.)

6. Find the value of $2\cdot\dot{4}28571\dot{4}$ of £3 8*s.* $4\frac{3}{4}$*d.* − $5\cdot\dot{2}9\dot{7}$ of £1 3*s.* $1\frac{1}{2}$*d.* (P.T.)

7. Find, to the nearest penny, the value of £$62\cdot3\dot{1}4\dot{5}$ × 15·1874, avoiding unnecessary work.

8. $\frac{\cdot475}{1\cdot2\dot{6}}$ of $\frac{35\cdot\dot{5}}{\cdot0625}$ of $\frac{5\cdot625}{8\cdot\dot{8}}$ of £1 15*s.* (L.M.)

9. Find the value of $\frac{\cdot475}{1\cdot2\dot{6}}$ of $\frac{35\cdot\dot{5}}{\cdot0625}$ of $\frac{5\cdot675}{8\cdot\dot{8}}$ of £1 15*s.* (L.M.)

10. Add together £$3\cdot05425$, $12\cdot1\dot{2}$ of £2 5*s.*, and $7\cdot\dot{2}85714\dot{2}$ of a guinea. (C.P.)

11. Find the value of ·094921875 of £5 6*s.* 8*d.* − $\cdot05\dot{4}$ of 24*s.* 9*d.* + $\cdot5\dot{8}\dot{3}$ of 41*s.* 3*d.* (C.PR.)

12. Find the value in tons, cwts., qrs., etc., of $2\cdot1\dot{6}$ of 3 tons 12 cwt. + $\cdot\dot{1}42857\dot{1}$ of 1 cwt. 3 qrs. 22 lbs. + ·75 of a cwt. + $3\cdot8\dot{3}$ of 9 lbs. 13 oz. (P.T.)

13. Simplify $3\cdot13\dot{7}$ of a ton + $2\cdot9\dot{3}\dot{5}$ of a cwt. − $\cdot3\dot{8}\dot{1}$ of a pound. (C.P.)

14. Find the value of $\frac{3}{4}$ of $\frac{1}{9\frac{1}{2}}$ of £1 18*s.* + $\frac{2}{3}$ of ·375 of 15*s.* + $\frac{5}{14}$ of $\cdot32\dot{8}$ of 11*s.* 3*d.* (C.W.).

15. Find the value of $2{\cdot}7875$ days $- 6{\cdot}56$ hrs. $+ {\cdot}3\dot{2}\dot{7}$ of 2 hrs. 1 min. (C.P.)

16. Find the values of ${\cdot}375$ of a guinea $- {\cdot}0\dot{2}\dot{7}$ of £2 15*s.*; and of $\frac{{\cdot}2\dot{1}3\dot{3}4\dot{8}}{{\cdot}326}$ of 2 cwt. 102 lbs. $+ {\cdot}3\dot{4}0\dot{5}$ of $1\frac{2}{3}$ qrs. (P.P.)

17. Reduce ${\cdot}9\dot{6}\dot{2}$ of ${\cdot}2\dot{5}\dot{9}$ of ${\cdot}1\dot{6}3\dot{6}$ of ${\cdot}\dot{0}32967\dot{}$ of ${\cdot}6875$ of 27 mi. to yards. (C.S.)

18. Find the number of poles in $\frac{{\cdot}9\dot{4}}{3{\cdot}86\dot{3}}$ of 6 ac. 1 rd. 35 po. (C.P.)

19. Find the value of $\frac{{\cdot}\dot{4}61538\dot{} \times 29{\cdot}25}{65{\cdot}52 \times {\cdot}\dot{1}42857\dot{}}$ of 2 cwt. 1 qr. 8 lbs. (C.M.)

20. Find the value of $3{\cdot}3\dot{4}2857\dot{1}$ of £${\cdot}4\dot{9}$, first converting to vulgar fractions. (I.I.)

EXERCISE XXVIII.

Repeating Decimals.

Reduction of One Compound Quantity to the Decimal Fraction of Another.

1. Express (1) a guinea as a decimal of a sovereign; (2) a sovereign as the decimal of a guinea. (P.E.A.)

2. Express as a decimal of £1 each of the English coins; and say what advantage there is in the use of the decimal method in solving money problems. (C.W.)

3. Reduce $\frac{196}{275}$ lb. and $\frac{66}{101}$ qr. each to the decimal of a cwt., and add the results together. (C.W.)

4. Reduce $1\frac{3}{4}d.$ to the decimal of 1*s.* 4*d.*; and $1\frac{3}{4}$ oz. to that of 15 lbs. avoirdupois. (C.P.)

5. What decimal fraction is $2\frac{1}{4}d.$ of 7*s.* 6*d.*, and what decimal of a cwt. is 64 lbs.? (S.K.)

6. Express 18 cwt. 1 qr. 3 lbs. 8 oz. as a vulgar fraction, and also as a decimal fraction, of 2 tons 11 cwt. 2 qrs. 7 lbs. (C.H.L.)

7. Express an ounce avoirdupois as decimal of an ounce Troy, assuming that a pound avoirdupois contains 7000 grains Troy. (P.E.A.)

8. Express 4 lbs. 3 oz. 12 dwts. 6 grs. Troy weight as the decimal of a pound avoirdupois. (C.S.)

9. Reduce 5 dys. 16 hrs. to the decimal of a week. (C.PR.)

10. Reduce 40 yds. $1\frac{1}{2}$ ft. to the decimal of 2 mi. (C.P.)

11. Express 3 rds. 18 po. $16\frac{1}{2}$ sq. yds. as a decimal of 9 acs. 2 rds. 4 po. (O. and C.H.C.)

12. Find the value of ${\cdot}04375$ of a ton, and express ${\cdot}039\dot{5}$ of $5\frac{1}{2}$ lbs. as a decimal of $\frac{7}{11}$ cwt. (C.S.L.)

13. Reduce (£14·15 − £5·3$\dot{6}$) to the decimal of (£2·$\dot{8}$ − £2·4$\dot{3}$). (C.M.)

14. Express £91 12*s.* 8*d.* as a fraction of £151 15*s.* $4\frac{1}{4}d.$, and 2 days 5 hrs. $19\frac{1}{2}$ min. as a decimal of a week. (C.P.)

15. Express 4*s.* $3\frac{1}{2}d.$ as the decimal of £19 6*s.* 3*d.*, and find the value of $\cdot 458\dot{3}$ of £52 13*s.* 6*d.* (C.P.)

16. Express as a decimal of 14*s.* 7*d.* the difference between $\cdot 08\dot{3}$ of a guinea and ·0875 of 15*s.* (C.M.)

17. Express the smallest of the following sums as the decimal of the largest: $\frac{\frac{1}{7}+\frac{1}{8}}{\frac{1}{8}\div\frac{1}{7}}$ of 2*s.* $0\frac{1}{2}d.$, $\frac{\frac{23}{44}}{2\frac{1}{11}}$ of $\cdot\dot{5}7142\dot{8}$ of a guinea, $\frac{8}{9}$ of $\frac{1\cdot53}{\cdot17}$ of a penny, $\cdot 0769\dot{2}\dot{3}$ of £6 10*s.* (C.W.S.)

18. Express as the decimal of a florin the difference between $\frac{11}{15}$ of a sovereign and $\frac{13}{14}$ of a guinea. (P.E.A.)

19. Reduce $\cdot 50\dot{2}$ of $7\cdot\dot{1}42857$ of £5 $-\frac{1}{63}$ of £20 to the fraction of $£\cdot 02\dot{5} + \frac{1}{7}$ of $\cdot 01\dot{4}$ of 5*s.* (C.P.)

20. Find the value of 3·075 of a ton + $\cdot\dot{1}42857$ of a cwt. + $2\frac{17}{28}$ cwt., and reduce the result to the decimal of 9 tons 12 cwt. 3 qrs. (C.M.)

21. Express as a recurring decimal (without converting the decimals into vulgar fractions) the sum of $£\cdot\dot{5}\dot{7}$, $£\cdot 00\dot{2}0\dot{4}$, $£\cdot 9\dot{0}1\dot{2}$, and find the value of the sum to the nearest penny. (C.J.)

22. Express 2·625 of $125\frac{5}{7}$ yds. as the decimal of the sum of ·1875 of $1\frac{3}{5}$ mi. and $\frac{3}{13}$ of $\cdot 4\dot{3}$ fur. (K.S.S.)

23. Subtract $\cdot 00\dot{5}$ of 2 cwt. 1 qr. from 1·4 of 5 lbs., and give the answer as a decimal of 12 lbs. (C.W.)

24. Express the difference between $\cdot 1458\dot{3}$ of £100 and $\cdot 2875416\dot{6}$ of 1000*s.* as a decimal of one million pence. (P.P.)

25. Reduce to its simplest form $\frac{2\cdot3888 \text{ . . . of £27 } 3s.\ 9d.}{3\cdot9090 \text{ . . . of £19 } 7s.\ 9d.}$. (S.K.)

CHAPTER XIV.

A FEW APPLICATIONS OF ALGEBRA TO ARITHMETIC IN THE SOLUTION OF VULGAR AND DECIMAL FRACTIONS.

An application of Algebra to Arithmetic is often very useful in the solution of complex fractions, both vulgar and decimal.

Example.—Simplify—

$$\frac{\cdot 321 \times \cdot 321 - \cdot 179 \times \cdot 179}{\cdot 321 - \cdot 179} \text{ of £10.} \quad \text{(C.S.)}$$

Let $\cdot 321 = a$, and $\cdot 179 = b$.

Substituting, we have—

$$\frac{a^2 - b^2}{a - b} = a + b = \cdot 321 + \cdot 179 = \cdot 5$$

$\cdot 5$ of £10 = £5 *Ans.*

Example.—Simplify—

$$\frac{\left(1 + \frac{1}{10 + \frac{1}{10}}\right) \times \left(1 + \frac{1}{10 + \frac{1}{10}}\right) - \left(1 - \frac{1}{10 + \frac{1}{10}}\right) \times \left(1 - \frac{1}{10 + \frac{1}{10}}\right)}{\left(1 + \frac{1}{10 + \frac{1}{10}}\right) + \left(1 - \frac{1}{10 + \frac{1}{10}}\right)}. \quad \text{(P.T.)}$$

Let $a = 1$, and $b = \frac{1}{10 + \frac{1}{10}}$.

Re-writing the above, we have—

$$\frac{(a + b)(a + b) - (a - b)(a - b)}{(a + b) + (a - b)} = \frac{a^2 + 2ab + b^2 - a^2 + 2ab - b^2}{2a}$$

$$= \frac{4ab}{2a} = 2b = 2 \times \frac{1}{10 + \frac{1}{10}} = \frac{2}{\frac{101}{10}}$$

$$= \frac{20}{101} \quad \textit{Ans.}$$

Example.—Find the value of—

$$\frac{\left(\frac{1}{2}\right)^4 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{6}\right)^2}{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)}. \quad \text{(L.M.)}$$

Let $a = \frac{1}{2}$, and $b = \frac{1}{3}$.

Then $ab = \frac{1}{6}$

The algebraic statement may now be substituted for the above—

$$\frac{a^4 + b^4 + a^2b^2}{a^2 + b^2 + ab} = \frac{(a^2 + b^2 + ab)(a^2 + b^2 - ab)}{a^2 + b^2 + ab} = a^2 + b^2 - ab$$

Substituting—

$$a^2 + b^2 - ab = (\tfrac{1}{2})^2 + (\tfrac{1}{3})^2 - (\tfrac{1}{2} \times \tfrac{1}{3}) = \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{6} = \tfrac{7}{36} \quad Ans.$$

Example.—*Solve*—

$$\frac{\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}} - \frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{2} + \frac{1}{3}}}{\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}} + \frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{2} + \frac{1}{3}}}$$

Let $a = \frac{1}{2}$, and $b = \frac{1}{3}$.

Then we can re-write the above as—

$$\frac{\frac{a+b}{a-b} - \frac{a-b}{a+b}}{\frac{a+b}{a-b} + \frac{a-b}{a+b}} = \frac{\frac{(a+b)^2 - (a-b)^2}{a^2 - b^2}}{\frac{(a+b)^2 + (a-b)^2}{a^2 - b^2}} = \frac{\frac{4ab}{a^2 - b^2}}{\frac{2a^2 + 2b^2}{a^2 - b^2}}$$

$$= \frac{2ab}{a^2 + b^2} = \frac{2 \times \frac{1}{2} \times \frac{1}{3}}{(\frac{1}{2})^2 + (\frac{1}{3})^2}$$

$$= \frac{\frac{1}{3}}{\frac{1}{4} + \frac{1}{9}} = \frac{\frac{1}{3}}{\frac{13}{36}} = \frac{12}{13} \quad Ans.$$

Example.—*Express*—

$$\frac{(\cdot25 \times \cdot25) - (\cdot24 \times \cdot24)}{\cdot49} \text{ cub. yds.} - \frac{\cdot49}{(\cdot25 \times \cdot25) - (\cdot24 \times \cdot24)}$$

of $\cdot00\dot{1}\dot{6}$ *cub. ft. as a fraction of a cubic foot.*

This may be solved upon the algebraic model—

$$A^2 - B^2 = (A + B)(A - B)$$

Let $\cdot25 = A$, and $\cdot24 = B$.

Then we have—

$$\frac{(A^2 - B^2) \times 27}{\cdot49} \text{ cub. ft.} - \frac{\cdot49 \times \cdot0016}{A^2 - B^2} \text{ cub. ft.}$$

$$\frac{(A + B)(A - B) \times 27}{\cdot49} \text{ cub. ft.} - \frac{\cdot49 \times \cdot00\dot{1}\dot{6}}{(A + B)(A - B)} \text{ cub. ft.}$$

$$\frac{\cdot49 \times \cdot01 \times 27}{\cdot49} \text{ cub. ft.} - \frac{\cdot49 \times \cdot00\dot{1}\dot{6}}{\cdot49 \times \cdot01} \text{ cub. ft.}$$

$$\cdot27 \text{ cub. ft.} - \cdot1\dot{6} \text{ cub. ft.}$$

$$\cdot10\dot{3} \text{ cub. ft.} \quad Ans.$$

The student might work the foregoing examples by the ordinary method, in order to assure himself of the brevity and superiority of the algebraic method shown above.

EXERCISE XXIX. 27

Algebraic Processes in Solution of Complex Fractions, etc.

1. $\dfrac{61^2 - 60^2}{105^2 - 104^2} - \dfrac{62^2 - 61^2}{116^2 - 112^2}$. (C.M.)

2. Find the value of $\dfrac{\cdot321 \times \cdot321 - \cdot179 \times \cdot179}{\cdot321 - \cdot179}$ of £5. (C.PR.)

3. Find the price of $\dfrac{(\frac{2}{3} \times \frac{1}{5}) - (\frac{2}{3} \times \frac{1}{5})}{(\frac{2}{3} \text{ of } \frac{2}{3} \text{ of } \frac{1}{3} \text{ of } \frac{2}{3}) - (\frac{2}{3} \text{ of } \frac{2}{3} \text{ of } \frac{2}{3} \text{ of } \frac{1}{3})}$ of 1 lb. at a penny an ounce. (P.T.)

4. $\dfrac{(\cdot0\dot1 \times \cdot0\dot1) + (\cdot02 \times \cdot02)}{(\cdot0\dot1 + \cdot02) \times (\cdot0\dot1 + \cdot02) + (\cdot02 - \cdot0\dot1) \times (\cdot02 - \cdot0\dot1)}$. (P.T.)

5. $\dfrac{(1 - \frac{1}{4})(1 + \frac{1}{4})}{(1 + \frac{1}{4} \times \cdot0\dot1)^2 - (\frac{1}{4} + \cdot0\dot1)(\frac{1}{4} + \cdot0\dot1)} \div \frac{1}{2}\left(\dfrac{1}{1 - \cdot0\dot1} + \dfrac{1}{1 + \cdot0\dot1}\right)$.

6. $\dfrac{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} - 3 \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}}{\frac{1}{2} \text{ of } \frac{1}{2} + \frac{1}{3} \text{ of } \frac{1}{3} + \frac{1}{4} \text{ of } \frac{1}{4} - (\frac{1}{3} \text{ of } \frac{1}{4} + \frac{1}{4} \text{ of } \frac{1}{2} + \frac{1}{2} \text{ of } \frac{1}{3})}$. (P.T.)

7. $\dfrac{\{(\frac{1}{3} \times \frac{1}{3}) + (\frac{1}{3} \times \frac{1}{10}) + (\frac{1}{10} \times \frac{1}{10})\} \times \{(\frac{1}{3} \times \frac{1}{3}) - (\frac{1}{3} \times \frac{1}{10}) + (\frac{1}{10} \times \frac{1}{10})\}}{(\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}) + (\frac{1}{3} \times \frac{1}{3} \times \frac{1}{10} \times \frac{1}{10}) + (\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10})}$. (P.T.)

8. Simplify and express as a decimal—

$$\frac{\left(1 + \dfrac{1}{10 + \frac{1}{10}}\right) \times \left(1 + \dfrac{1}{10 + \frac{1}{10}}\right) - \left(1 - \dfrac{1}{10 + \frac{1}{10}}\right) \times \left(1 - \dfrac{1}{10 + \frac{1}{10}}\right)}{\left(1 + \dfrac{1}{10 + \frac{1}{10}}\right) + \left(1 - \dfrac{1}{10 + \frac{1}{10}}\right)}.$$ (P.T.)

9. $\{(\frac{1}{10})^2 - \frac{1}{16}\} \times \left(\dfrac{\frac{1}{16} + 1}{(\frac{1}{16})^2}\right) \div \dfrac{(\frac{1}{16})^4 - 1}{\frac{1}{16}\{(\frac{1}{16})^2 + 1\}}$.

10. $\dfrac{(\frac{5}{8} \times \frac{5}{8} + 2 \times \frac{5}{8} \times \frac{3}{5} + \frac{3}{5} \times \frac{3}{5}) - (\frac{2}{7} \times \frac{2}{7} + 2 \times \frac{2}{7} \times \frac{1}{9} + \frac{1}{9} \times \frac{1}{9})}{(\frac{5}{8} \times \frac{5}{8} + 2 \times \frac{5}{8} \times \frac{2}{7} + \frac{2}{7} \times \frac{2}{7}) - (\frac{3}{5} \times \frac{3}{5} + 2 \times \frac{3}{5} \times \frac{1}{9} + \frac{1}{9} \times \frac{1}{9})}$

11. $\frac{1}{(\frac{1}{3}-\frac{1}{4})(\frac{1}{2}-\frac{1}{7})}+\frac{1}{(\frac{1}{4}-\frac{1}{5})(\frac{1}{3}-\frac{1}{5})}+\frac{1}{(\frac{1}{3}-\frac{1}{4})(\frac{1}{4}-\frac{1}{5})}$.

12. $\frac{\frac{7}{8}-\frac{3}{5}}{(\frac{11}{12}\times\frac{11}{12})-(\frac{7}{8}\times\frac{7}{8}-2\times\frac{7}{8}\times\frac{3}{5}+\frac{3}{5}\times\frac{3}{5})}+\frac{\frac{3}{5}-\frac{11}{12}}{(\frac{7}{8}\times\frac{7}{8})-(\frac{3}{5}\times\frac{3}{5}-2\times\frac{3}{5}\times\frac{11}{12}+\frac{11}{12}\times\frac{11}{12})}$

$+\frac{\frac{11}{12}-\frac{7}{8}}{(\frac{7}{8}\times\frac{3}{5})-(\frac{11}{12}\times\frac{11}{12}-2\times\frac{7}{8}\times\frac{11}{12}+\frac{7}{8}\times\frac{7}{8})}$.

13. $\frac{(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2})}{(\frac{1}{2}-\frac{1}{3})(\frac{1}{2}-\frac{1}{4})}+\frac{(\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3})}{(\frac{1}{3}-\frac{1}{4})(\frac{1}{3}-\frac{1}{5})}+\frac{(\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4})}{(\frac{1}{4}-\frac{1}{5})(\frac{1}{4}-\frac{1}{5})}$.

14. $\frac{5\frac{1}{3}(4\frac{1}{3}-3\frac{1}{4})(3\frac{1}{4}-2\frac{1}{5})-3\frac{1}{4}(2\frac{1}{5}-5\frac{1}{3})(5\frac{1}{2}-4\frac{1}{3})}{4\frac{1}{3}(3\frac{1}{4}-2\frac{1}{5})(2\frac{1}{5}-5\frac{1}{3})-2\frac{1}{5}(5\frac{1}{2}-4\frac{1}{3})(4\frac{1}{3}-3\frac{1}{4})}$.

15. $\frac{(2\frac{1}{3}\times2\frac{1}{3})+(1\frac{1}{3}\times1\frac{1}{3})}{(2\frac{1}{3}\times2\frac{1}{3})-(1\frac{1}{3}\times1\frac{1}{3})}-\frac{(2\frac{1}{2}\times2\frac{1}{2})-(1\frac{1}{3}\times1\frac{1}{3})}{(2\frac{1}{2}\times2\frac{1}{2})+(1\frac{1}{3}\times1\frac{1}{3})}$.
$\frac{2\frac{1}{2}+1\frac{1}{3}}{2\frac{1}{2}-1\frac{1}{3}}-\frac{2\frac{1}{2}-1\frac{1}{3}}{2\frac{1}{2}+1\frac{1}{3}}$

16. $\frac{1+\frac{2\times\frac{5}{8}}{\frac{6}{7}}+\frac{(\frac{5}{8}\times\frac{5}{8})}{(\frac{6}{7}\times\frac{6}{7})}}{1-\frac{(\frac{5}{8}\times\frac{5}{8})}{(\frac{6}{7}\times\frac{6}{7})}}+\frac{1-\frac{(\frac{5}{8}\times\frac{5}{8})}{(\frac{6}{7}\times\frac{6}{7})}}{1-\frac{2\times\frac{5}{8}}{\frac{6}{7}}+\frac{(\frac{5}{8}\times\frac{5}{8})}{(\frac{6}{7}\times\frac{6}{7})}}$.

17.
$$\frac{(6\frac{1}{4}\times6\frac{1}{4})-(5\frac{1}{3}\times5\frac{1}{3}-2\times5\frac{1}{3}\times\frac{3}{4}+\frac{3}{4}\times\frac{3}{4})}{(6\frac{1}{4}\times6\frac{1}{4}+2\times6\frac{1}{4}\times\frac{3}{4}+\frac{3}{4}\times\frac{3}{4})-(5\frac{1}{3}\times5\frac{1}{3})}+\frac{(5\frac{1}{3}\times5\frac{1}{3})-(\frac{3}{4}\times\frac{3}{4}-2\times\frac{3}{4}\times6\frac{1}{4}+6\frac{1}{4}\times6\frac{1}{4})}{(6\frac{1}{4}\times6\frac{1}{4}+2\times6\frac{1}{4}\times5\frac{1}{3}+5\frac{1}{3}\times5\frac{1}{3})-(\frac{3}{4}\times\frac{3}{4})}$$
$$+\frac{(\frac{3}{4}\times\frac{3}{4})-(6\frac{1}{4}\times6\frac{1}{4}-2\times6\frac{1}{4}\times5\frac{1}{3}+5\frac{1}{3}\times5\frac{1}{3})}{(5\frac{1}{3}\times5\frac{1}{3}+2\times5\frac{1}{3}\times\frac{3}{4}+\frac{3}{4}\times\frac{3}{4})-(6\frac{1}{4}\times6\frac{1}{4})}.$$

18. $\frac{\{\frac{1}{2}-(\frac{1}{4}-\frac{1}{5})\}\{\frac{1}{2}+(\frac{1}{4}-\frac{1}{5})\}}{(\frac{1}{2}+\frac{1}{3})^2-\frac{1}{16}}+\frac{\{\frac{1}{4}-(\frac{1}{5}-\frac{1}{6})\}\{\frac{1}{4}+(\frac{1}{5}-\frac{1}{6})\}}{(\frac{1}{2}+\frac{1}{4})^2-\frac{1}{9}}$
$+\frac{\{\frac{1}{3}-(\frac{1}{2}-\frac{1}{4})\}\{\frac{1}{3}+(\frac{1}{2}-\frac{1}{4})\}}{(\frac{1}{4}+\frac{1}{5})^2-\frac{1}{4}}$.

19. $\frac{(4\times\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5})+(7\times\frac{3}{5}\times\frac{3}{5}\times\frac{1}{2}\times\frac{1}{2})+(2\times\frac{3}{5}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2})}{(6\times\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5})-(\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}\times\frac{1}{2})+(12\times\frac{3}{5}\times\frac{3}{5}\times\frac{1}{2}\times\frac{1}{2})}$
$+(3\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2})$
$+(\frac{3}{5}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2})+(6\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2})$

CHAPTER XV.

THE METRIC SYSTEM.

Why so called.—The Metric System is so called because its unit is the *Metre.*

Why introduced.—The different units of weight and measurement employed in different countries had always been a great obstacle to commerce, and various attempts (all more or less unsuccessful) had been made from the time of Charlemagne onwards to unify the conflicting systems. The idea gradually gained ground that a *natural* unit should be adopted, and in 1791 a French Commission recommended that the standard of length should be $\frac{1}{10000000}$ part of the distance from the Equator to the North Pole. This standard of length was subsequently called a *metre.*

It was argued that the distance from the Equator to the North Pole would not vary, and if a small fraction of this were taken, a natural and unvarying unit could be obtained; bars of metal with the exact length of the unit marked on them could be made; and that even if the standard bars were lost or destroyed, it would be possible to recover the standard length by once more calculating the ten-millionth part of the distance from the Equator to the North Pole.

The greatest possible pains were taken to measure this distance, and in 1794 a platinum bar showing the exact length of a metre was placed in the *Bureau des Archives* in Paris.

The measurements made in 1791-4 were, however, not quite correct, and subsequent investigations have shown that the distance between the North Pole and the Equator is nearer 10,000,856 than 10,000,000 metres. So the metre is only an arbitrary standard after all.

The **Metre** may be defined as the distance between two lines marked on a bar of platinum kept at the *Bureau des Archives* in Paris.

Where used.—In practically all civilized countries of the world except Russia, the United States, and the British Empire. It is universally adopted by all scientists.

Russia and the United States possess a decimal coinage, but not a decimal system of weights and measures. In 1898 the British Parliament legalized the metric system, but it is, unfortunately, little used. For Decimal Coinages, see Chapter XXXVII.

Measure of Length.—The unit is the *metre.*

To denote the multiples and sub-multiples of the metre (and indeed of all the derived units) the following prefixes are used:—

1. **Multiples**—

Deca (Gk. *deka*, ten) stands for 10 times
Hecto (Gk. *hekaton*, hundred) stands for 100 „
Kilo (Gk. *kilioi*, thousand) „ „ 1000 „
Myria (Gk. *murios*, ten thousands) „ 10000 „
} the unit.

2. **Sub-multiples**—

Deci (Lat. *decem*, ten) stands for $\frac{1}{10}$
Centi (Lat. *centum*, hundred) stands for $\frac{1}{100}$
Milli (Lat. *mille*, thousand) „ „ $\frac{1}{1000}$
} of the unit.

Note (1) that the *multiples* are of *Greek*, the *sub-multiples* of *Latin* derivation.
(2) that all the sub-multiples end in i.
(3) that *deca* must be carefully distinguished from *deci.*

We can now write a **Metric Long Measure Table.**

10 millimetres (mm.) = 1 centimetre
10 centimetres (cm.) = 1 decimetre
10 decimetres (dm.) = 1 metre
10 metres (m.) = 1 decametre
10 decametres (Dm.) = 1 hectometre
10 hectometres (Hm.) = 1 kilometre
10 kilometres (Km.) = 1 myriametre (Mm.)

Kilo	Hecto	Deca	Unit	Deci	Centi	Milli
1000	100	10	1	·1	·01	·001
Km.	Hm.	Dm.	m.	dm.	cm.	mm.

Note that capital letters are used in the contractions for multiples, and small letters for sub-multiples.

English Equivalents.

1 metre = 39·37 inches
= 3 ft. + 3 in. + $\frac{1}{3}$ in. approximately
1 decimetre = 4 inches nearly
1 kilometre = $\frac{5}{8}$ mile approximately.

A kilometre was designed to be a centesimal minute of latitude.
A cricket pitch is about 20 metres long.

Writing our metrical table of length in another form, we are able to make an important deduction.

10 mm. = 1 cm.	= ·3937079... in.
100 mm. = 10 cm. = 1 dm.	= 3·937079... ,,
1000 mm. = 100 cm. = 10 dm. = 1 m.	= 39·37079... ,,
10000 mm. = 1000 cm. = 100 dm. = 10 m. = 1 Dm.	= 393·7079... ,, etc.

From this table, it will appear that *the reduction of a quantity from a lower denomination to a higher one, or from a higher denomination to a lower one, is effected by simply moving the decimal point to the* left *or to the* right *respectively for each denomination.*

Ascending reduction is effected by simply moving the decimal point to the *left*; *descending* reduction by moving the decimal point to the *right*.

The relative merits of the Metric and the English Systems will be best shown by working examples side by side.

METRIC SYSTEM.

Reduce 3762843 *millimetres to kilometres.*

3762843 mm. = 3·762843 Km., or 3 Km. 7 Hm. 6 Dm. 2 m. 8 dm. 4 cm. 3 mm. *Ans.*

[By moving the decimal point *one* place to the left we get *centimetres*, by moving it *two* places we get *decimetres*, and so on, until by moving it six places to the left we get *kilometres*.]

Work done.—A decimal point moved *six* places to the left.

ENGLISH LONG MEASURE.

Reduce 3762843 *inches to feet, yards, etc.*

12 | 3762843
3 | 313570 ft. 3 in.
5½ | 104523 yds. 1 ft. 3 in.
2 | 2
11 | 209046
40 | 19004 po. 1 yd. 1 ft. 3 in.
8 | 475 fur. 4 po. 1 yd. 1 ft. 3 in.
59 mi. 3 fur. 4 po. 1 yd. 1 ft. 3 in. *Ans.*

Work done.—5 divisions, 2 multiplications, involving at least 50 figures.

The superiority of the metric system is indisputable. Its great advantage rests on the fact that the multiples and sub-multiples are those of the decimal system, *i.e.* of the ordinary system of notation. In the metric system the "Compound Rules" virtually disappear, and become merely applications of the four simple rules.

Measure of Area, or Square Measure.

100 sq. millimetres (sq. mm. or $mm.^2$)	= 1 sq. centimetre (sq. cm. or $cm.^2$)
100 sq. centimetres	= 1 sq. decimetre (sq. dm. or $dm.^2$)
100 sq. decimetres	= 1 **sq. metre** (sq. m. or $m.^2$) or 1 centiare (ca.)
100 sq. metres	= 1 sq. decametre (sq. Dm.) or 1 **Are** (a.)
100 sq. decametres or 100 ares	= 1 sq. hectometre (sq. Hm.) or 1 **hectare** (Ha.)
100 sq. hectometres	= 1 **sq. kilometre** (sq. Km. or $Km.^2$)

Note (1) that the same relationship exists between long and square measure in the metric system as in the English system; the denominations proceed by *hundreds*, and not by *tens*, because $10^2 = 100$.

(2) that meadows, fields, etc., are measured with the *are*, of which the *hectare* and the *centiare* are the only derivatives in use.

(3) that the area of a country is expressed in square kilometres.

English equivalent.—$40\frac{1}{2}$ ares = 1 acre.

Measure of Volume, or Cubic Measure.

1000 c. millimetres = 1 **c. centimetre** (c.cm. or $cm.^3$)
1000 c. centimetres = 1 **c. decimetre** (c.dm. or $dm.^3$)
1000 c. decimetres = 1 **c. metre** (c.m. or $m.^3$)

A cubic metre of firewood is called a **Stere.**

Measure of Capacity, or Fluid Measure.

10 millilitres (ml.)	= 1 **centilitre** (cl.)
10 centilitres	= 1 **decilitre** (dl.)
10 decilitres	= 1 **litre** (l.)
10 litres	= 1 decalitre (Dl.)
10 decalitres	= 1 **hectolitre** (Hl.)

A litre is a cubic decimetre.

In order to thoroughly familiarize himself with the exact dimensions of a decimetre, a square decimetre, and a cubic decimetre or litre, the student is recommended to work the following simple exercise :—

(*a*) Cut a strip of paper a *decimetre* long. Compare its length with 4 inches (see p. 154).

(*b*) Cut out a *square* strip one decimetre in the side. We now have a *square decimetre.*

(*c*) Make a cardboard cube (open at the top), which is a decimetre in the side. The cubic contents of this box is a cubic decimetre, or a litre.

(*d*) Fill the box with sand.

(*e*) Pour the sand into a quart measure, and note that the pot is about $\frac{7}{8}$ full.

English Equivalent.—1 litre = $1\frac{7}{8}$ pints.

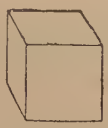
TABLE OF WEIGHT.

10 milligrammes (mg.)	= 1	centigramme (cg.)
10 centigrammes	= 1	decigramme (dg.)
10 decigrammes	= 1	**gramme** (g.)
10 grammes	= 1	decagramme (Dg.)
10 decagrammes	= 1	hectogramme (Hg.)
10 hectogrammes	= 1	**kilogramme** (Kg.)
10 kilogrammes	= 1	myriagramme (Mg.)
10 myriagrammes	= 1	**quintal** (Ql.)
10 quintals	= 1	tonne (T.)

Relation between Volume and Weight.

A cubic centimetre of distilled water at 4° C. and 760 mm. pressure weighs 1 gram.

The student should construct a small cube with internal dimensions of 1 c.cm.



1 cubic centimetre.

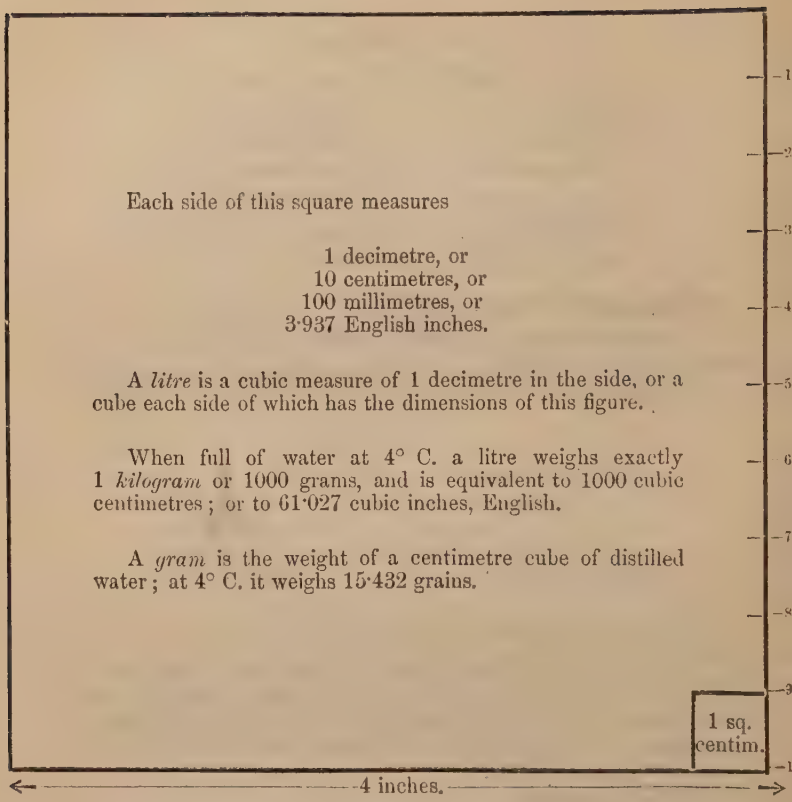
1-gram weight.

A thousand cubic centimetres (*i.e.* one cubic decimetre or litre) weigh a thousand grams, or one kilogram. A litre of water thus weighs a kilogram. In the English system no such simple relationship exists, but it may be remarked *en passant* that a gallon of water weighs about 10 lbs. The simple relationship existing between the measurement of length, volume, and weight is one of the characteristic advantages of the Metric System.

English equivalents.

1 gram	=	·0353 oz. avoir.
1 kilogram	=	$2\frac{1}{5}$ lbs.
1 cwt.	=	50·8 Kg.

The following diagram, taken from Miller's "Inorganic Chemistry," shows the connection which exists between the various measures in the metric system :—


Each side of this square measures
1 decimetre, or
10 centimetres, or
100 millimetres, or
3·937 English inches.
A *litre* is a cubic measure of 1 decimetre in the side, or a cube each side of which has the dimensions of this figure.
When full of water at 4° C. a litre weighs exactly 1 *kilogram* or 1000 grams, and is equivalent to 1000 cubic centimetres; or to 61·027 cubic inches, English.
A *gram* is the weight of a centimetre cube of distilled water; at 4° C. it weighs 15·432 grains.
1 sq. centim.
4 inches.

The entire square is the 100th part of a square metre, and contains 15·5 sq. inches, or ·1076 of a square foot nearly.

Angular Measure.

100 seconds (″) = minute (′)
100 minutes = 1 grade (g.)
100 grades = 1 right angle

The principal unit is the right angle, as in English angular measure.

A right angle contains 10,000 centesimal minutes. This fact is really the foundation of the metric system. If, on a great circle of the earth, the arc between two radii inclined to each other at an angle of 1′ be measured, the

length of the arc would be found to be a *kilometre*, so that a kilometre was designed to be the centesimal minute of latitude.

Examples.—*Reduce* 9 *kilometres to metres.*

Since 1000 metres = 1 kilometre, we merely add 3 noughts
$\therefore$ 9 Km. = 9000 m. *Ans.*

Express ·0015625 *gram in milligrams.*

Since 1000 milligrams = 1 gram, we move the decimal point 3 places to the *right*
·0015659 gram = 1·5659 mg. *Ans.*

Divide 57 *kilometres* 16 *metres by* 8.

57 Km. 16 m. = 57·016 Km.

8 | 57·016

7·127 Km. or 7 Km. 1 Hm. 2 Dm. 7 m. *Ans.*

But there is no need to write it in the second way, because the position relative to the decimal point shows the value of the digits.

The Advantages of the Metric System.

(1) A simple relationship exists between the units of length, area, volume, and weight.

(2) The multiples and sub-multiples of the units are in the ordinary scale of notation, *i.e.* the decimal system, consequently—

(*a*) Reduction involves practically no labour, for it consists merely of multiplying or dividing by a power of 10.

(*b*) English Compound Rules are avoided; all operations are performed by *simple* addition, subtraction, multiplication, or division of decimals.

(3) When once the meanings of the names of the different units and the multiples and sub-multiples are known, the "tables" of the Metric System impose no tax on the memory.

(4) It is becoming the universal system, and our commerce would be facilitated by its introduction.

The Reasons urged against the introduction of the Metric System are the practical difficulties attendant upon the substitution of the metric weights and measures for our present ones. All our pound weights, all our yard measures, all our pint measures would become practically useless, and great expense would be incurred in buying the new weights and measures. It has been estimated that the introduction of the Metric System would cost *millions* of money.

To change from the English to the Metric System.

(*a*) Express the compound quantity in the English system decimally in terms of some unit whose equivalence in the metric system is known.

(*b*) Multiply by that equivalence in the metric system.

Example.—*Find, correct to three places of decimals, the number of kilogrammes in 23 tons 7 cwt. 3 qrs. 14 lbs., taking one ton to contain 1016·0475 Kg.* (C.P.)

Reducing 23 tons 7 cwt. 3 qrs. 14 lbs. to the decimal of a ton (see p. 139), we have—

$$\begin{aligned} 23 \text{ tons} &= 23\cdot \\ 7 \text{ cwt.} &= \cdot 35 \\ 3 \text{ qrs.} = 3 \times \cdot 0125 &= \cdot 0375 \\ 14 \text{ lbs.} = 2 \times 3 \times \cdot 0010416 &= \cdot 00625 \\ & \hline \\ & 23\cdot 39375 \text{ tons} \end{aligned}$$

$$23\cdot 39375 \times 1016\cdot 0475 = 23769\cdot 161\ldots$$

Ans. 23769·161 . . . Kg.

To change from the Metric to the English System.

(*a*) Express the quantity in some metric unit whose equivalence to an English unit is known.

(*b*) Multiply by that equivalent.

(*c*) Reduce the product to a compound quantity in the metric system.

Example.—*Assuming that 39 in. = 99 cm., express 13 mi. in kilometres, metres, and centimetres.* (P.T.)

$$13 \text{ mi.} = 13 \times 1760 \times 3 \times 12 \text{ in.}$$

$$39 \text{ in.} = 99 \text{ cm.}$$

$$13 \text{ mi.} = \frac{99 \times 13 \times 1760 \times 3 \times 12}{39} \text{ cm.} = 2090880 \text{ cm.}$$

$$= 209088 \text{ dm.} = 20908\cdot 8 \text{ m.} = 2090\cdot 88 \text{ Dm.}$$

$$= 209\cdot 088 \text{ Hm.} = 20\cdot 9088 \text{ Km.}$$

Ans. 20 Km. 908 m. 80 cm.

THEORY.

1. What is the fundamental unit of the metric system?

2. Assuming that we know what a *metre* is, define a *centimetre* and a *kilometre.* (P.T.)

3. What are the units of weight and of capacity in the French system of measurement? (C.W.)

4. Compare as nearly as you can the following measures: The metre and the yard; the litre and the gallon; the kilogramme and the pound avoirdupois; the are and the acre. (S.A.)

5. What are the meanings of the prefixes deci-, kilo-? (S.A.)

6. Describe briefly (i.) the advantages, and (ii.) the disadvantages, of the decimal system of coinage and the metric system of weights and measures. (C.P.)

7. Distinguish between a *dekametre* and a *decimetre.* (C.M.)

8. A gramme being the weight of a cubic centimetre of water, and a

litre being a cubic decimetre, show that the weight of a litre of water is a kilogramme. (O.W.)

9. What practical difficulties impede the adoption of the decimal system of weights and measures in England? (C.W.)

10. If a system of decimal coinage, weights, and measures is at some future time established in Great Britain, what modifications must be made in the arithmetic taught in schools? For instance, what method would then be used to solve questions which are now worked by Practice? (K.S.)

11. What advantages are claimed for the metric system? (P.T.)

12. What is the connection between a *metre* and a *gramme*? (C.M.)

13. Name the countries where the metric system is in general use. (S.A.)

14. Explain how the units of weight, capacity, and surface have been obtained from the metre. (S.A.)

EXERCISE XXX.

The Metric System.

Money.

1. Find the sum of 40 fr. 80 c. + 119 fr. 6 c. + 88 fr. 5 c. + 53 fr. 4 c. + 606 c. + 207 fr. 28 c.

2. What is the sum of 53 fr. 4½ c. + 8·16 fr. − 23·40 fr. + 17·07½ fr. + 226·30 fr. − 36¼ fr. + 95·02 fr.?

3. What sum is required to pay the wages of 7 men and 3 boys, if each of two of the men receive 27·75 fr., each of the other 5 men 24·50 fr., and each of the 3 boys 5·60 fr.?

4. The gross amount of an invoice is the sum of 47 fr. 43 c. + 159 fr. 16 c. + 360 c. + 32¾ fr. + 18 fr. 8 c., and a discount is allowed, making the net amount 254½ fr. How much is the discount?

5. If 37 fr. 47 c. be divided between A and B so that B may have twice as much as A, what will A receive?

6. How many kilogrammes of coffee at 3 fr. 75 c. per kilogramme will amount to 53 fr. 55 c.?

7. Three bottles of champagne and five bottles of Bordeaux cost as much as five bottles of champagne and one of Bordeaux. If a bottle of champagne costs 11 fr. 30 c., what will be the cost of a bottle of Bordeaux? (P.T.)

8. Divide a franc among A, B, and C so as to give A 20 c. less than B and C together, and so as to give B 1·4 times as much as C. (P.T.)

Avoirdupois Weight.

9. Express 9 g. 8 cg. 8 mg. as a decimal of 4 Kg. 9 dgs. (C.P.)

10. Find the total amount of 3 Kg. of tea at 6·60 fr. per kilogramme, 2 Hg. of coffee at 33 c. per hectogramme, and 2½ Kg. of sugar at 56 c. per kilogramme.

11. Find (to the nearest pound) the cost of 5122·7 Kg. at 3·43 fr. per kilogramme, reckoning 25·42 fr. to the pound. (C.P.)

12. Find the value of 47 Kg. 8 Dg. 4 g. of an article the price of which is £23·57 per kilogramme. Give the answer in £ *s. d.* (O.S.)

13. Assuming that a gramme is ·0022 of a pound avoirdupois, find the value of a ton of metal at the price of 2*s.* 1*d.* per kilogramme. (C.H.L.)

14. How many pounds are there in ·708624 of a ton? And how many kilogrammes are there in it if 100 Kg. equal 1·9684 cwt.? (S.K.)

Troy Weight.

15. If 1 oz. troy is equivalent to 31·1 g., determine the French measure in weight of 25 tons 2 cwt. 27 lbs. (C.P.)

16. If gold is worth 3 fr. 10 c. per gramme, find its value per ounce troy in English money, taking a franc as equal to 9¾*d.*, and a gramme as 15·43 grs. (S.K.)

17. The weight of a gold 10-fr. piece is 49·78 grs.; the weight of a silver 5-fr. piece is 385·8 grs., of which the equivalent in pure silver is ·782 oz. troy. If the relative value of coined gold and silver be 19·29 : 1, find the intrinsic value of a gold 10-fr. piece, the intrinsic value of pure silver being 4*s.* 2*d.* per oz. troy. (C.S.)

Long Measure.

18. How many steps, each 65 cm. long, does a French soldier take in marching 273 Km.? Five German steps are equal to six French steps. How many steps will a German soldier take in marching the same distance? (C.P.)

19. How many kilometres are there in 153428 mm.? And how many decametres in 452 mym.? (C.P.)

20. How many decametres are there in 1000 dm.? And how many centimetres in 140 Hm.? How many kilometres may be measured from 393·708 mi.? (1 m. = 39·3708 in.) (C.P.)

21. The diameter of a 5-c. piece is exactly 25 mm. How many, placed in a straight line and touching one another, will extend to the distance of 2·25 m.? (C.P.)

22. A horseman thinks that he travels 15 Km. in an hour, but his watch registers 59¾ min. when in reality an hour elapses. How far does he really travel in an hour? Give your answer correct to a metre. (P.T.)

23. From a length of 3·7 m. how many lengths of 7·1 cm. can be cut off, and what will be the length of the remainder? (C.J.)

24. A rod, whose length is 25 m. 50 cm., is measured with a metre rule (divided into centimetres), and appears to be 22 m. 35 cm. long. Supposing that the divisions of the rule are all equal, find to the nearest millimetre the error in each. (C.S.)

25. The armour-plate of a vessel is described as being 45 cm. thick. What is this equivalent to in English inches? (1 m. = 39·37 in). (C.P.)

26. How many metres (1 yd. = ·9144 m.) are there in 4 mi. (K.S.)

27. Assuming that 39 in. = 99 cm., express 26 mi. in kilometres, metres, and centimetres. (P.T.)

28. A map is drawn on such a scale that a kilometre is represented by a centimetre. What length represents a mile on the same scale? Give your answer correct to three places of decimals of an inch. (P.T.)

29. Find the difference in yards between 5 mi. and 8 Km. (A kilometre = 39370·8 in.) (C.W.)

30. Express 787·4 mi. in kilometres and metres. (1 Km. = 1000 m. = 39370 in.) (C.W.)

31. Find the number of metres in a mile, given that a metre is 39·37 in.; also find the number of miles in ten million metres. (L.M.)

32. A kilometre being 1093·638 yards, find to four places of decimals how many kilometres there are in 100 English miles. (S.K.)

33. Find how many feet and how many metres there are in ·581257 of a mile, having given that 1 m. = 1·093638 yds. (S.K.)

34. Find the value of 585 cm. of cloth at 5*s.* 5*d.* per metre. (C.PR.)

35. Find to the nearest centimetre the length of silk at 7 fr. 25 c. per metre which can be bought for 381 fr. 50 c. (C.J.)

36. A metre being equal to 39·371 in., and a franc being equal to 9·38 pence, what is the value in English money of a yard of silk worth $7\frac{1}{2}$ fr. a metre? (S.K.)

37. A merchant buys 485 m. 70 cm. of silk, and pays for it 3764 fr. 20 c. Find the average cost per metre in francs and centimes (the answer to be correct to the nearest centime). (O.J.)

38. The fare for a certain railway journey is 18 fr. 70 c.; while for another journey, 87 Km. longer than the former, the fare is 24 fr. 20 c. Find the length of each journey, the rate per Km. being the same for both. (C.S.L.)

39. If one can travel 264 Km. by railway in France for 39 fr., find what should be the fare in English money for a distance of 77 mi. at the same rate, supposing that a kilometre is 1092 yds. and that 25 fr. are equal in value to £1. (O.S.L.)

40. Suppose a metre to be 40 in. and 25 fr. to be £1. Which is the higher price for the same cloth, 4 fr. a metre (of length) or 3*s.* a yard, and by how much in centimes per metre and in pence per yard? (P.T.)

41. Find, to the nearest farthing, the value of a bar of gold, $2\frac{1}{3}$ m. long, at £1 per inch. (1 metre = 39·37079 in.) (C.M.)

42. Having given that the circumference of the earth is 40 million metres, express it in miles (1 yard = ·91 of a metre). (C.H.L.)

43. A metre is 39·37 in. If this is a ten-millionth part of the distance of the equator from the pole, how far is it round the world, the measurement being taken along a meridian? (Give the result in miles, furlongs, yards, feet, and inches.) (P.T.)

44. The metre was originally determined so that forty million metres should be equal to the earth's circumference (a metre is 39·37 in.). Compute from these data the mean radius of the earth in miles, being given that the circumference of a circle is to its diameter as 355 to 113. (L.M.)

45. The circumference of the earth is 40000000 m., the length of a metre being 39·3709 in. Calculate the diameter of the earth in miles, assuming the ratio of the circumference to its diameter to be 355 : 113. (P.P.)

46. A metre is defined to be $\frac{1}{10000000}$ of the one-fourth part of the circumference of the earth, and is equal to 39·37079 in. Find the circumference of the earth in miles. (O.S.L.)

47. Assuming that the earth's polar circumference is 40000000 m., and that a metre is 39·3708 in., calculate the earth's polar circumference in miles, correct to the nearest unit. (L.M.)

48. A man buys 143·60 m. of ribbon in Paris at 65 c. per metre, and offers a £5 note in payment, which the shopman agrees to accept at the rate of 25·12 fr. to the £. How much change will the purchaser receive? (O.S.L.)

49. The railway fare from London to Dover (70 mi.) being 14*s.* 7*d.*,

and that from Paris to Boulogne (240 Km.) being 36·16 fr., compare the cost of travelling in England and France, the French metre being equal to 3 ft. 3·37068 in., and £1 being worth 25·44 fr. (C.S.)

50. Find a decimal fraction, correct to the first three places, such that, if the price of a material in francs per metre be multiplied by it, the product will be the price of the same material in shillings per yard, having given that 1 pound = 25 fr. and 1 m. = 39·37 in. (L.M.)

51. The height of a certain mountain is marked on a map in metres, and in order to form a rough estimate of its height in feet I multiply by 10 and then divide by 3. I afterwards find that the height so determined is too great by 126 ft. What is the height of the mountain in feet, assuming that a metre is equal to 39·37 in.? (C.H.L.)

52. Find to the nearest franc the cost of making a railway 27345 m. long, the cost per kilometre being 303625 fr. (O. and C.L.C.)

53. How much per kilogram is a centime per milligram? If it be true that to convert metres into feet we affix 0, divide by 3, and subtract $\frac{1}{40}$ of the result, how long is a metre? (S.A.)

Area.

54. What is the area in square metres of the walls, floor, and ceiling of a room which is 11 m. long, 8·25 m. wide, and 5·75 m. high?

55. What is the charge for painting the walls of a room 14·25 m. long, 8·75 m. wide, and 11 m. high, at 3·25 fr. the square metre?

56. What is the cost of papering the walls of a room 11·75 m. long, 9·25 m. wide, and 5·64 m. high, at 25 c. the square metre?

57. Write down (i.) $7\frac{1}{4}$ eg. as a decimal of a decagramme; (ii.) 22 sq. dm. $5\frac{1}{2}$ sq. cm. as a decimal of a square metre. (C.P.)

58. Express (i.) 175 sq. cm. as a decimal of a square metre; and (ii.) 23 c.cm. as a decimal of a cubic decametre. (C.P.)

59. Express the area of a rectangle 5 m. 7 cm. long and 4 m. 9 dm. wide, as a decimal of a square kilometre. (C.P.)

60. If a French are is a square with a base of 10 m., and a metre is taken as equal to 39·36 English inches, show that there are $40\frac{1}{2}$ French ares in an English acre, within a small fraction of an are, or within a very few square feet. (C.P.)

61. What is the content in square metres of a plot of ground 485 dm. square? Determine the length in decametres of each side of a square plot containing 34741·2321 sq. dm. (C.P.)

62. A metre is 39·3708 in. Find to three places of decimals the number of acres in a hectare, which is 10000 sq. m. (C.M.)

63. The annual rent in France of a hectare of land is 130 fr. Find the rent of one acre in English money, it being given that 25 fr. are equal to £1, and 100 Ha. equal to 247 ac. (C.P.)

64. Suppose that a metre is really 39·4 in. Then what percentage of 1600 sq. in. must be deducted in order to arrive at the value of a square metre in inches? Give (1) the true percentage, (2) the percentage to two places of decimals. (P.T.)

65. Three brothers sell a farm of 40 Ha. for 3275 fr. the hectare. What should each receive? How much is it in English money at 25 fr. for the sovereign? (C.M.)

66. A plot of land is valued at £1936 per acre. What is its worth in francs per square metre? (£1 = 25¼ fr.; a metre = 39⅜ in.) (C.P.)

67. If tar-paving costs 2*s.* 6*d.* per square yard, what amount in francs

will pay for a pavement 1 Km. long by 15 m. broad? (A linear foot being taken as = 3·05 dm. and a franc = 10*d*.) (C.P.)

68. Having given that a metre is 39·3708 in., express a square foot in terms of a square metre, the approximation being correct to within one-thousandth of the whole. (C.H.L.)

69. If a metric system of area were adopted, so that 2 ac. 2 rds. 6 per. would be represented by 5·12, express the unit of measurement in square yards and decimal parts of a square yard. (C.W.)

70. Find the cost of papering a room 12 m. long, 6·2 m. broad, and 4·25 m. high, with paper of which the breadth is 70 cm. and the price is 2 fr. per piece of 8 m., 21 sq. m. being allowed for windows and doors. (C.S.L.)

71. Assuming that a square metre is 1·196 of a square yard, find the number of acres in a square kilometre, neglecting fractions of an acre. (O. and C.H.C.)

72. Find to the nearest centime the cost of fencing a square field, of which the area is 7000 sq. m., at a cost of 3 fr. 25 c. per metre. (C.S.L.)

73. Find to the second place of decimals the number of square yards in 7289 sq. m. Find to the nearest penny the value of 900 Kg. of a material which costs £25 14*s*. 6*d*. a ton. N.B.—1 m. = 3·28092 ft.; 1 Kg. = 2·2046 lbs. (S.K.)

74. A rectangular tract of land, 1½ Km. long and 5 Dm. broad, is to be planted with shrubs at all intersections of straight lines, which are themselves 2 m. apart and parallel to the sides of the rectangle respectively, its four corners and its sides being similarly occupied. How many shrubs will be required? (C.P.)

75. Determine how many hectares there are in 750 ac. of land. (Are = square decametre; metre = 39·38 in.) (C.P.)

76. An acre is ·40467 Ha., and a pound sterling is 25·25 fr. An estate measuring 1927 Ha. is sold for 10100000 fr. What is the selling price per acre in £ *s*. *d*.? (S.K.)

77. Two plots of ground lie four square. Find (i.) the area of the first in hectares and decimals of hectares, if each of its sides measures 523 m. 7 dm. 5 cm.; (ii.) the length in yards of each side of the second, if its area be 10621081 sq. rds. or po. (C.P.)

78. Given that a metre is 3·3708 in. longer than a yard, find which is greater, 10 sq. m. or 12 sq. yds.; and express the difference between these areas as a decimal of a square metre correct to three places. (S.K.)

79. A centimetre being ·3937 in., find to four significant figures how many acres are in a square kilometre. (L.M.)

80. The length of a rectangular field is to its breadth as 3 to 2, and its area is 11094 sq. m. Find the cost of surrounding it with a fence at 2·25 fr. per metre length of boundary. (S.K.)

Cubic.

81. Express fully the different denominations involved in 255·09230471 c. dm. (*i.e.* write down how many cubic metres, etc., are involved in the decimal). (C.P.)

82. If 1 gramme = 15 grains, and a cubic foot of water weighs 1000 oz. avoirdupois, what will be its weight in grammes? (1 lb. avoirdupois = 7000 grains.) (P.T.)

83. Find the weight in kilograms of the volume of water contained in a cistern 4·35 m. long, 3·64 m. wide, and 2·85 m. deep in kilograms, supposing the water to be at 4° Centigrade.

84. How many cubic decimetres of air are contained in a room which measures 8·25 m. long, 5·64 m. wide, and 3·65 m. high?

85. How many cubic metres of water pass under a bridge in a minute where the river is 26·25 m. wide, 3·44 m. deep, and is running at 3·2 Km. per hour?

86. If a cubic foot of water weighs 1000 oz., express a kilogramme in pounds correct to one place of decimals. (Assume a metre = $3\frac{1}{3}$ ft.) (C.M.)

87. Assuming a metre to be $39\frac{3}{8}$ in., find the nearest whole number of litres (cubic decimetres) in 1 c. ft. (P.E.A.)

88. If a metre be taken as 40 in., and a cubic centimetre of water weighs one gramme, what, stated metrically, is the weight of a cubic inch of water? (P.T.)

89. If a cubic decimetre of ice weighs 918 g., what is the weight, to the nearest kilogramme, of a rectangular sheet of ice 53 m. long, 27 m. wide, and 39 mm. thick? (C.P.)

90. A metre is the 10000000 part of a quadrant of the meridian, and a minute of latitude contains 6080 ft. A cubic foot of water weighs 1000 oz., and a cubic centimetre of water weighs one gramme. Express a kilogramme in ounces to one place of decimals. (K.S.)

CHAPTER XVI.

PRACTICE, INVOICES, AND BILLS.

Practice is the method of finding by means of aliquot parts the cost of any given quantity when the cost of a unit of any denomination is given.

An aliquot part (or sub-multiple) of a quantity is a part which is contained in such quantity an exact number of times. See also p. 29.

As its name signifies, Practice is a convenient method of calculation frequently adopted in *practice* by business men.

Practice is of two kinds—

Simple Practice, in which the given quantity the cost of which is to be found is *simple*: *e.g.* Find by practice the cost of 278 articles @ £14 18*s.* 10*d.* each.

All examples in simple practice may be worked by compound multiplication, and many examples may be worked more expeditiously by decimalization of money (p. 137).

Compound Practice, in which the given quantity the cost of which is to be found is *compound*: *e.g.* Find by practice the cost of 4 tons 7 cwt. 1 qr. @ £2 7*s.* 6*d.* per ton.

General Method of Procedure in Simple Practice.—Separate the *given value* into portions such that the first is some unit or multiple of the unit, and each of the others is an aliquot part of this unit or of some preceding portion.

Skill in working practice consists largely in ability to separate the parts of the given value which are less than unity into aliquot parts.

The following *Aliquot Parts* should be especially noticed :—

Of a Pound.		*Of a Shilling.*
6*s.* 8*d.* is $\frac{1}{3}$	1*s.* 8*d.* is $\frac{1}{12}$	1½*d.* is $\frac{1}{8}$
3*s.* 4*d.* is $\frac{1}{6}$	1*s.* 4*d.* is $\frac{1}{15}$	
2*s.* 6*d.* is $\frac{1}{8}$	1*s.* 3*d.* is $\frac{1}{16}$	

Example.—*Find the cost of* 923 articles @ £2 15*s.* 10¼*d. each.*

	£	*s.*	*d.*			£	*s.*	*d.*
10*s.* = ½ of £	½ of 923	0	0					
			2					
	1846	0	0	cost	@	2	0	0
5*s.* = ½ of 10*s.*	½ of 461	10	0	„	@	0	10	0
6*d.* = 1/10 of 5*s.*	1/10 of 230	15	0	„	@	0	5	0
3*d.* = ½ of 6*d.*	½ of 23	1	6	„	@	0	0	6
1*d.* = ⅓ of 3*d.*	⅓ of 11	10	9	„	@	0	0	3
¼*d.* = ¼ of 1*d.*	¼ of 3	16	11	„	@	0	0	1
		19	2¾	„	@	0	0	0¼
Ans.	£2577	13	4¾	„	@	£2	15	10¼

Example.—*Find the cost of* 426 *articles* @ £2 18*s.* 4*d. each.*

Since £2 18*s.* 4*d.* is 1*s.* 8*d.* less than £3, the shortest method is to find the cost at £3 each and subtract the cost at 1*s.* 8*d.* each.

	£	*s.*	*d.*			£	*s.*	*d.*	
1*s.* 8*d.* = 1/12 of £1	426	0	0 =	cost	@	1	0	0	each
			3						
	1278	0	0 =	„	@	3	0	0	„
	35	10	0 =	„	@	0	1	8	„
Ans.	£1242	10	0 =	„	@	£2	18	4	„

The above method is sometimes called Practice by the Method of Subtraction.

In **Compound Practice** the *compound quantity* is treated in the same way as the given value is treated in Simple Practice.

Example.—*Find the value of* 10 *cwt.* 3 *qrs.* 16 *lbs.* @ £12 15*s.* 6½*d. per cwt.*

	£	*s.*	*d.*		cwt.	qrs.	lbs.
2 qrs. = ½ cwt.	½ of 12	15	6½	value of	1	0	0
			10				
	127	15	5	„	10	0	0
1 qr. = ½ of 2 qrs.	½ of 6	7	9¼	„	0	2	0
14 lbs. = 1/7 of 1 qr.	1/7 of 3	3	10⅝	„	0	1	0
2 lbs. = 1/7 of 14 lbs.	1/7 of 1	11	11 5/16	„	0	0	14
		4	6 5/112	„	0	0	2
Ans.	£139	3	6 53/56	„	10	3	16

An Invoice, or **a Bill,** is a written statement made out by the seller and delivered to the buyer, in which the latter is

informed of the *quantity*, *kind*, and *cost* of the goods supplied to him.

A *bill* is handed to the customer by the seller with the goods. An *invoice* is a statement despatched when the goods are sent to the customer.

Each separate amount in a bill or in an invoice is called an *Item*.

Item (*Latin* = likewise, also) was once written at the commencement of the second and following lines of a bill; hence each line in a bill is now called an item.

SPECIMEN OF AN INVOICE OR BILL.

London,
June 18, 1903.

Messrs. Taylor & Stevens.
Bought of FARREN & Co.,
3, Ludgate Hill, E.C.

	£	*s.*	*d.*
8 yards silk @ 5*s.* 6*d.* per yard	2	4	0
12 yards calico @ 6½*d.* ,, ...		6	6
3 yards velvet @ 7*s.* 6*d.* ,, ...	1	2	6
10 yards muslin @ 1*s.* 3*d.* ,, ...		12	6
	£4	5	6

If the goods are paid for at the same time, the invoice is receipted, *i.e.* the seller or his representative writes, "Received" or "Paid" on the invoice, and adds his signature and the date of the receipt. If the amount is £2 or over, a penny stamp must be affixed, and this stamp should be cancelled by writing either the name or date upon it.

If, however, credit is given (*i.e.* if the buyer does not pay "ready" money), a statement is sent, usually at the end of each quarter, showing the dates and amounts of the invoices unpaid. This statement is called an Account.

SPECIMEN OF AN ACCOUNT.

London,
June, 1903.

Messrs. Taylor & Stevens.
Dr. to FARREN & Co.,
3, Ludgate Hill, E.C.

		£	*s.*	*d.*
April 16	To goods as per Invoice	3	7	6
May 7	,, ,, ,, ...	2	17	4
June 18	,, ,, ,, ...	4	5	6
		10	10	4

A Detailed Account is one in which each invoice or bill is reproduced in full.

THEORY.

1. What is meant by "aliquot parts"? Exemplify the use of them in finding the price of 1856⅞ articles at 1½ guins., and at £3 3*s.* 10*d.* each. (C.W.)

2. Explain the principle upon which sums are worked in practice, and find by that rule the amount which is lost by the creditors of a bankrupt who can pay only 12*s.* 10½*d.* in the £ on liabilities amounting to £3548 6*s.* 8*d.* (C.W.)

3. Distinguish between simple and compound practice.

4. Distinguish between an *invoice* and a *bill*, an *account* and a *detailed account*.

5. What is meant by aliquot parts? Explain their use in abbreviating calculations of price. (C.W.)

EXERCISE XXXI.

Simple Practice.

1. Find by practice the cost of 2592 articles at 16*s.* 10½*d.* each. (C.W.)

2. Show that not more than four aliquot parts are required for finding by practice the value of 3178 articles at £1 13*s.* 9¾*d.* each. (K.S.)

3. Find by practice the value of 420 articles at £3 6*s.* 7½*d.* each. Using the above result, find the value of 424 articles at the same price. (P.T.)

4. What sum was paid for 459021 oxen at £15 18*s.* 6*d.* each? (C.P.)

5. Find by practice the value of 21840 articles at 18*s.* 10½*d.* each. (O.J.)

6. What will be the cost of laying a cable across the Atlantic if 2939 mi. of cable are required, and the price, including the expense of laying, is £148 13*s.* 6*d.* per mile? (C.H.L.)

7. Determine by two different methods the cost of 14732 articles at £1 17*s.* 2½*d.* each. (C.P.)

8. Find the cost of 1225 articles at 16*s.* 8*d.* a score. (C.P.)

9. How much will 56375 articles cost at £2 15*s.* 9*d.* per hundred? (C.P.)

10. Find what must be paid to 721 labourers for 7 weeks' work at 17*s.* 4½*d.* per week each. (P.T.)

11. Obtain to the nearest penny the cost of 258700 cub. ft. of gas at 2*s.* 4½*d.* per thousand cubic feet. (C.P.)

12. Find by practice the rent of 2543 ac. at an average value of £1 14*s.* 1½*d.* per acre. (P.T.)

13. Find the cost of 4013½ cub. yds. of timber at £2 16*s.* 6½*d.* per cubic yard. (C.P.)

14. Determine the value of seven million one hundred and twenty-eight trinkets at 1*s.* 4½*d.* the twelve dozens. (C.P.)

15. Find by practice the dividend on £3245 15*s.* at 13*s.* 9*d.* in the £. (R.M.A.)

EXERCISE XXXII. 32

COMPOUND PRACTICE.

Avoirdupois.

1. Find the value of 3 tons 3 cwt. 3 qrs. 14 lbs. of coal at £1 3*s.* 4*d.* per ton. (O.S.L.)

2. Find the cost of 3 tons 11 cwt. 3 qrs. 25 lbs. of tea at £11 6*s.* 4*d.* per cwt. (C.P.)

3. Find the value of 7 tons 3 cwt. 2 qrs. 21 lbs. of wire at £8 6*s.* 8*d.* per ton. (C.W.)

4. Find the cost of 3 qrs. 8 lbs. 10 oz. at £1 8*s.* per cwt. (I.I.)

5. Find the cost of 12 tons 2 cwt. 3 qrs. of coal at 16*s.* 9*d.* per ton, with 2*d.* a cwt. additional for cartage. (C.P.)

6. Find to the nearest penny the value of 13 tons 17 cwt. 3 qrs. 17 lbs. at £9 13*s.* 9*d.* per ton. (S.A.)

Troy.

7. Find the cost of a nugget of gold weighing 3 lbs. 11 oz. 8 dwts. 4 grs. at £3 17*s.* 6*d.* per ounce. (C.PR.)

8. Find to the nearest penny the value of a mass of pure gold whose weight is 10 oz. 13 dwts. 14 grs. at £4 4*s.* 11½*d.* per ounce. (C.S.L.)

9. Find the value of 6 lbs. 9 oz. 12 dwts. 16 grs. of silver at 3*s.* 9*d.* per ounce. (O.S.L.)

10. Find the value of 361 oz. 10 dwts. of gold at £3 17*s.* 10½*d.* per ounce. (O.S.L.)

11. Find the value of a silver tankard weighing 1 lb. 10 oz. 17½ dwts. at 5*s.* 6*d.* an ounce. (P.E.A.)

Length, Area, and Volume.

12. Find the cost of 2 fur. 7 po. fencing at 1½ yd. of £29 6*s.* 8*d.* per mile. (C.S.L.)

13. Find the length of an imaginary straight line made up of 745½ straight lines, each 3 mi. 5 fur. 37 po. 5 yds. long. (C.P.)

14. Find the rent of 24 ac. 3 rds. 16 per. at 2½ guins. per acre. (K.S.)

15. Find the value of 449 ac. 3 ro. 15 po. of land at £3 12*s.* 8*d.* an acre. (O. and C.H.C.)

16. Find the value of 13 ac. 3 rds. 36 per. at £18 6*s.* 9*d.* per acre. (S.A.)

17. Find the value of 63 ac. 3 rds. 24 po. 15 yds. at £393 5*s.* per acre. (C.W.)

18. Find the value of a crop on 10 ac. 2 rds. 27 po. of land at £12 16*s.* 8*d.* per acre. (C.P.)

19. Find the rent of 643 ac. 1 rd. 27 po. of land at £1 16*s.* 8*d.* per acre. (C.PR.)

20. Obtain the value of 240 c. yds. 7 c. ft. 144 c. in. at £4 15*s.* 6*d.* the cubic yard (to the nearest penny). (C.P.)

EXERCISE XXXIII.

Bills.

1. Make out a bill for 175 needles at 8*s.* 4*d.* per 100; 8 doz. buttons at 1*s.* 1½*d.* the gross; ⅜ yd. of velvet at 4*s.* 2*d.* the yard; 3½ oz. thread at 4*s.* 8*d.* the pound. (P.T.)

2. Make out the following account, showing all the working: 210 eggs at 9 for a shilling; 39½ lbs. of butter at 1*s.* 3*d.* per pound; 21½ lbs. of tea at 1*s.* 9*d.* per pound; 132 buns at 22 for a shilling. (C.W.)

3. Make out a bill for the following articles: 17½ yds. of silk at 11*s.* 7½*d.* a yard; 648 buttons at 1*s.* 9½*d.* a gross; 15½ yds. of velvet at 1*s.* 11½*d.* a yard; and 153 reels of cotton at 11*d.* a dozen. (O.S.L.)

4. Make out and receipt the following bill: 9½ yds. of flannel at 1*s.* 5½*d.* a yard; 26 yds. of calico at 2*s.* 1¾*d.* a yard; 23 yds. of muslin at 3*s.* 7*d.* a yard; 18 yds. of linen at 2*s.* 9¼*d.* a yard; 15 yds. of ribbon at 11½*d.* a yard; half a dozen pairs of gloves at 2*s.* 3½*d.* per pair. (C.W.)

5. Make out a bill for the following: 2½ doz. hats at 3*s.* 6*d.* each; 6¾ doz. collars at 5*s.* 6*d.* per dozen; 5¼ doz. neckties at 1*s.* 1*d.* each; 4 pairs of gloves at 2*s.* 5¾*d.* per pair; 16 pairs of socks at 13*s.* per dozen. (P.T.)

6. Make out a bill for 7¾ lbs. of butter at 1*s.* 7*d.* per pound; 9¼ lbs. of tea at 2*s.* 4*d.* per pound; 13½ lbs. of sugar at 2½*d.* per pound; 7 lbs. of cheese at 1*s.* 1½*d.* per pound; 4½ lbs. of raisins at 5½*d.* per pound; and 19 jars of marmalade at 14 jars for 1 guinea. (K.S.)

7. If I take £6 10*s.* to the shop, and buy and pay for the following goods, what amount do I bring away?—10 reams of note-paper at 9*d.* per 5 quires; 1 ream of foolscap at 9*d.* a quire (20 quires = 1 ream); 2½ reams of blotting-paper at 10*d.* a quire; 5 packages of envelopes, containing 250 each, at 2½*d.* per packet of 25; 3 boxes of pens at 1*s.* 4*d.* per box; ¾ gross of penholders at 1*s.* 1½*d.* per dozen; ½ gross of pencils at 7½*d.* per dozen; 200 visiting cards, with plate, for 4*s.* 6*d.* (C.P.)

8. A cheque for £3 6*s.* is tendered in payment of a grocer's bill for 9 lbs. of tea at 2*s.* 11*d.* per pound; 3½ st. of sugar at 2¾*d.* per pound; 5 lbs. of coffee at 1*s.* 6½*d.* per pound; 6 qts. of vinegar at 4½*d.* per pint; and 2 lbs. of gelatine at 3½*d.* per ounce. How much honey at 9½*d.* per pound can be bought with the change? (C.J.)

9. Find the total cost of the following to the nearest penny: ⅓ lb. of tea at 2*s.* 0½*d.* a pound; 3½ lbs. of sugar at 1*s.* for 5 lbs. Why would you get a different result if you found each sum separately to the nearest penny and added the two together? (P.T.)

10. Make out this bill: 119¾ lbs. of beef at 1*s.* 2*d.* a pound; 12½ doz. skewers at 1*s.* 3*d.* a dozen; 1001 plates at 2½*d.* each; 1210 oysters at 2*s.* 3*d.* a score; 1 cwt. of butter at 11½*d.* a pound. How can you tell, without working out completely the above items, that the following results for the respective items are wrong?—(1) £5 19*s.* 10½*d.*; (2) 15*s.* 9¾*d.*; (3) £10 18*s.* 6*d.*; (4) £6 17*s.* 2*d.*; (5) £5. (C.M.)

CHAPTER XVII.

RATIO.

Ratio is the relation existing between two quantities or numbers of the same kind with respect to the number of times the one is contained in the other.

1. If one quantity, A, contains 12 equal parts, and another, B, 4 of those parts, the ratio of A to B is said to be as 12 to 4. In fact, ratio is a comparison as to how many times the one contains the other, and hence it may also be defined as *the fraction that one number is of another*; *e.g.* $\frac{12}{4}$. Thus, 12 : 4 is $\frac{12}{4}$ or 3, meaning that 12 is 3 times 4.

2. The quantities must be of *the same kind*. There is no ratio between 4 apples and 12 guns. Ratio has no definite meaning unless the compared quantities are of the same kind.

As a result of this limitation, the terms of a ratio are usually considered to be abstract numbers, for it is possible to pass from *quantities* of the same kind to the *numbers* which measure them, and conversely when necessary.

3. If necessary, the quantities must also be brought to the *same denomination*. To express the ratio between £5 6*s.* 7*d.* and £7 6*s.* 5*d.*, both quantities must be reduced to pence.

4. The number of times the first number contains the second is called the *measure* or value of the ratio; *e.g.* $\frac{12}{4} = 3$, and 3 is said to be the *measure* of the ratio between 12 and 4.

5. Although ratio is the relative magnitude of two quantities or numbers, ratio itself, just like a fraction, is a *number*, and not a quantity.

6A. *Methods of Expression.*—£2 is $\frac{2}{3}$ of £3, or £3 is $\frac{3}{2}$ of £2. The ratio between these two sums of money may be expressed (*a*) as 2 is to 3, (*b*) as 2 : 3, or (*c*) as $\frac{2}{3}$. The sign (:) may be viewed as the division sign (÷) with the line omitted, and thus marks the relation between ratio and fractions.

6B. *Ratio and Fractions.*—For practical purposes, these may be considered as two views of the same thing. But there are differences.

1. A *fraction* implies *partition*, whereas a *ratio* does not necessarily imply division, but *comparison*.

2. A *fraction* is *numerical*, whilst a *ratio* may exist between two magnitudes which are *not numerical*.

7. In a ratio the first number is called the *antecedent* (that which

goes before), and the second the *consequent* (that which goes after); *e.g.* in the ratio 12 : 4, 12 is the antecedent and 4 is the consequent.

Kinds of Ratio.—Every ratio must contain two terms, viz. the Antecedent and the Consequent. The various relations existing between these give rise to several forms or kinds of ratio.

1. *Ratio of Greater Inequality.*—The antecedent is *greater* than the consequent; *e.g.* 7 : 5. Such a ratio is said to be in descending order of magnitude. The fraction expressing such a ratio is always *improper.*

2. *Ratio of Less Inequality.*—The antecedent is *less* than the consequent; *e.g.* 5 : 7. Such a ratio is said to be in *ascending* order of magnitude, and the fraction equivalent to the ratio is always *proper.*

3. *Ratio of Equality.*—The antecedent is *equal* to the consequent; *e.g.* 6 : 6. The measure of the ratio is always unity, for $\frac{6}{6} = 1$.

4. *Duplicate Ratio.*—The ratio of the *squares* of two numbers is called the duplicate ratio of the numbers.

e.g. Simple ratio, 7 : 5
Duplicate ratio, $7^2 : 5^2$ or 49 : 25

5. *Triplicate Ratio.*—The ratio of the *cubes* of two numbers is called the triplicate ratio of the numbers; *e.g.*—

Simple ratio, 2 : 3
Triplicate ratio, $2^3 : 3^3$ or 8 : 27

6. *Subduplicate Ratio.*—The ratio of the *square roots* of any two numbers; *e.g.*—

Simple ratio, 16 : 9
Subduplicate ratio, 4 : 3

7. *Subtriplicate Ratio.*—The ratio of the *cube roots* of any two numbers; *e.g.*—

Simple ratio, 27 : 64
Subtriplicate ratio, 3 : 4

8. *Compound Ratio.*—The antecedents, or the consequents of two or more ratios are called homologous terms, and compound ratio is formed by the multiplication of these homologous terms; *e.g.* consider the ratios—

$$7 : 3;\ 5 : 2;\ 11 : 4$$

Then the compound ratio becomes $(7 \times 5 \times 11) : (3 \times 2 \times 4)$, which may be expressed thus, $\frac{7}{3} \times \frac{5}{2} \times \frac{11}{4}$.

$$\frac{\text{Product of the antecedents}}{\text{Product of the consequents}} = \frac{7 \times 5 \times 11}{3 \times 2 \times 4}$$

It will be noticed that the given ratios may be expressed as fractions, hence *the compounding of ratios is equivalent to the multiplication of fractions.*

9. *Inverse Ratio.*—If the antecedent and consequent of one ratio become respectively the consequent and antecedent of another, the second is called the inverse ratio of the first; *e.g.*—

First ratio, 9 : 11
Inverse ratio, 11 : 9

Some Properties of Ratio.

1. The measure of a ratio is always an abstract number.

A ratio may be formed by the division—

(*a*) Of one abstract number by another; *e.g.*—

Abstract numbers $\frac{24}{6} = 4$ (abstract)

(*b*) Of one concrete number by another of the same kind; *e.g.*—

Concrete numbers $\frac{£24}{£6} = 4$ (abstract)

2. Since a ratio may be expressed as a fraction, the truths of fractions are also the truths of ratio.

(*a*) If *both terms* of a ratio be multiplied or divided by the same number, the ratio is unaltered; *e.g.*—

2 : 3 is the same ratio as $5 \times 2 : 5 \times 3$

for $\frac{2}{3} = \frac{5 \times 2}{5 \times 3}$

again, $\frac{2}{3} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{5} \times \frac{5}{3}$

(*b*) If the *antecedent* of a ratio be increased or diminished, the ratio is increased or diminished.

I. *Antecedent increased.*

(*a*) A ratio of greater inequality is diminished: 11 : 7 or $\frac{11}{7}$ is diminished by adding 4 to both terms—

for $\frac{11}{7} = \frac{11}{7} \times \frac{11}{11} = \frac{121}{77}$

and $\frac{11+4}{7+4}$ or $\frac{15}{11} = \frac{15}{11} \times \frac{7}{7} = \frac{105}{77}$

and $105 < 121$

The denominators must be kept alike for the purpose of comparison.

The second ratio is the smaller.

(*b*) A ratio of less inequality is increased; *e.g.* 3 : 4 or $\frac{3}{4}$ is increased by adding 4 to each term—

for $\frac{3}{4} = \frac{3}{4} \times \frac{8}{8} = \frac{24}{32}$

and $\frac{3+4}{4+4}$ or $\frac{7}{8} = \frac{7}{8} \times \frac{4}{4} = \frac{28}{32}$

The second ratio is the larger.

II. *Antecedent diminished.*

(*a*) A ratio of greater inequality is increased; *e.g.* 12 : 6 or $\frac{12}{6}$ is increased by subtracting 4 from each term—

for $\frac{12}{6} = \frac{12}{6} \times \frac{2}{2} = \frac{24}{12}$

and $\frac{12-4}{6-4}$ or $\frac{8}{2} = \frac{8}{2} \times \frac{6}{6} = \frac{48}{12}$

The second ratio is the larger.

or measure of $\frac{12}{6} = 2$

,, ,, $\frac{8}{2} = 4$

The second measure is the greater.

(*b*) A ratio of less inequality is diminished; *e.g.* 3 : 4 or $\frac{3}{4}$ is diminished by subtracting 1 from each term—

$$\text{for } \tfrac{3}{4} = \tfrac{3}{4} \times \tfrac{3}{3} = \tfrac{9}{12}$$
$$\text{and } \frac{3-1}{4-1} \text{ or } \tfrac{2}{3} = \tfrac{2}{3} \times \tfrac{4}{4} = \tfrac{8}{12}$$

The second ratio is the smaller.

III. *Consequent increased.*

If the *consequent* of a ratio be increased, the ratio is diminished. This is obvious, for an increased consequent means an increased denominator, and consequently a smaller quotient; *e.g.*—

$$3 : 4 = \tfrac{3}{4}$$
and $3 : 6 = \tfrac{3}{6}$, or $\tfrac{1}{2}$, which is less than $\tfrac{3}{4}$

IV. *Consequent diminished.*

If the *consequent* be diminished, the ratio is increased, for a diminished consequent means a diminished denominator, and therefore a larger measure or quotient; *e.g.*—

$$3 : 4 = \tfrac{3}{4}$$
$$3 : 2 = \tfrac{3}{2} = \tfrac{6}{4}, \text{ and } \tfrac{6}{4} > \tfrac{3}{4}$$

V. $$\text{Since } \frac{\text{antecedent}}{\text{consequent}} = \text{ratio}$$
$$\therefore \text{antecedent} = \text{consequent} \times \text{ratio}$$
$$\text{and consequent} = \text{antecedent} \div \text{ratio}$$

e.g. consider the ratio $\frac{3}{4}$

$$\text{then ratio} = \frac{3 \text{ (antecedent)}}{4 \text{ (consequent)}}$$
$$\text{and } 3 = 4 \times \text{ratio}$$
$$\text{and } 4 = 3 \div \text{ratio}$$

To find the Ratio of Two Quantities.

Example.—*Express as a decimal the ratio of* 4*s.* 6*d.* *to* 12*s.*

Reduce to a common base (sixpences); find what fraction the first quantity is of the second, and reduce it to its lowest terms.

Now, 4*s.* 6*d.* = 9 sixpences; and 12*s.* = 24 sixpences,
$\therefore$ ratio is 9 : 24
and $9 : 24 = \frac{9}{24} = \frac{3}{8} = .375$ *Ans.*

Note.—Since a vulgar fraction may be expressed as a decimal, a ratio may also be expressed as a decimal.

To divide a Given Number in a Given Ratio.

Example.—*Divide* 144 *into two parts in the ratio of* 4 : 5.

1. We may consider the given number (144) made up of (4 + 5) groups.
Hence number of groups = 144 ÷ 9 = 16
And each group contains (4 + 5) shares
Hence the number of shares will be 4 × 16, or 64, and 5 × 16, or 80
Ans. 64 and 80.

Given the Antecedent and Measure, to find the Ratio.

Example.—*Find a ratio equal to* $\frac{3}{4}$ *whose antecedent is* 7.

This simply means, find a fraction whose numerator is 7, equal to $\frac{3}{4}$.

$$\frac{3}{4} \times \frac{7}{7} = \frac{21}{28} = \frac{21 \div 3}{28 \div 3} = \frac{7}{9\frac{1}{3}}$$

Then the required ratio is $\frac{7}{9\frac{1}{3}}$ or $7 : 9\frac{1}{3}$ *Ans.*

Given the Difference of the Two Terms and the Measure, to find the Ratio.

Example.—*Find a ratio whose terms differ by* 30, *and the measure of which is* $\frac{2}{5}$.

In the ratio as 3 : 5 the difference is 2
" " $(3 \times 2) : (5 \times 2)$ the difference is 4
" " $(3 \times 3) : (5 \times 3)$ " " 6

i.e. the difference is twice as great as the multiplier of the ratio. Then to obtain a difference of 30, we must multiply by 15.

Then as $(3 \times 15) : (5 \times 15)$
or as 45 : 75 is the ratio required. *Ans.*

To divide a Number into Two Parts, so that Some Multiple of the One Part, together with Some Multiple of the Other Part is equal to Some Given Number.

If two fractions or ratios are equal, we can multiply both by any number and add and subtract the products to form new numerators or denominators, either separately or together, without altering the value of the fractions, provided we treat both fractions or ratios alike.

Example.—*Divide* 33 *into* 2 *parts, so that* 7 *times one part and* 11 *times the other are together equal to* 312.

33 is the sum of the 2 parts, so that when 7×33 is taken away from 312, there is taken away 7 times each part. There remains then $(11 - 7)$, or four times one part to be subtracted, and this must be equal to the number left after the first subtraction, viz. 81. That part, therefore, equals $81 \div 4$ or $20\frac{1}{4}$. The other part is found by subtracting $20\frac{1}{4}$ from 33, and equals $12\frac{3}{4}$.

Ans. $12\frac{3}{4}$; $20\frac{1}{4}$.

Proof.—
$$7 \times 12\frac{3}{4} = 89\frac{1}{4}$$
$$\text{and } 11 \times 20\frac{1}{4} = 222\frac{3}{4}$$
$$\text{and } 89\frac{1}{4} + 222\frac{3}{4} = 312$$

Given the Ratio of A to B, and of B to C, to find the Ratio of A to C.

Example.—*A and B are in the ratio of* 5 : 6; *B and C of* 7 : 8. *What is the ratio of A to C?*

A : B 5 : 6 B : C 7 : 8	Multiply the first ratio by 7, and the second by 6, so as to make B equal in each ratio	A : B 35 : 42 B : C 42 : 48

Then A : C as 35 : 48 *Ans.*

To find a Continued Ratio.

Example.—*The ratio of A to B is as* 3 : 4; *of B to C as* 5 : 6; *of C to D as* 7 : 8. *Find the continued ratio of A, B, C, and D.*

$$\begin{aligned}
&(a)\ \text{A} : \text{B} = 3 : 4 = 105 : 140\\
&(b)\ \text{B} : \text{C} = 5 : 6 = 140 : 168\\
&(c)\ \text{C} : \text{D} = 7 : 8 = 168 : 192\\
&\therefore\ \text{A} : \text{B} : \text{C} : \text{D} = 105 : 140 : 168 : 192
\end{aligned}$$

Notes.—(1) The first antecedent (105) is the product of the antecedents (3, 5, 7); the last consequent (192) is the product of the consequents (4, 6, 8).

(2) The consequent of one ratio becomes the antecedent of the next, and determines the number by which the consequent in each case is to be multiplied; *e.g.* in (*b*) $\frac{140}{5} = 28$, and 6 is multiplied by 28; in (*c*) $\frac{168}{7} = 24$, and 8 is multiplied by 24.

(3) If several quantities have ratios to one another, those ratios may be expressed in the form of a *continued ratio*.

Example.—*If* ·2 *of one number be equal to* ·04 *of another, what is the ratio of the two numbers?* (K.S.)

$$\begin{aligned}
\tfrac{2}{10}\text{ of one number} &= \tfrac{4}{100}\text{ of the other}\\
i.e.\ \tfrac{1}{5}\quad\ \ ,,\quad\ \ ,, &= \tfrac{1}{25}\quad\ \ ,,\quad\ \ ,,\\
\therefore\ \text{ratio of the numbers is } &5 : 25\\
\text{or } &1 : 5
\end{aligned}$$

Ans. 1 : 5.

Example.—*The sum of two numbers is* $9\frac{5}{8}$, *and their difference is* $4\frac{5}{8}$. *Find the ratio of the numbers.* (C.W.)

$$\begin{aligned}
\text{Sum of the numbers} &= 9\tfrac{5}{8}\\
\text{difference}\quad\ \ ,, &= 4\tfrac{5}{8}\\
\therefore\ 2 \times \text{smaller} &= 9\tfrac{5}{8} - 4\tfrac{5}{8} = 5\\
\text{and smaller} &= 5 \div 2 = 2\tfrac{1}{2} = 2\tfrac{4}{8}\\
\therefore\ \text{larger} &= 9\tfrac{5}{8} - 2\tfrac{4}{8} = 7\tfrac{1}{8}\\
\therefore\ \text{ratio of numbers is } &2\tfrac{4}{8} : 7\tfrac{1}{8}\\
,,\quad\ \ ,, &\quad\ \ \tfrac{20}{8} : \tfrac{57}{8}\\
,,\quad\ \ ,, &\quad\ \ 20 : 57
\end{aligned}$$

Ans. 20 : 57.

Example.—*Find a fraction which shall bear the same ratio to* $\frac{3}{5}$ *that* $\frac{3}{7}$ *does to* $\frac{4}{9}$. (C.S.)

$$\frac{\text{Fraction}}{\frac{3}{5}} = \frac{\frac{3}{7}}{\frac{4}{9}}$$

$$\therefore\ \text{fraction} = \frac{3}{7} \times \frac{9}{\cancel{4}_{2}} \times \frac{\overset{1}{\cancel{2}}}{5} = \frac{27}{70}$$

Ans. $\frac{27}{70}$.

Example.—*A warehouse consists of five floors; the rent of each floor is ·9 times that of the floor below; the rent of the middle floor is £33 15s. Find the ratio of the rents of the highest and lowest floors, and find also the rent of the lowest floor.* (K.S.)

The rents of the floors will be respectively—

$$1,\ \tfrac{9}{10},\ (\tfrac{9}{10})^2,\ (\tfrac{9}{10})^3,\ (\tfrac{9}{10})^4$$

Now, $(\tfrac{9}{10})^2 = £33\tfrac{3}{4} = £1\tfrac{35}{4}$

$$\therefore \text{ rent of lowest floor} = \frac{£1\overset{5}{\cancel{35}} \times \overset{5}{\cancel{10}} \times \overset{5}{\cancel{10}}}{\underset{2}{\cancel{4}} \times \overset{3}{\cancel{9}} \times \underset{3}{\cancel{9}}} = £1\tfrac{25}{3} = £41\tfrac{2}{3}$$

and rent of highest floor $= (\tfrac{9}{10})^4 \times £1\tfrac{25}{3}$

$$= £\frac{9 \times 9 \times 9 \times \overset{3}{\cancel{9}} \times \overset{5}{\cancel{25}}}{10 \times \underset{2}{\cancel{10}} \times \underset{2}{\cancel{10}} \times \underset{2}{\cancel{10}} \times \underset{1}{\cancel{3}}} = £2\tfrac{187}{80}$$

$\therefore$ ratio is $2\tfrac{187}{80} : 1\tfrac{25}{3}$

$6561 : 10000$

$\cdot 6561 : 1$

Ans. (*a*) ·6561 : 1; (*b*) £41 13*s.* 4*d.*

Example.—*Find two numbers in the ratio of* $3\tfrac{1}{3}$ *to* $2\tfrac{2}{3}$, *such that when each is diminished by* $11\tfrac{1}{3}$, *they shall become in the ratio of* $4\tfrac{1}{3}$ *to* $3\tfrac{1}{3}$. (L.M.)

Let N be the greater number; then—

$\frac{2\frac{2}{3}}{3\frac{1}{3}}$ of N is the other, *i.e.* $\frac{4}{5}$N

$$\therefore \frac{N - \frac{34}{3}}{\frac{4}{5}N - \frac{34}{3}} = \frac{4\frac{1}{3}}{3\frac{1}{3}} = \frac{13}{10}$$

$$\therefore 10N - \frac{340}{3} = \frac{52}{5}N - \frac{442}{3}$$

$$\frac{102}{3} = \frac{2}{5}N$$

$$\therefore \frac{5}{\underset{1}{\cancel{2}}} \times \frac{\overset{17}{\cancel{102}}}{\underset{1}{\cancel{3}}} = N$$

$$\text{and } 85 = N$$

$$\text{and other number} = \tfrac{4}{5}N = 68$$

Ans. 85 and 68.

THEORY.

1. Define "ratio." (K.S.)

2. Show that the ratio of two numbers may be expressed by a fraction. (K.S.)

3. How is the value of a ratio obtained? Illustrate your answer by an example. (I.I.)

4. In ratio the quantities to be compared must be of the same kind. Explain this, and state the result of this limitation.

5. What do you understand by the "measure" of a ratio? Define antecedent and consequent.

6. Give and explain the various methods of expressing a ratio.

7. Define, with examples, the following forms of ratio: equality, greater inequality, less inequality.

8. Explain the meaning, with examples, of duplicate ratio, triplicate ratio, subduplicate ratio, and subtriplicate ratio.

9. A ratio is always abstract. Explain this statement.

10. Since a ratio may be expressed as a fraction, the truths of fractions are also the truths of ratio. Demonstrate this.

11. Take a ratio and show what effect each of the following alterations has upon its value: (*a*) the increasing of the antecedent; (*b*) the decreasing of the antecedent; (*c*) the increasing of the consequent; (*d*) the decreasing of the consequent.

12. What is a continued ratio? Show how it differs from a compound ratio.

EXERCISE XXXIV. 3 4

Ratio.

1. (*a*) Which is the greater ratio, and by how much 2 : 3 or 9 : 13? (*b*) Alter the first of the above four numbers so as to equalize the ratios. (C.M.)

2. Find the number that is to $7\frac{2}{3}$ in the ratio of £3 1*s.* 3*d.* to £4 13*s.* 11*d.* (S.K.)

3. Find the ratio which $\frac{2}{3}$ of £27 1*s.* $5\frac{3}{4}$*d.* bears to ·6 of £42 10*s.* $10\frac{3}{4}$*d.* (S.K.)

4. Divide £47 17*s.* $6\frac{1}{4}$*d.* by ·3727272 . . . , and express in a decimal form the ratio of £143 to the quotient. (S.K.)

5. Express the ratio of 17 hrs. 45 min. 28 sec. to 124 hrs. 18 min. 16 sec. as a vulgar fraction in its lowest terms. (S.K.)

6. Find a fraction which shall bear the same ratio to $\frac{7}{26}$ that $\frac{11}{13}$ bears to the third part of $\frac{13}{39}$. (C.S.)

7. If ·2 of one number is equal to ·04 of another, what is the ratio of the two numbers? (K.S.)

8. Express $1 - \frac{\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}}{\frac{2}{3} \times 1\frac{2}{5}}$ as a simple ratio. (K.S.)

9. Reduce to fractions in their lowest denomination the sums $\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7}$; $\frac{1}{3} + \frac{3}{8} + \frac{2}{5} + \frac{5}{12} + \frac{3}{7}$; and $\frac{2}{9} + \frac{1}{4} + \frac{4}{15} + \frac{5}{18} + \frac{3}{7}$; and determine the ratio which the first bears to each of the others. (L.M.)

10. Two numbers are in the ratio of 24 and 30, and their L.C.M. is 720. What are the numbers? (P.T.)

11. Express in their simplest forms: (*a*) The ratio of $2\frac{1}{4}$ to $7\frac{1}{3}$; (*b*) the ratio of $\frac{3}{4}$ of 53 cwt. 3 qrs. 3 lbs. to ·4 of 65 cwt. 0 qrs. 11 lbs. (S.K.)

12. Express in their simplest forms: (*a*) The ratio of 4·333 . . . to $10\frac{1}{9} + 3{\cdot}0333\ldots$; (*b*) the ratio of $\frac{5}{6}$ of £5 9*s.* 8*d.* to 1·424242 . . . of £4 16*s.* 3*d.* (S.K.)

13. Express in their simplest forms as vulgar fractions: (*a*) The ratio of 2·12 of £1 15*s.* 5*d.* to $\frac{18}{17}$ of £4 10*s.* 8*d.*; (*b*) the ratio of $\frac{2}{7}$ of 2·050505 . . . to $\frac{198}{539}$ of 2·1060606 . . . (S.K.)

14. In a certain town there are 15933 white and 50055 black inhabitants. Express in its simplest form the ratio of the number of the white inhabitants to the whole population. (C.H.L.)

15. A walks $5\frac{1}{3}$ mi. in $2\frac{1}{3}$ hrs., B walks 1210 yds. in $11\frac{1}{4}$ min. What ratio do their speeds bear to each other? (C.S.)

16. If 5 c. in. of mercury weigh 2·45 lbs., and 2 c. in. of cast iron weigh ·52 lb., what ratio does the density of mercury bear to that of cast iron? (S.K.)

17. If 226 members of Parliament represent 12832803 persons, what is the ratio (of which the first term is unity) between representation and population? How many members *at least* should a population of 189860 have on the same principle? (C.P.)

18. Two persons buy postage stamps at 12 for a shilling; one retails them at 11 for a shilling, the other at 13*d.* a dozen. Compare their gains in selling £11 worth. (C.W.)

19. A and B buy oranges at 10 for 1*s.*; A retails them at 9 for 1*s.* and B at 1*s.* 5*d.* per dozen. Compare the gains of A and B in selling an equal number of oranges. (P.T.)

20. Compare the savings of A and B at the end of the half-year, if A spends only £1 18*s.* 4*d.* a week out of an income of £250 a year (52 weeks), and B spends only 6*s.* 8*d.* a day out of a weekly income of five guineas. (C.W.)

21. Find the ratio of the rates for carriage on two railways, one of which charges 22*s.* 6*d.* for the carriage of $37\frac{1}{2}$ cwt. 44 mi., and the other 31*s.* 6*d.* for the carriage of 48 cwt. $52\frac{1}{2}$ mi. (C.S.L.)

22. If the price of coal be 42*s.* per ton, and the value of coal to the same weight of coke be in the ratio of 5 : 4, calculate the value of 10 chaldrons of coke if 3 chaldrons = 2 loads, and each load weighs 21 cwt. (C.W.)

23. If A's rate of profit is $\frac{4}{5}$ of B's, and for every guinea gained by B, C gains £1, compare the profits of A and C. (P.T.)

24. A warehouse consists of 7 floors; the rent of each floor is ·875 times that of the floor below; the rent of the fourth is £17 3*s.* Compare the rents of the highest and lowest floors, and find that of the lowest. (K.S.)

25. In a school the ratio of children in the 1st class to the whole school is 1 in 7; in the 2nd class 5 in 12; in the 3rd class 8 in 21; the remaining children, 15 in number, are infants. Find the total number of children in the school. (P.T.)

26. If the ratio of threepenny to fourpenny pieces in a given sum which consists entirely of those coins were altered from 3 : 7 to 7 : 3, the sum would be diminished by £20. Find the sum. (K.S.)

27. An estate is divided between two claimants in the ratio of $5\frac{1}{4}$ to ·025; the money value of the smaller share is £238 2*s*. Express in guineas the corresponding value of the larger share. (K.S.)

28. 16 men and 5 boys together dig a trench 130 yds. long, 3 yds. wide, and 2 yds. deep in 4 days; 12 men and 9 boys dig one 122 yds. long, 3 yds. wide, and 3 yds. deep in 6 days. Compare the rates of working of a man and a boy. (C.PR.)

29. Each person in Great Britain pays 9*s*. 6*d*., and in France 17 fr. 22 c., to the support of the army; the size of the armies is as 3 : 10, and the populations equal. Taking 25 fr. 20 c. to the £, what is the ratio of the cost of a British to that of a French soldier? (S.A.)

30. The weight of a gold 10-fr. piece is 49·78 grs.; the weight of a silver 5-fr. piece is 385·8 grs., of which the equivalent in pure silver is ·782 oz. Troy. If the relative value of coined gold and silver be 19·29 : 1, find the intrinsic value of a gold 10-fr. piece, the intrinsic value of pure silver being 4*s*. 2*d*. per ounce Troy. (C.S.)

31. A person finds that a hat which cost him 15*s*. lasted 7 months, and that another for which he paid 11*s*. lasted only 5 months. Which of the two qualities will it be more advantageous to him to buy in future? Show how your answer is obtained. (P.T.)

32. Determine the ratio of a pound avoirdupois to a pound Troy in its simplest form. (1 lb. avoir. = 7000 grs.) (P.E.A.)

33. The value of gold is fixed at 77*s*. $10\frac{1}{2}$*d*. per ounce. What should be the price of silver per ounce if its value were fixed in the ratio of 1 to $15\frac{1}{2}$ to that of gold? (L.M.)

34. If a steamer makes the passage from New York to Liverpool (say 2760 mi.) in 9 days 14 hrs., and a train goes from London to Edinburgh (say 405 mi.) in 18 hrs., compare the rates of the steamer and the train. (C.S.)

35. The railway fare from Calais to Paris, a distance of 240 km., is 38·16 fr.; and from London to Dover, a distance of 70 mi., is 14*s*. 7*d*. Compare the cost of travelling in both countries, having given that a metre = 39·3708 in., and 25·44 fr. = £1. (C.S.)

36. The cost of making one bushel of wheat into bread (including the baker's profit) is 2*s*. 3*d*. If the price of wheat falls from 60*s*. a quarter to 34*s*. a quarter, in what ratio ought the size of the 4*d*. loaf to be altered? (1 qr. = 8 bush.) (C.W.)

37. In two successive years the numbers of acres under wheat in Great Britain were 3125997 and 2914065; the average yields per acre were $27\frac{1}{2}$ bush. and $29\frac{1}{2}$ bush., and the prices per bushel were 4*s*. 9*d*. and 4*s*. 3*d*. respectively. Find the ratio of the values of the crops. (C.M.)

38. In an orphan asylum for a certain quarter of the year in which the average number of boys was 104, and that of girls 72, the whole cost of keep was £669 12*s*., while for a single month in which the number of boys was 91, and that of girls 60, the cost was £191 17*s*. What is the ratio of the cost of a girl to that of a boy? (C.H.L.)

39. A earns £1 in 3 days, A and B earn £9 in 12 days, B and C earn £35 10*s*. in 42 days. Compare rates of wages of A, B, and C.

40. The value of a Spaniard's work to that of a Frenchman's being as 3 : 4, and their rate of daily wages being 5*s*. and 7*s*. respectively, find the gain obtained by employing 100 Spaniards for 21 days instead of Frenchmen. (K.S.)

CHAPTER XVIII.

SIMPLE PROPORTION.

If we take two simple ratios, *e.g.* 2 : 3 and 4 : 6, and two other simple ratios, *e.g.* 2 : 3 and 3 : 4, and examine the pairs, we note a difference between them. The first two ratios are equal, for $\frac{2}{3} = \frac{4}{6}$, and the ratios may be written thus :—2 : 3 = 4 : 6. But the second set of ratios does not present the same equality, for $\frac{2}{3}$ does not equal $\frac{3}{4}$.

In the first pair we discover an *equality of ratios*, and this equality of ratios is called Proportion. *Proportion may thus be defined as an Equality of Ratios.*

Notes.—(1) In a ratio two *terms* are compared.
(2) In a proportion two (or more) *ratios* are compared.

Methods of expressing a Proportion.

(*a*) 6 : 12 = 20 : 40

(*b*) $\frac{6}{12} = \frac{20}{40}$

(*c*) As 6 : 12 :: 20 : 40

Notes.—(1) The last method is the usual way, and is read, As 6 *is to* 12 *so is* 20 *to* 40.
(2) The four dots :: are said to represent the four ends of the two lines expressing equality (:=:).

Nomenclature of Proportion.—Consider the proportion

6 : 12 :: 20 : 40

1. *Terms.*—Each of the four quantities in a proportion is called a *Term*, *e.g.* 6, 12, 20, 40.

2. *Extremes.*—The outside terms, 6 and 40, are called *Extremes.*

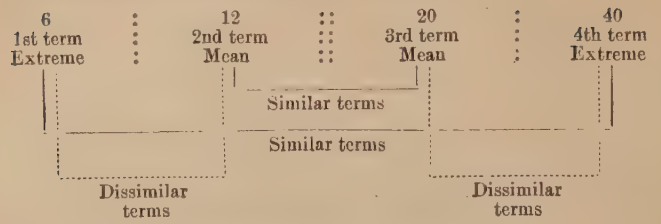
3. *Means.*—The inside terms, 12 and 20, are called *Means.*

4. *Order of Terms.*—6 is called the *first* term, 12 the *second*, 20 the *third*, and 40 the *fourth*.

5. *Similar Terms.*—The two Extremes or the two Means are called *Similar Terms.*

6. *Dissimilar Terms.*—Terms composed of one Mean and one Extreme are called *Dissimilar Terms.*

These facts may be graphically shown thus—



Two Great Truths of Proportion.

1. *The product of the extremes equals the product of the means.*

General Proof.

If $a : b :: c : d$

then $\frac{a}{b} = \frac{c}{d}$

for multiplying each ratio by bd,

we get $\frac{a}{\cancel{b}} \times \cancel{b}d = \frac{c}{\cancel{d}} \times b\cancel{d}$

$\therefore a \times d = c \times b$

Special Proof.

If $6 : 12 :: 20 : 40$

then $\frac{6}{12} = \frac{20}{40}$

for $\frac{6}{\cancel{12}} \times \cancel{12} \times 40 = \frac{20}{\cancel{40}} \times 12 \times \cancel{40}$

$\therefore 6 \times 40 = 20 \times 12$

This is the great truth of Proportion, for by it, given any *three* terms of a proportion, we can always find the *fourth*.

Let E′ and E″ = the two extremes, and M′ and M″ = the two means.

Then $E' \times E'' = M' \times M'' =$ the great general truth of Proportion.

and $E' = \frac{M' \times M''}{E''}$

,, $E'' = \frac{M' \times M''}{E'}$

} *i.e.* either extreme = the product of the means divided by the other extreme.

,, $M' = \frac{E' \times E''}{M''}$

,, $M'' = \frac{E' \times E''}{M'}$

} *i.e* either mean = the product of the extremes divided by the other mean.

Applying this principle, we can easily supply the missing term in any of the following:—

(*a*) : 12 :: 20 : 40

(*b*) 6 : : : 20 : 40

(*c*) 6 : 12 : : : 40

(*d*) 6 : 12 :: 20 :

As we shall subsequently see, it is *usual* to assign the unknown quantity to the 4th term, but this is merely a convention; the "unknown quantity" might be assigned to any other term.

2. *If the 3rd and 4th terms of a proportion are in* {*ascending* / *descending*} *order of magnitude, then the 1st and 2nd terms must be in* {*ascending* / *descending*} *order of magnitude.*

$\underbrace{6 : 12}_{\text{Ascending ratio}} :: \underbrace{20 : 40}_{\text{Ascending ratio}}$	$\underbrace{12 : 6}_{\text{Descending ratio}} :: \underbrace{40 : 20}_{\text{Descending ratio}}$

The following truth is a rider of No. 1 :—

If the product of two numbers be equal to the product of two other numbers, the four factors can be arranged in a proportion.

To do this, the two factors of one product must be the extremes, and the two factors of the other product the means, or conversely. In this way we may get eight different proportions; *e.g.* consider the case of 4×9 and 3×12.

$4 : 3 :: 12 : 9$	$12 : 9 :: 4 : 3$
$4 : 12 :: 3 : 9$	$3 : 9 :: 4 : 12$
$9 : 3 :: 12 : 4$	$12 : 4 :: 9 : 3$
$9 : 12 :: 3 : 4$	$3 : 4 :: 9 : 12$

Now, four things can be arranged in twenty-four different ways, hence there might be twenty-four proportions instead of eight. But the tests are not true for the other sixteen cases; *e.g.* 4, 3, 9, and 12 are *not* a proportion.

General Proof.	*Arithmetical Proof.*
Since $a \times d = c \times b$	Since $4 \times 9 = 3 \times 12$
then $\frac{a \times \cancel{d}}{b \times \cancel{d}} = \frac{c \times \cancel{b}}{d \times \cancel{b}}$	then $\frac{4 \times \cancel{9}}{12 \times \cancel{9}} = \frac{3 \times \cancel{12}}{9 \times \cancel{12}}$
and $\frac{a}{b} = \frac{c}{d}$	and $\frac{4}{12} = \frac{3}{9}$
$\therefore a : b :: c : d$	$\therefore 4 : 12 :: 3 : 9$

The other seven proportions can be similarly demonstrated or proved.

It would now be well to work two examples in proportion to illustrate the application of the two great truths given above.

Example.—*If 3 ac. 24 po. can be rented for* £14, *how much land of the same character may be rented for* £20? (P.T.)

Step 1.—Classify the terms in "like pairs" with a view to determining the ratio.

£14 }	{ 3 ac. 24 po.
£20 }	{ the required number of acres.

Step 2.—In accordance with custom, place the unknown quantity in the 4th term.

: : :: : the required number of acres.

Step 3.—Then, since the 3rd and 4th terms of a proportion form a ratio, and ratio exists only between quantities of a like kind, the quantity 3 ac. 24 po. ($3\frac{3}{20}$ ac.) must go in the 3rd term.

: :: $3\frac{3}{20}$ ac. : the required number of acres.

Step 4.—But $3\frac{3}{20}$ ac. is the amount of land rented for £14, and the "required number of acres" is rented for £20. The amount of land rented for £20 will be *greater* than the amount rented for £14; hence "*the required number of acres*" will be *greater* than $3\frac{3}{20}$ ac.; therefore the ratio is in ascending order.

: :: $3\frac{3}{20}$ ac. : required number of acres

{amount of land rented for £14} : {amount of land rented for £20}

Less : Greater

An ascending ratio

Step 5.—But if the 3rd and 4th terms are in *ascending* ratio, then the 1st and 2nd terms must also be in *ascending* ratio (Truth 2, p. 181). Then arranging £14 and £20 in an ascending ratio, we have—

£14 : £20

Step 6.—And putting these quantities into the 1st and 2nd terms respectively, we have—

£14 : £20 :: $3\frac{3}{20}$ ac. : required number of acres.

Step 7.—Now, Truth 1 (p. 180) states that the product of the extremes is equal to the product of the means, and since we cannot multiply concrete quantities together, and since the ratio between £14 and £20 is the same as the ratio between 14 and 20, we rewrite thus—

14 : 20 :: $3\frac{3}{20}$ ac. : required number of acres.

i.e. required number of ac. $\times$ 14 = $3\frac{3}{20}$ ac. $\times$ 20

$$\therefore \text{required number of ac.} = \frac{3\frac{3}{20} \text{ ac.} \times 20}{14} = \frac{63}{20} \times \frac{20}{14} \text{ ac.}$$

$$= \frac{9}{2} \text{ ac.} = 4\frac{1}{2} \text{ ac.}$$

Ans. $4\frac{1}{2}$ ac.

Example.—*If* 12 *men can perform a piece of work in* 8 *days, in what time will* 48 *men perform the same?*

1. Classify the terms.

{12 men, 48 men} {8 days, the required number of days}

2. In accordance with custom, place the unknown quantity in the 4th term.

: :: : the required number of days.

3. The quantity 8 *days* must go into the 3rd term to complete the ratio.

: :: 8 days : required number of days.

4. Now, 8 *days* is the time 12 men take to do the piece of work, and "*the required number of days*" is the time 48 men take. More time must be taken

by 12 men than by 48 men to do the work, hence 8 *days* is greater than "*the required number of days*," and the ratio is of the *descending order*.

:	::	8 days	:	required number of days
		time taken by 12 men	:	time taken by 48 men
		greater time	:	*less* time

descending ratio

5. Then, as the 3rd and 4th terms are in descending ratio, the 1st and 2nd terms must be also in descending ratio. Arranging 12 men and 48 men in descending ratio, we have—

48 men : 12 men

6. Putting these quantities in the first and second terms, we have—

48 men : 12 men :: 8 days : required number of days, or,
48 : 12 :: 8 days : required number of days.

7. Applying Truth 1 (p. 180), we obtain—

$$\text{required number of days} \times 48 = 12 \times 8 \text{ days}$$

$$\therefore \text{required number of days} = \frac{12 \times 8}{48} = 2$$

Ans. 2 days.

From the above explanations, the student may exercise his ingenuity in discovering the reasons for the following "Rule of Thumb" for working proportion :—

Rule.—(1) Place the odd quantity in the 3rd term.
(2) Arrange the other two terms in the following order :—
(*a*) If the answer is to be *greater* than the 3rd term, place the larger quantity in the 2nd term.
(*b*) If the answer is to be less than the 3rd term, place the *smaller* quantity in the 2nd term.
(3) Multiply the 2nd and 3rd terms together and divide by the 1st term.

Kinds of Proportion.—An examination of Step 4 in both of the above examples reveals a difference which is of some theoretical, but of little practical importance. In the first example, the money varies *directly* as the amount of land ; in the second example, the number of men varies *indirectly* or *inversely* as the time taken. The first example is said to be in *direct*, and the second in *inverse* proportion.

1. **Direct Proportion.**—If in a proportion, an increase or decrease of the antecedent or consequent in the first ratio leads to a corresponding increase or decrease in the antecedent or consequent in the second ratio, the four terms are said to be *directly* proportional, *e.g.*—

If 6 lbs. of tea cost 12*s.*, 10 lbs. will cost 20*s.* The ratio of the lbs. $\frac{6}{10}$ is *directly* equal to the ratio of the shillings ($\frac{12}{20}$), for more tea costs more money, and less tea costs less money. Similarly—

(*a*) The rent of a house varies directly as the time it is occupied.
(*b*) The rent of land as the position, area, time.
(*c*) The cost of carriage as the weight and distance.
(*d*) The price of bread as the price of flour; and so on.

2. Inverse Proportion.—If in a proportion an increase in the first antecedent means a decrease in the second antecedent, or a decrease in the first antecedent means an increase in the second antecedent, and similarly for the consequents, then the four terms are said to be *inversely* proportional; *e.g.*—

If it requires 6 men to do a piece of work in 12 days, it would require 8 men to do it in 9 days. Similarly—
(*a*) For a given piece of work more workmen means less time.
(*b*) In a carpet, the greater the breadth, the less the length required.
(*c*) In provisioning a garrison, the greater the number of men, the less time the provisions will last.
(*d*) In running a mile, the faster the pace, the less the time.

The Rule of Three.—Proportion is often called the Rule of Three, because in every problem in Proportion *three terms are given*, and the fourth has to be found (p. 180).

The Nature of the Rule of Three.—It will be observed that every sum in simple proportion involves a theorem and a problem.

The Theorem.—The theorem involves the application of Truth 2 (p. 181), and requires the discovery of the relation that exists between the unknown quantity (the answer) and the known number of the same denomination (usually the 3rd term).

The Problem.—The problem involves the application of Truth 1 (p. 180), and is merely a process of multiplication and division, *i.e.* having three terms given, the fourth is easily found.

The two are closely connected together, for the problem depends on the theorem, since the theorem arranges the terms to express the required relation. The theorem is the chief difficulty in all cases of proportion.

Proportion and Fractions.—Since ratios may be expressed as fractions, some properties of fractions are great aids to the better understanding of direct and inverse proportion.

1. **Like Denominators.**—If two fractions have like denominators, the fractions vary *directly* as their numerators; *i.e.* they are to each other as their numerators; *e.g.*—

$$\tfrac{4}{9} : \tfrac{7}{9} :: 4 : 7$$

Obviously, since the denominators are alike, the fractions are greater or smaller, according as their numerators are greater or smaller (p. 61).

2. **Like Numerators.**—If two fractions have like numerators, but

unlike denominators, then they are to each other in the *inverse* ratio of their denominators; *i.e.* they vary inversely as their denominators; *e.g.*—

$$\tfrac{8}{11} : \tfrac{8}{15} :: 15 : 11, \text{ or } \tfrac{8}{11} : \tfrac{8}{15} :: \tfrac{1}{11} : \tfrac{1}{15}$$

We know that with the same numerator a fraction is greater or less according as its denominator is less or greater. We know, also, that the four terms are proportionals, for $\frac{8}{11} \times 11 = \frac{8}{15} \times 15$, therefore the fractions vary inversely as their denominators.

The Unitary Method.—This method is so called because the *unit* is the medium through which the problem is solved.

Thus, in the example below, we first find the value of a unit (in this case the length of time *one* man takes to do the work), and then proceed to find the value of any number of units.

The method is also known as the *Method of Unity*, the *Method of Analysis*, and the *Method of First Principles.*

It has been asserted that the Rule of Three involves an absurdity, on the ground that it necessitates the multiplying of one concrete quantity by another—an impossibility. It is said that the Unitary Method was introduced to avoid this weakness.

The objection rested on a misconception of the nature of proportion. Proportion deals only with ratios; * ratios are always abstract, and we *may* multiply one abstract number by another. There is the same ratio between 3 boys and 12 boys as between 6 lbs. and 24 lbs., and these terms can be stated as a true proportion; *e.g.*—

$$3 : 12 :: 6 : 24$$

We do not multiply *boys* and *lbs.*, and no assertion of equality is made between them; the equality asserted is that existing between the two ratios $\frac{3}{12}$ and $\frac{6}{24}$, *i.e.* between the ratios abstracted from the concrete quantities.

We will now proceed to work two examples in the Rule of Three by the Unitary Method.

Example.—*If* 25 *men do a piece of work in* 15 *days, in what time will* 15 *men do the same amount of work?*

If 25 men do the work in 15 days
then 1 man does „ „ 15×25 days
and 15 men do „ „ $\frac{15 \times 25}{15} = 25$ days
Ans. 25 days.

The change from the quantity 25 men to the quantity 15 men is made through *a common unit* (1 man). Hence the name of the method.

* *Some good authorities do not think it necessary to restrict the idea of proportion to numerical ratios.*

Example.—*If* $\frac{5}{7}$ *lb. cost* $\frac{4}{9}$*s., what will* $\frac{3}{5}$ *cwt. cost?* (P.T.)

If $\frac{5}{7}$ lb. cost $\frac{4}{9}$*s.*

Then $\frac{1}{7}$ lb. ,, $\frac{4}{9 \times 5}$*s.*

And $\frac{7}{7}$,, (or 1 lb.) ,, $\frac{4 \times 7}{9 \times 5}$*s.*

And 112 lb. (or 1 cwt.) ,, $\frac{4 \times 7 \times 112}{9 \times 5}$*s.*

And $\frac{3}{5}$ cwt. ,, $\frac{4 \times 7 \times 112 \times \frac{3}{5}}{\frac{9}{3} \times 5 \times 5}$*s.* $= \frac{3136}{75}$*s.* $= £2\ 1s.\ 9\frac{1}{25}d.$

Ans. £2 1*s.* $9\frac{1}{25}d.$

Note.—To be logical in this method, the division should be performed first, and then the multiplication; but it is often convenient to reverse the process, especially in a case of Inverse Proportion.

Method of Work.—This involves a *Statement* and a *Choice of Unit*, then multiplication and division, or *vice versâ*.

1. *The Statement.*—The quantity which is of the same denomination as the answer should always be placed *last* in the *statement.*

2. *Choice of Unit.*—The best unit should be chosen, and a little practice will help in its detection. As a rule this unit is the G.C.M. of the two quantities, or that common measure which is most readily recognized; *e.g.* the unit for £25 7*s.* 6*d.* and £100 would be 2*s.* 6*d.*

Value of the Method.—It is said to be very simple, and the claim is based on the absence of mechanical rules. It is wide in its application, solving problems in all those rules like Interest, etc., which are mere applications of Proportion. It is the method of common sense, and it requires only a repeated application of the same process of reasoning. It affords good mental exercise, and exhibits more clearly the reason of the operation performed. It is a logical method, as each succeeding step depends on some preceding one. It is also said to train in habits of neatness and exactness in the method of work. It is an advantage to know both the "Rule of Three," and the Method of Unity, but for practical purposes the latter is inferior to the former. In long or involved problems, if all the reasoning is set down, the Method of Unity becomes cumbrous and slow; whereas, much of the reasoning is done mentally in the "Rule of Three."

The Query Form of Statement.—This form is really a species of shorthand. All the conditions of the problem are placed before the eye at a glance, and the terms that are to form the given ratio are brought together.

Example.—*If* 20 *loaves cost* 8*s.* 4*d.*, *what will* 60 *cost at the same rate?*

If 20 loaves cost 8*s.* 4*d.*
then 60 ,, ,, ?

The form of reasoning is similar to the ordinary method.

The Common Term.—Apparently, more than three terms are sometimes given in a problem in proportion. In such

cases one term will be found common to both conditions, and it can then be omitted ; *e.g.*—

Example.—*If* 72 *men for* 13 *days' work receive* £119 7*s.* 6*d.*, *what ought* 54 *men to receive?*

Here the term 13 *days' work* is common to both conditions, and so can be disregarded. The statement then becomes—

72 men : 54 men :: £119 7*s.* 6*d.* : required amount.

Having considered direct and inverse proportion, we can now consider other kinds, and the truths connected with them. They possess some theoretic value, but are of little practical importance.

Inverted Proportion.—*If the ratios of a proportion are inverted, the four terms are still proportionals.* The antecedent becomes the consequent and the consequent the antecedent in each ratio.

Consider the eighth proportion given on p. 181. It is an Inverted Proportion of the first.

General Proof.

Given $a : b :: c : d$
then $b : a :: d : c$
For since $a \times d = b \times c$

$$\therefore \frac{a \times d}{a \times c} = \frac{b \times c}{a \times c}$$

$$\text{and } \frac{d}{c} = \frac{b}{a}$$

i.e. $b : a :: d : c$

Arithmetical Proof.

Given $4 : 3 :: 12 : 9$
then $3 : 4 :: 9 : 12$
For since $4 \times 9 = 3 \times 12$

$$\therefore \frac{\cancel{4} \times 9}{\cancel{4} \times 12} = \frac{3 \times \cancel{12}}{4 \times \cancel{12}}$$

$$\text{and } \tfrac{9}{12} = \tfrac{3}{4}$$

i.e. 3 : 4 :: 9 : 12

Compounded Proportion.—*If four numbers are proportionals, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

General Proof.

If $a : b :: c : d$
then $(a + b) : b :: (c + d) : d$
For since $\frac{a}{b} = \frac{c}{d}$

$$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$$

$$i.e.\ \frac{a+b}{b} = \frac{c+d}{d}$$

or $a + b : b :: c + d : d$

Arithmetical Proof.

If $6 : 10 :: 12 : 20$
then $(6 + 10) : 10 :: (12 + 20) : 20$
For since $\frac{6}{10} = \frac{12}{20}$

$$\therefore \tfrac{6}{10} + 1 = \tfrac{12}{20} + 1$$

$$i.e.\ \frac{6+10}{10} = \frac{12+20}{20}$$

or $6 + 10 : 10 :: 12 + 20 : 20$

i.e. $16 : 10 :: 32 : 20$

Divided Proportion.—*If four numbers are proportionals, the difference of the first two is to the second as the difference of the last two is to the fourth.*

The proof is similar to that for compounded proportion, except that unity is subtracted instead of being added. The general and arithmetical statements are as follows :—

If $a : b :: c : d$	If $24 : 8 :: 12 : 4$
then $a - b : b :: c - d : d$	then $(24 - 8) : 8 :: (12 - 4) : 4$

Converse Proportion.—*If four terms are proportionals, then the first is to the sum (or difference) of the two first as the third is to the sum (or difference) of the two last.*

General Proof.

If $a : b :: c : d$
then $a : a \pm b :: c : c \pm d$

For by inverted proportion $\frac{b}{a} = \frac{d}{c}$,

and by compounded and divided proportion $\frac{a \pm b}{b} = \frac{c \pm d}{d}$

$\therefore$ by multiplication—

$$\frac{(a \pm b)\cancel{c}}{\cancel{a}\cancel{b}} = \frac{(c \pm d)\cancel{d}}{\cancel{c}\cancel{d}}$$

$$\therefore a \pm b : a :: c \pm d : c$$

and by inverted proportion—

$$a : a \pm b :: c : c \pm d$$

Arithmetical Proof.

If $24 : 8 :: 12 : 4$
then $24 : 24 \pm 8 :: 12 : 12 \pm 4$

For $\frac{8}{24} = \frac{4}{12}$

and $\frac{24 \pm 8}{8} = \frac{12 \pm 4}{4}$

$$\therefore \frac{(24 \pm 8)\cancel{8}}{24 \times \cancel{8}} = \frac{(12 \pm 4)\cancel{4}}{12 \times \cancel{4}}$$

i.e. $24 \pm 8 : 24 :: 12 \pm 4 : 12$
and $24 : 24 \pm 8 :: 12 : 12 \pm 4$

Corollary.—*If four numbers are proportionals, the sum of the first two is to their difference as the sum of the last two is to their difference.*

General Proof.

If $a : b :: c : d$
then $(a + b) : (a - b) :: (c + d) : (c - d)$

Since $\frac{a + b}{b} = \frac{c + d}{d}$ (compounded)

and $\frac{a - b}{b} = \frac{c - d}{d}$ (divided)

Dividing the first by the second—

$$\frac{a + b}{a - b} = \frac{c + d}{c - d}$$

i.e. $a + b : a - b :: c + d : c - d$

Arithmetical Proof.

If $21 : 7 :: 42 : 14$
then $(21 + 7) : (21 - 7) :: (42 + 14) : (42 - 14)$

Since $\frac{21 + 7}{7} = \frac{42 + 14}{14}$

and $\frac{21 - 7}{7} = \frac{42 - 14}{14}$

then $\frac{21 + 7}{21 - 7} = \frac{42 + 14}{42 - 14}$

i.e. $21 + 7 : 21 - 7 :: 42 + 14 : 42 - 14$
or $28 : 14 :: 56 : 28$

If there be a number of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any one of the antecedents is to its consequent.

General Proof.

$$\left\{\begin{array}{l}\text{If } a:b::c:d::e:f \\ \text{then } \dfrac{a+c+e}{b+d+f}=\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}\end{array}\right.$$

Since $a \times d = b \times c$, and $a \times f = b \times e$

$$\text{then } \left.\begin{array}{l}(a\times d)+\\(a\times f)\end{array}\right\} = \left\{\begin{array}{l}(b\times c)+\\(b\times e)\end{array}\right.$$

and adding $(a \times b)$ to each side—

$$\left.\begin{array}{l}(a\times b)+(a\times d)\\+(a\times f)\end{array}\right\} = \left\{\begin{array}{l}(a\times b)+(b\times c)\\+(b\times e)\end{array}\right.$$

$$\therefore a(b+a+f)=b(a+c+e)$$

$$\therefore \frac{a}{b}=\frac{a+c+e}{b+d+f}$$

Arithmetical Proof.

$$\left\{\begin{array}{l}\text{If } 3:4::12:16::15:20 \\ \text{then } \dfrac{3+12+15}{4+16+20}=\dfrac{3}{4}=\dfrac{12}{16}=\dfrac{15}{20}\end{array}\right.$$

Since $3 \times 16 = 4 \times 12$, and $3 \times 20 = 4 \times 15$

$$\left.\begin{array}{l}(3\times 16)+\\(3\times 20)\end{array}\right\} = (4\times 12)+(4\times 15)$$

and adding (3×4) to each side, we get—

$$\left.\begin{array}{l}(3\times 4)+(3\times 16)\\+(3\times 20)\end{array}\right\} = \left\{\begin{array}{l}(3\times 4)+(4\times 12)\\+(4\times 15)\end{array}\right.$$

and taking 3 out of each bracket on the one side, and 4 out of each bracket on the other side, we get—

$$3(4+16+20)=4(3+12+15)$$

i.e. $3 \times 40 = 4 \times 30$

or $120 = 120$

Continued Proportion.—Quantities or numbers are said to be in continued proportion when the consequent of one ratio becomes the antecedent of the next; *i.e.* when the first term is to the second as the second is to the third, as the third is to the fourth, and so on; *e.g.*—

$$a:b::b:c::c:d::d:e$$
$$2:4::4:8::8:16::16:32$$

If the ratios are expressed as fractions, it will be seen that any two consecutive fractions have a common item; *e.g.*—

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}, \text{ or } \tfrac{2}{4}=\tfrac{4}{8}=\tfrac{8}{16}=\tfrac{16}{32}$$

Mean Proportional.—*In any continued proportion, the second term is called a Mean Proportional to the third;* e.g. if $a:b::b:c$, then b is said to be a mean proportional to a and c. Hence, if in any proportion one mean is equal to the other, each of the means is called a Mean Proportional to the Extremes.

Example.—*Find a mean proportional to* 4 *and* 16.

This means: Find a number which bears the same ratio to 4 that 16 bears to the required number.

As 4 : required number :: required number : 16

i.e. required number squared $= 4 \times 16 = 64$

$\therefore$ the required number $= \sqrt{64} = 8$ *Ans.*

The proportion thus becomes as 4 : 8 :: 8 : 16. Hence the following **Rule**:—

To find a mean proportional to two given numbers, multiply them together, and take the square root of the product.

Third Proportional.—*If* $a : b :: b : c$, *then* c *is called the third proportional to* a *and* b.

Example.—*Find a third proportional to* 50 *and* 120.

The statement is, as 50 : 120 :: 120 : required number

$$\therefore \text{ third proportional} = \frac{\overset{24}{\cancel{120}} \times \overset{120}{\cancel{120}}}{\underset{5}{\cancel{50}}} = 288 \quad \textit{Ans.}$$

the full proportion is then 50 : 120 :: 120 : 288

Notes.—(1) A third proportional, like a mean proportional, is found only in a continued proportion.

(2) In each case, although there are *four* terms, there are only *three* different quantities.

(3) "Mean Proportional" and "Third Proportional" are interdependent, for where the one exists there the other is found also.

Fourth Proportional.—This is not a case of continued proportion, but the usual case of the Rule of Three. We have the first three terms in a proportion given, and it is required to find the fourth; *e.g.*—

$$6 : 12 :: 20 : x$$

Some Examples in Proportion.

Example.—*If* 77 *tailors can execute an order for regimental clothing in* 30 *days, how many more tailors must be engaged to complete the order* 8 *days sooner?* (P.T.)

By the Unitary Method.

If in 30 dys. the work is done by 77 tailors
then in 1 dy. ,, ,, 77×30 ,,
and in (30−8) or 22 dys. ,, $\frac{77 \times 30}{22}$,,

$= 105$ tailors

$\therefore$ number of additional tailors $= 105 - 77 = 28$

Ans. 28.

By Proportion.

22 dys. : 30 dys. :: 77 men : x men

$$\therefore x = \frac{\overset{15}{\cancel{30}} \times \overset{7}{\cancel{77}}}{\underset{2}{\cancel{22}}} = 105$$

$\therefore$ number of extra men $= 105 - 77 = 28$

Ans. 28.

Example.—*If either* 5 *oxen or* 7 *horses eat up the grass of a field in* 87 *days, in what time will* 2 *oxen and* 3 *horses eat up the same quantity?* (C.S.)

5 oxen = 7 horses

$\therefore$ 1 ox $= \frac{7}{5}$,,

and 2 oxen + 3 horses $= (2 \times \frac{7}{5} + 3)$ horses $= \frac{29}{5}$ horses $= 5\frac{4}{5}$ horses

We then get the following statement :—

$\frac{29}{5}$ horses : 7 horses :: 87 days : required number of days

$$\therefore \text{required number of days} = \frac{7 \times \overset{3}{\cancel{87}} \times 5}{\underset{29}{\cancel{29}}} = 105$$

Ans. 105 days.

Notes.—(1) In mixed denominations of this nature, it is necessary to reduce the terms (oxen, horses) to the same denomination (horses), and then to proceed as in simple proportion.

(2) If it is thought undesirable to speak of the fraction of a horse, the objection can be met by multiplying all the terms of the proportion by 5. We then get the following statement :—

29 horses : 35 horses :: 435 days : 5 × required number of days

$$\therefore 5 \times \text{required number of days} = \frac{35 \times 435}{29} \text{ days}$$

$$\text{and required number of days} = \frac{\overset{7}{\cancel{35}} \times \overset{15}{\cancel{435}}}{\underset{1}{\cancel{29}} \times \underset{1}{\cancel{5}}} = 105$$

THEORY.

1. When are four numbers said to be in proportion? (C.W.)

2. What is meant by ratio, and how does it differ from proportion? (L.M.)

3. Explain the term "proportion." In working a problem in proportion, is it necessary or merely convenient to make the unknown quantity the fourth term? (P.T.)

4. What is meant by the rule of three direct and the rule of three inverse? (K.S.)

5. When are three numbers said to be in continued proportion? (P.T.)

6. Define "mean proportional," and give an example. (K.S.)

7. Explain the meaning of the term "unitary" in the unitary method of working rule-of-three sums. (C.M.)

8. How do you find the first term of a proportion when the remaining three are given? (I.I.)

9. Give the rule for finding either extreme of a proportion of four terms when the other three are known; also the rule for finding either mean when the other three terms are known. (I.I.)

10. Explain why, when you work a proportion sum, you multiply the second and third terms together and divide by the first. (P.T.)

11. Give a simple method for showing that the product of the extremes is equal to the product of the means. (K.S.)

12. Define a proportion of four numbers, and find how many proportions can be formed if the product of the four whole numbers is 144. (C.S.)

13. Is it true that $3 : 4 :: \cdot0\dot{3} : \cdot0\dot{4}$? (C.W.)

14. Show that the four quantities, 9 yds. 1 ft. 5 in., 2 yds. 1 ft. 9 in., 7 lbs. 15 oz. 14 drms., 2 lbs. 2 oz. 14 drms., are proportionals. (P.T.)

15. Show in what way the following is incorrect: £2 10*s.* : £125 :: £5 : 250. (P.T.)

16. The three terms of a rule-of-three sum in proper order are £4, £5, and 7 tons 3 cwt. What alteration must be made before the ordinary rule, "multiply the second and third terms together," etc., can be applied correctly? (K.S.)

17. What two things are declared equal in the following sentence?—1 : $\frac{4}{11}$ :: $\frac{1}{2}$ ac. : 1320 sq. yds. Prove that they are not equal, and alter the "$\frac{1}{2}$ ac." so as to make the sentence true. (P.T.)

18. A labourer working $7\frac{1}{2}$ hrs. a day does a job in 9 days. How many hours a day must he work to get it done in $7\frac{1}{2}$ days? Show that it is unnecessary to *state* the sum. (P.T.)

19. (*a*) Write *in words only* the sentence 11 : 3 :: 22 : 6; (*b*) write the same sentence without words and without dots, using fractions. (P.T.)

20. Determine whether it is true that $5d. : 2\frac{1}{7}d. :: 2\frac{1}{7}d. : \frac{15}{49}d.$ Prove that by adding a farthing to one of the three sums, 1*s.* $3\frac{1}{2}d.$, 2*s.* $2\frac{1}{4}d.$, and 3*s.* $7\frac{3}{4}d.$, we can make the second a mean proportional between the other two. Also, without multiplying out, prove that there is only one of the three sums to which a farthing can be added with this result. (Reduction is allowed.) (K.S.)

21. Express in words the arithmetical truth embodied in the following statements, and give numerical examples of each: (1) If $a : b :: c : d$, then $a : c :: b : d$, and $a \times d = b \times c$, and $\frac{b \times c}{a} = d$; (2) If a is less than b, then the fraction $\frac{a}{b}$ is less than $\frac{a+1}{b+1}$, but greater than $\frac{a-1}{b-1}$. (K.S.)

22. Show why it is of advantage to be acquainted with both methods, "first principles" (method of unity) and the "rule of three." (K.S.)

23. Show how the unitary method may be abbreviated by the use of ratios. Illustrate this by this example: If 5 cwt. 7 lbs. cost £3 12*s.*, what is the price of 21 cwt. 1 qr. 14 lbs.? (K.S.)

24. By what different methods may the following sum be worked? Give your reasons for preferring the method you adopt.—Estimate the cost of constructing a telegraph for a distance of 84·4125 mi. at the rate of £4 16*s.* $7\frac{1}{2}d.$ per chain. (K.S.)

25. Do one of the following sums by proportion, and explain clearly why the other cannot be done by proportion: (*a*) A clerk's salary begins at 18*s.* per week, and is increased by 1*s.* per week after each completed year of service. What is the rate of salary during the fifth year of service? (*b*) A watch, ticking at a uniform rate throughout the year 1899, ticked twelve million and fifteen times in April. How often did it tick in February? (P.T.)

26. Show clearly how a ratio is measured, and the relation that proportion bears to ratio, giving an example in illustration of your answer. Suppose an express train travels 15 mi. in 20 min., and an ordinary train 20 mi. in $\frac{3}{4}$ hr., state the ratio of the speed of the express to that of the ordinary train. (P.T.)

27. State the rule of three, and explain clearly in each of the following cases whether it is applicable: (*a*) If the weight of a globe of 100 sq. in. in surface be 6 lbs., what is the weight of a globe of the same

material of 150 sq. in. in surface? (*b*) If a train, moving with a uniform velocity of 1 ft. per second, traverses a certain distance in 1 min., how long would it have taken to traverse the same distance had the velocity been $1\frac{1}{2}$ ft. per second? (*c*) If a train starting from rest, and moving with a uniform increase of velocity of 1 ft. per second, traverses a certain distance in 1 min., how long would it have taken to traverse the same distance had the uniform increase in velocity been 12 ft. per second? (C.S.)

EXERCISE XXXV.

Simple Proportion.

Complete the following proportions:—

1. 287 : 492 :: [ ] : 5376. (C.W.) **2.** 64 : [ ] :: 576 : 225. (K.S.)
3. [ ] : 17 :: 804 : 67. (K.S.) **4.** 832 : 122 :: [ ] : 2440. (K.S.)

Find the fourth proportional to—

5. 18; 35; $12\frac{6}{7}$. (C.H.L.) **6.** $\frac{2}{9}$; $\frac{1}{4}$; $\frac{3}{8}$. (C.W.)

7. ·45; ·8; ·369. (O.S.L., K.S.) **8.** $\frac{2}{3}$ of $\frac{3}{4}$ cwt.; $21\frac{5}{8}$ lbs.; $\frac{\frac{3}{4}-\frac{2}{3}}{\frac{8}{9}-\frac{2}{3}}$ ton. (C.W.)

Find a third proportional to—
9. (*a*) 5; ·05 (L.M.); (*b*) 7; 100; (*c*) $\frac{4}{5}$; $\frac{8}{15}$.

10. If 6 tons 16 cwt. of coals, bought at the pit's mouth for £4 5*s.*, are carried 108 miles at a charge of $\frac{3}{4}d.$ per ton per mile, what is the total cost per ton on delivery? (O.S.L.)

11. If a human heart expels 5 oz. of blood at each beat, and 28000 lbs. in the course of a day, how many beats would it make in a life of 72 yrs. long, each year consisting of $365\frac{1}{4}$ days? (K.S.)

12. In a book on arithmetic an example was printed thus: "If $42\frac{3}{4}$ yds. cost £12, how much will 114 yds. cost?" the denominator of the fraction being accidentally omitted. The answer at the end of the book was £32. Find the missing denominator. (P.T.)

13. A tree casts a shadow 16 ft. 6 in. long, while the shadow of a neighbouring tower is 49 ft. 6 in. If the height of the tower be 94 ft. 6 in., what is the height of the tree? (P.T.)

14. If 3 sheep per acre are sufficient for prairie land, how many may feed upon a plot 240 mi. long by 18 mi. broad? (C.P.)

15. If 12 guins. worth of water leak out of a reservoir in a week, what value would be lost at the same rate of leakage in 2 wks. 1 day 20 hrs. $42\frac{1}{2}$ min.? (C.S.L.)

16. If a penny weighs $\frac{1}{3}$ oz., find the weight in tons of a sum of £30800 in pence. (C.PR.)

17. Work the following sum by the rule of three and by "first principles" (the method of unity): If the carriage of 10 cwt. 14 lbs. for $79\frac{1}{4}$ mi. cost £7 17*s.* 6*d.*, what will it cost to have 1 ton 1 cwt. 1 qr. conveyed the same distance? (K.S.)

18. A soldier who enlisted on the day that he turned 18 years of age, will have, on retiring, a pension proportioned to the length of his

service. If he retires at 56 years of age, the pension will be $9\frac{1}{2}d.$ a day. At what age must he retire in order to have a pension of $8d.$ a day? (P.T.)

19. Silver was in demand at $41\frac{1}{8}d.$, showing a rise of $\frac{1}{16}d.$ per ounce. What was the value of 1000 oz. before the rise in the price? (P.T.)

20. If $(2\frac{1}{2}$ of $\frac{2}{3} \div \frac{5}{8}$ of $3\frac{1}{3})$ of an acre of land be worth 35 guins., what will 7 times $\left(\frac{7\frac{3}{8} - 4\frac{1}{4}}{\frac{1}{2} \text{ of } 3\frac{5}{6}} \times \frac{7\frac{3}{8}}{31\frac{1}{5}}\right)$ of $2\frac{1}{2}$ ac. be worth at the same rate? (P.T.)

21. Determine the height of Mont Blanc in old Parisian feet; its height being given as 15784 in English feet, and the ratio of the English to the old Parisian foot being as 1·0297 to ·9662. (L.M.)

22. The weight of a cubic inch of water is 253·17 grs.; that of a cubic inch of air ·31 gr. Find to three places of decimals how many cubic inches of water are equal in weight to 1 c. ft. of air. (C.S.L.)

23. If 1 c. ft. of water is equal to 6·25 gals. or 62·321 lbs., find the number of gallons and the weight of water contained in a cistern 4 ft. long by $2\frac{1}{2}$ ft. broad and 3 ft. deep. (O.R.)

24. If $\cdot\dot{2}$ of an acre of building land fetch $\cdot1\dot{5}\dot{6}$ of £198, what ought $\cdot2857\dot{1}\dot{4}$ of 7 ac. to fetch at the same rate? Give the proper name to each of the decimal fractions in this question. (K.S.)

25. If $\cdot857142\dot{8}$ of $1\cdot\dot{9}2\dot{5}$ of 13 cwt. 2 qrs. of bread costs £16 18*s.*, what should be the price of a 2-lb. loaf? (C.S.)

26. A boy is paid at a fixed rate per inch for the distance he jumps beyond 8 ft. If he is paid $5\frac{1}{2}d.$ for a jump of 9·26 ft., what should he be paid for a jump of $9\cdot\dot{2}\dot{7}$ ft.? (P.T.)

27. If the weights of the same bulks of mercury and glycerine be as 11 to 1, what will be the height of a glycerine barometer when the mercurial barometer stands at $29\cdot4\dot{5}$ in.? (C.M.)

28. A boy thinks that whenever there are three numbers in a sum, they are three terms of a proportion. He is asked, "If a man's income, after deducting $27\cdot9d.$ in the £, is £700, what would it have been after deducting twice as much?" His answer is "£1400." Is his answer too great or too little, and by how much? (P.T.)

29. Sound travels at the rate of 1140 ft. per second. An engine travelling towards a signal box at the rate of 50 mi. an hour, whistles when 684 yds. away. How far is the engine from the box when the signalman hears the whistle? (P.T.)

30. A snail starts on June 1 to climb a wall at the rate of 2 ft. per day, but during each night he slips down 18 in. while he sleeps. If the wall be 12 ft. high, on the evening of what day will he first reach the top? (P.T.)

31. A steamer which starts regularly at 11.35 a.m., and should perform a certain journey in $4\frac{1}{4}$ hrs., is fined so much a minute for arriving late. When it arrives at 5.10 p.m. it is fined £2 6*s.* $8d.$ What is the steamer fined when it arrives at 6 p.m.? (P.T.)

32. If 2 cub. in. of mercury weigh 1 lb. avoirdupois, and if 100 cub. in. of air weigh 31 grs., find how many miles in height a column of air will be which has the same weight as, and stands on a base of the same area as, a column of mercury 29·388 in. high. (C.S.L.)

33. A girl was absent from illness 39 days out of a term of 63 days, the school fees were £4 19*s.* $6d.$, and 12*s.* were deducted from the fees. What fraction of the proportionate amount did the school authorities remit? (S.A.)

34. The death rate in a town is 16·25 per 1000 of population for a half-year of 26 wks., and 803 persons die in a year of 365 days. What is the population? (C.S.)

35. Cleopatra's Needle on the Thames Embankment, which is erected upon a pedestal 16 ft. high, casts a shadow of 23 yds. $1\frac{2}{3}$ ft.; the pedestal at the same time throws one of 4 yds. $2\frac{2}{3}$ ft. What is the total height of the Needle and the pedestal? (P.P.)

36. The number of grains in 1 lb. troy is to the number in 1 lb. avoirdupois as 144 : 175. Determine the weight avoirdupois that will equal 21 lbs. 1 oz. 15 dwts. troy. (C.P.)

37. A certain piece of excavation cost £1347. How much would it have cost if the wages of the workmen per day had been increased by $\frac{1}{8}$, and the length of the working day increased by $\frac{1}{20}$?

38. What must be the cost per 1000 of producing oysters if a dozen can be sold for $9\frac{1}{2}d.$ at a profit of $3d.$? (K.S.)

39. Watching Helen and Annie at their sewing, I found that Helen had made 92 stitches more than Annie, and that together they had made 1104 stitches in 9 min. 12 sec. How many could Helen make in 13 min.? (K.S.)

40. If 35 men can do a piece of work in 45 days, and if 7 men drop off every 15 days, find what time will elapse before the work is finished. (C.S.)

41. If 6 men and 2 boys build a wall in 27 days, in how many days will 27 men and 9 boys build it? (P.T.)

42. If it cost as much to feed 3 men as to feed 4 boys, and for 3 boys the cost is found to be $19s.$ $2\frac{1}{4}d.$ per week, how much per week will it be for 51 men.? (C.J.)

43. If 3 men with 4 boys earn £5 $16s.$ in 8 days, and 2 men with 3 boys earn £4 in the same time, in what time will 6 men and 7 boys earn 20 guins.? (C.P.)

44. A town is provisioned for a siege of 6 wks., its population being 20000. How many people must leave the town in order that, at the same rate of consumption per head, the provisions may last for 8 wks.? (P.T.)

45. A fort had provisions for 200 soldiers for 30 days. After 5 days 50 more men entered the fort. How long will the food last at the same rate? (C.W.)

46. A besieged town has provisions to last 3 wks. Its population is 22400. How many people must be sent away in order that the provisions may last for 7 wks.? (P.T.)

47. A garrison of 2000 men has provisions for 54 days; at the end of 15 days a reinforcement arrives, and it is found that there are now provisions for only 20 days. What was the reinforcement? (K.S.S.)

48. A garrison of 1000 men, provisioned for 60 days, was reinforced at the end of 18 days, and the provisions were exhausted at the end of 30 days from that time. Of how many men did the reinforcement consist? (K.S.)

49. A garrison of 1800 men is provisioned for 12 wks. How long will the provisions last if 600 more men are introduced, but the daily allowance per man is diminished by one-third? (C.W.S.)

50. A besieged town, containing 5000 people, has provisions for 30 days at the rate of 12 oz. for each person per day. Suppose a quarter of the population leave the town, and each person left is allowed 15 oz. per day, determine the time the provisions will now last. (P.T.)

CHAPTER XIX.

COMPOUND PROPORTION.

Definition.—Compound proportion may be defined as the equality of ratios, one of the ratios being compounded.

The student should carefully distinguish between—
(*a*) Compound ratio (p. 170).
(*b*) Compounded proportion (p. 187).
(*c*) Compound proportion, the subject of this chapter.

For the solution of questions in this rule, it is necessary only to compound two or more ratios and then proceed as in simple proportion. The study of an example in compound proportion will show it to be composed of a certain number of simple proportions, with a common third term, all combined into one operation.

Example.—*If it cost* £11 5*s. to provide oats for* 7 *horses for* 20 *days, when oats are at* 16*s.* 6*d. per quarter, how much will it cost for* 55 *horses for* 12 *weeks, when oats are* 13*s.* 6*d. per quarter?* (K.S.)

This problem can be analyzed into three simple proportions.
(*a*) If it cost £11 5*s.* to provide oats for 7 *horses*, how much will it cost to provide oats for 55 *horses*?
(*b*) If it cost £11 5*s.* to provide oats for 20 *days*, how much will it cost to provide oats for 12 *weeks*?
(*c*) If it cost £11 5*s.* to provide oats when oats are 16*s.* 6*d. per quarter*, how much will it cost when oats are 13*s.* 6*d. per quarter*?

Stating the above as three simple proportions, we obtain—

(*a*) 7 horses : 55 horses :: £11$\frac{1}{4}$: required number of pounds.
(*b*) 20 days : 84 days :: £11$\frac{1}{4}$: „ „ „
(*c*) 16$\frac{1}{2}$*s.* : 13$\frac{1}{2}$*s.* :: £11$\frac{1}{4}$: „ „ „

Arranging in the conventional form for compound proportion, we have—

$$\left.\begin{array}{r} \text{horses } 7 : 55 \\ \text{days } 20 : 84 \\ \text{shillings } 16\frac{1}{2} : 13\frac{1}{2} \end{array}\right\} :: £11\tfrac{1}{4} : £x$$

$$x = £\frac{55 \times 84 \times 13\frac{1}{2} \times 11\frac{1}{4}}{7 \times 20 \times 16\frac{1}{2}} = £\frac{1212\,15}{4} = £303\ 15s.$$

Ans. £303 15*s.*

Note.—The compound ratio is $\frac{5.5}{7} \times \frac{8.4}{2.0} \times \frac{13\frac{1}{2}}{16\frac{1}{2}}$; the simple ratio is $\frac{x}{11\frac{1}{4}}$. It will be seen that in the compound ratio three antecedents are compounded for a new antecedent, and three consequents for a new consequent.

Rule.—From a study of the above example, we may deduce the rule—

1. Write down in the third term that quantity which is of the same kind as the answer.
2. Take any two other terms of like kind, and with the third term make a simple proportion, and proceed in the same way with every other pair of like terms.
3. Arrange all the antecedents under one another in a column, and all the consequents similarly in another column.
4. Multiply all the second terms (the consequents) and the third term together, and divide the product by all the first terms (the antecedents).

Notes.—(1) Any first term may be cancelled into any second term, or into the third term, at any stage of the multiplication or division.

(2) The first and second terms in each statement must, when of different denominations, be reduced to the same denomination; and the common third term, when compounded, must be reduced to the lowest denomination, unless it can be conveniently expressed as a fraction of a higher denomination.

Proof of Compound Proportion.—If the corresponding ratios of two or more proportions are compounded, the resulting ratios will form a proportion.

Arithmetical Proof.

For $3 : 4 = 6 : 8$, or $\frac{3}{4} = \frac{6}{8}$
and $5 : 9 = 10 : 18$, or $\frac{5}{9} = \frac{10}{18}$
and $2 : 3 = 14 : 21$, or $\frac{2}{3} = \frac{14}{21}$

$\therefore \frac{3}{4} \times \frac{5}{9} \times \frac{2}{3} = \frac{6}{8} \times \frac{10}{18} \times \frac{14}{21}$

i.e. $(3 \times 5 \times 2) : (4 \times 9 \times 3) :: (6 \times 10 \times 14) : (8 \times 18 \times 21)$

i.e. $30 : 108 :: 840 : 3024$

i.e. $5 : 18 :: 5 : 18$

General Proof.

If $a : b = c : d$ and $e : f = g : h$ and $k : l = m : n$ then $aek : bfl = cgm : dhn$

For since $a : b :: c : d \ \therefore \ \frac{a}{b} = \frac{c}{d}$

Similarly, $\frac{e}{f} = \frac{g}{h}$ and $\frac{k}{l} = \frac{m}{n}$

$\therefore \frac{a}{b} \times \frac{e}{f} \times \frac{k}{l} = \frac{c}{d} \times \frac{g}{h} \times \frac{m}{n}$

i.e. $\frac{aek}{bfl} = \frac{cgm}{dhn}$

or $aek : bfl :: cgm : dhn$

Double Rule of Three.—Compound proportion is often so named, but the name is not a good one, for it is sometimes incorrect. It is often Double, and often something else—Treble or Quadruple Proportion.

Compound Proportion and the Unitary Method.

Problems in compound proportion may be solved by the

Unitary Method, which affords good mental training, especially in involved cases.

Example.—*If* 180 *men in* 6 *days, working* 10 *hrs. each day, can dig a trench* 200 *yds. long,* 3 *yds. wide, and* 2 *yds. deep, in how many days, of* 8 *hrs. each, will* 100 *men dig a trench* 360 *yds. long,* 4 *yds. wide, and* 3 *yds. deep?*

First Solution.

Time for 180 men to dig $200 \times 3 \times 2$ c. yds. at 10 hrs. a day $= 6$ days

Time for 1 man to dig $200 \times 3 \times 2$ c. yds. at 10 hrs. a day $= 6 \times 180$ days

Time for 100 men to dig $200 \times 3 \times 2$ c. yds. at 10 hrs. a day $= \frac{6 \times 180}{100}$ days

Time for 100 men to dig 1 c. yd. at 10 hrs. a day $= \frac{6 \times 180}{100 \times 200 \times 3 \times 2}$ days

Time for 100 men to dig $360 \times 4 \times 3$ c. yds. at 10 hrs. a day $= \frac{6 \times 180 \times 360 \times 4 \times 3}{100 \times 200 \times 3 \times 2}$ days

Time for 100 men to dig $360 \times 4 \times 3$ c. yds. at 1 hr. a day $= \frac{6 \times 180 \times 360 \times 4 \times 3 \times 10}{100 \times 200 \times 3 \times 2}$ days

Time for 100 men to dig $360 \times 4 \times 3$ c. yds. at 8 hrs. a day $= \frac{6 \times 180 \times 360 \times 4 \times 3 \times 10}{100 \times 200 \times 3 \times 2 \times 8}$ days

$= 48\frac{3}{5}$ days

Ans. $48\frac{3}{5}$ days.

Second Solution.

*Time for 180 *men* to dig $200 \times 3 \times 2$ c. yds. at 10 hrs. a day $= 6$ days

Time for 1 *man* to dig $200 \times 3 \times 2$ c. yds. at 10 hrs. a day $= 6 \times 180$ days

Time for 1 man to dig 1 *c. yd.* at 10 hrs. a day $= \frac{6 \times 180}{200 \times 3 \times 2}$ days

*Time for 1 man to dig 1 c. yd. at 1 *hr.* a day $= \frac{6 \times 180 \times 10}{200 \times 3 \times 2}$ days

Time for 100 *men* to dig 1 c. yd. at 1 hr. a day $= \frac{6 \times 180 \times 10}{200 \times 3 \times 2 \times 100}$ days

Time for 100 men to dig $360 \times 4 \times 3$ *c. yds.* at 1 hr. a day $= \frac{6 \times 180 \times 10 \times 360 \times 4 \times 3}{200 \times 3 \times 2 \times 100}$ days

*Time for 100 men to dig $360 \times 4 \times 3$ c. yds. at 8 *hrs.* a day $= \frac{6 \times 180 \times 10 \times 360 \times 4 \times 3}{200 \times 3 \times 2 \times 100 \times 8}$ days

$= 48\frac{3}{5}$ days

Ans. $48\frac{3}{5}$ days.

In the first solution, as soon as *one* quantity in the *statement portion* of the problem is reduced to unity, the quantity in the *question portion* of the problem is introduced. In the second solution *all* the quantities in the *statement* portion of the problem are reduced to unity before any of the quantities in the question portion are introduced.

The Fractional Form.—The advocates of this method of doing problems in compound proportion claim that it is easy of comprehension,

* Subsequently the process might be reduced to these three statements, the intervening steps being omitted.

that its superiority is especially marked in complex problems, and that it does away with the unnecessary distinction between simple and compound. The student will, however, need to remember two rules of variation.

1. *Direct Variation.*—Two quantities are said to vary directly when one increases (or decreases) as the other increases (or decreases).
2. *Indirect Variation.*—Two quantities vary indirectly when one increases as the other decreases.

Example.—*If* 44 *men can build a wall* 308 *yds. long in* 14 *days of* 8 *hrs. each, how long will it take* 25 *men to build a wall* 200 *yds. long, working* 9 *hrs. a day?*

44 men build a wall 308 yds. long in 14 days of 8 hrs. each
25 „ „ 200 „ „ ? „ 9 „

then $\frac{\cancel{44}^{\,1} \times \cancel{14}^{\,1} \times 8 \times \cancel{200}^{\,\cancel{100}^{\,50}}}{\cancel{308}^{\,\cancel{154}^{\,77}} \times \cancel{25}^{\,5} \times 9}$ days $= 1\frac{28}{9}$ days $= 14\frac{2}{9}$ days. *Ans.*

To understand the above statement, we must note that the required number of days will vary—

(1) Indirectly as the number of men.
(2) Directly as the length.
(3) Indirectly as the hours per day.

Rule.—This can be deduced from the above example.

1. Write down the statement of given facts.
2. Underneath write the query part of the problem.
3. The formation of the fraction will depend upon the laws of variation.

(*a*) Place the odd term in the numerator.
(*b*) Take like terms in pairs, and see whether each pair varies directly or indirectly with the odd term.
(1) If directly, invert the pair of terms.
(2) If indirectly, place the figures without inverting them.

4. The fraction should then be simplified.

The Detection of Misplaced Terms.—Students, when working problems from an arithmetic, fail sometimes to get the right answer. Some then "dodge" for the answer by "trying it another way," *i.e.* they change the order of their terms. There is no need to indulge in this reprehensible practice if the following method of detection is mastered—

Take the right answer from the book and the wrong answer; form a ratio or fraction of the two answers, placing the incorrect one in the numerator and the correct one in the denominator; then the fraction thus formed will always be—

(1) A square quantity if one ratio is misplaced.
(2) The product of two square quantities if two ratios are misplaced.

Example.—*If the carriage of* $8\frac{1}{2}$ *tons for* 102 *miles cost* £7 4*s.* 6*d.*, *what weight can be carried* 63 *miles for* 5 *guins.?* (O.S.L.)

Correct solution—

$$\left.\begin{matrix}\text{miles } 63 : 102 \\ \text{sixpences } 289 : 210\end{matrix}\right\} :: 8\tfrac{1}{2} \text{ tons} : x \text{ tons}$$

$$x = \frac{\overset{2}{\cancel{102}} \times \overset{10}{\cancel{210}} \times \overset{17}{\cancel{289}}}{\underset{21}{\cancel{63}} \times \underset{17}{\cancel{289}}} = 10 \text{ tons}.$$

Incorrect solution—

$$\left.\begin{matrix}\text{miles } 102 : 63 \\ \text{sixpences } 289 : 210\end{matrix}\right\} :: 8\tfrac{1}{2} \text{ tons} : x \text{ tons}$$

$$x = \frac{\overset{21}{\cancel{63}} \times \overset{105}{\cancel{210}} \times \overset{17}{\cancel{289}}}{\underset{34}{\cancel{102}} \times \underset{17}{\cancel{289}}} = \tfrac{2205}{578} \text{ tons}$$

$$\text{Ratio of the two answers } \frac{2205}{578 \times 10} = \frac{441}{1156} = \frac{21 \times 21}{34 \times 34}$$

Since 21 appears in the numerator and not in the denominator, the fraction has been multiplied by 21×21.

And since 34 appears in the denominator and not in the numerator, the fraction has been divided by 34×34.

Hence the ratio 21 : 34 or 63 : 102 is misplaced in the incorrect solution.

THEORY.

1. Give the rule for stating a sum in the double rule of three. (C.W.)

2. State and explain the rule for working sums in compound proportion. (K.S., P.T.)

3. Explain fully how simple proportion differs from compound proportion. (C.W.S.)

4. "The double rule of three is a shorter method of working out such questions as would require two or more applications of the rule of three." Exemplify this statement. (C.W.)

5. How would you explain the following sum?—If 7 men can earn £9 9*s.* in 6 days, how much will 11 men earn in 4 days at the same rate? (K.S.)

6. Define compound proportion, and say why it is sometimes called the double rule of three. Comment on the accuracy of the latter title.

7. Explain clearly the *principle* on which the method of compound proportion is based.

8. Give a general proof of the method of compound proportion, and test your answer numerically.

9. What advantages are claimed for the fractional form in solving problems in compound proportion, and on what rules of variation does the fractional method depend?

10. A problem in compound proportion yields an inaccurate answer through misplaced terms. How would you detect the misplaced terms?

EXERCISE XXXVI. 35

Compound Proportion.

1. If £16 will maintain a family of 8 persons for 11 wks. 3 days, for how long a time (in similar circumstances) will £21 maintain a family of 12 persons? (K.S.)

2. A contractor, having undertaken to make a railway in one year, employs 750 men for $4\frac{1}{2}$ mths., and then finds he has done $\frac{3}{10}$ of his work. How many men must he employ for the remaining time in order to complete his contract as agreed? (C.W.)

3. A boat with 2720 sq. ft. of sail spread travels 51 knots in $8\frac{1}{2}$ hrs. when the wind gives a pressure of ·125 lb. per square foot. How far will it go in $2\frac{2}{3}$ hrs. with a sail area of 7650 sq. ft. and a wind pressure of ·5 lb. per square foot, assuming the speed to be proportioned to the area of the sail and the wind pressure? (C.W.)

4. The value of the wheat grown in a field was £302 8*s.* when the yield was 4 qrs. to the acre, and the wheat worth 7*s.* per bushel. Find the value of the wheat grown in the same field when the yield is 5 qrs. to the acre, and the wheat worth 23*s.* per quarter. (C.P.)

5. A garrison of 4500 men is provisioned for 15 wks. at the rate of 13 oz. per diem per man. How many men must leave, that the same provisions may last 27 wks. at 10 oz. per diem each man? (C.S.)

6. A garrison of 1140 men is provisioned for 40 days. After 16 days on full rations the garrison is reduced by 152 men, but the existing provisions are increased by $\frac{1}{12}$ by capture. For how many days can the reduced garrison now hold out? (O.J.)

7. A town, which is defended by 1200 men, with provisions enough to sustain them 42 days, supposing each man to receive 18 oz. a day, obtains an increase of 200 men to its garrison. What must now be the allowance to each man, in order that the provisions may serve the whole garrison for 54 days? (P.T.)

8. If 1200 men besieged in a town with provisions for 7 wks. at 12 oz. for each man daily, be reinforced by 600 men, and the whole force be put on 7 oz. a day, how long can the town hold out? (C.S.)

9. 600 men were besieged in a garrison town with provisions for 35 days, allowing each man 18 oz. a day. How long would the provisions last if the garrison were reinforced by 300 men, and the daily allowance were reduced to 15 oz. a day? (P.T.)

10. If 18 pumps can raise 2150 tons of water in 10 days, working 7 hrs. a day, in how many days will 16 pumps raise 1634 tons, working 9 hrs. a day? How many hours of work is your answer equal to? (K.S.)

11. If a travelling party of 5 persons spend £36 16*s.* 8*d.* in 17 days, how much would another party of 8 persons spend in 19 days, supposing the daily expenditure of each of the latter party to be half as much again as that of each one in the former party? (C.S.)

12. Taking two steps a second, each step being 2 ft. 9 in. long, I walk a certain distance in $3\frac{3}{5}$ hrs. In what time shall I walk the same distance if I take 132 steps a minute, and each step is 2 ft. $10\frac{1}{2}$ in. long? (K.S.)

13. If 75 men can perform a piece of work in 12 days of 10 hrs. each, how many men will perform a piece of work twice as large in a tenth

part of the time, if they work the same number of hours in the day, supposing that 2 of the second set can do as much work in an hour as 3 of the first set? (O.W.)

14. If 5 men can perform a piece of work in 12 days of 10 hrs. each, how many men will perform a piece of work 4 times as large in $\frac{1}{5}$ part of the time, if they work the same number of hours in the day, supposing that 2 of the second set can do as much work in an hour as 3 of the first set? (C.S.)

15. If the cost of printing a book of 320 leaves, with 21 lines on each page, and on an average 11 words in each line, be £19, find that of printing a book with 297 leaves, 28 lines on each page, and 10 words in each line. (O.S.L.)

16. If 7 reams of paper are required for 875 copies of a pamphlet of 22 pages, how many reams will be required for 1000 copies of 33 pages each, a page of the latter being $\frac{2}{3}$ the size of a page of the former? (C.S.)

17. Each page of a book of 560 pages contains 40 lines, and each line contains on an average 41 letters. How many pages will there be in another edition of the book consisting of pages of 60 lines, each line one quarter longer, and printed in type occupying more space on the line in the proportion of 6 to 5 than that used in the smaller edition? (C.S.)

18. If 20 men can mow a field of 7 ac. 3 rds. in $8\frac{1}{2}$ days of 10 hrs. each, how many fields of half the area can 255 men mow in 10 days of 8 hrs. each? (K.S.)

19. If 44 labourers can do a piece of work in 15 days of 10 hrs. each, how many navvies must be employed to do $\frac{3}{4}$ more work in 7 days of 11 hrs., supposing 3 navvies able to do the work of 5 labourers? (O.S.L.)

20. If 30 men working 8 hrs. per day at a uniform rate earn £72 in 9 days, how much will 63 men earn, who work for 13 hrs. per day at $\frac{2}{3}$ of the same uniform rate, in 10 days, the wages being proportional to the amount of work done? (C.P.)

21. If 10 men can put down a pavement 450 ft. long by 2 yds. broad in 9 days, working 10 hrs. a day, how many men can put down twice that amount of pavement, working for 15 days at 8 hrs. a day? (C.W.)

22. A person can walk a certain distance in $4\frac{1}{2}$ hrs., taking 3 steps of 32 in. each in 2 sec. How many steps of a yard each must he take in a minute to walk half that distance in 2 hrs.? (C.S.)

23. A copyist can transcribe 3 pages of a certain work in $1\frac{1}{2}$ hr. How long will it take 3 men, working only half as rapidly, to copy 36 pages of another volume, the pages of which contain $\frac{1}{3}$ as much again as those of the former? (O.S.L.)

24. If 7 men can plant 945 rows of potatoes in 3 days of 9 hrs. each, how many women will plant 1200 rows in 5 days of 8 hrs., if 4 women can do as much as 3 men?

25. Find how many days it will take 150 men to excavate a canal 1 Irish mile long, 21 ft. wide, and 9 ft. deep, 75 days having been required by 210 men to excavate a canal 600 yds. long, 24 ft. wide, and 6 ft. deep. (I.I.)

26. If 55 men in 6 days of 7 hrs. each can dig a trench 105 yds. long and $5\frac{1}{2}$ yds. wide, how many hours a day must 96 men work in order to dig in 8 days a trench 160 yds. long, 6 yds. wide, and of the same depth as the former one? (O.S.L.)

27. If 12 masons working 8 hrs. a day can build a wall 6 ft. high

and 102 yds. long in 9 days, how long will it take 14 masons working 9 hrs. a day to build a wall 8 ft. high and 119 yds. long? (C.W.)

28. If 18 men can dig a trench 200 yds. long, 3 yds. wide, and 2 yds. deep in 6 days of 10 hrs. each, in how many days of 8 hrs. each will 10 men dig a trench 100 yds. long, 4 yds. wide, and 3 yds. deep? (P.T.)

29. If 24 men in 6 days of 8 hrs. each can dig a trench 140 yds. long, 4 yds. wide, and 2 yds. deep, in how many days of 9 hrs. will 45 men dig a trench 3 yds. wide, 4 yds. deep, and 490 yds. long? (C.S.)

30. If 6 compositors in 16 days of $10\frac{1}{2}$ hrs. each can set in type 720 pages, each of 60 lines with 40 letters in a line, in how many days of 7 hrs. each will 9 compositors set 960 pages, each of 45 lines, with 50 letters in a line? (C.J.)

31. If 48 pioneers in 5 days of $12\frac{1}{2}$ hrs. can dig a trench 139·75 yds. long, $4\frac{1}{2}$ yds. wide, and $2\frac{1}{2}$ yds. deep, how many hours per day must 360 pioneers work during 42 days in order to dig a trench $4910\frac{1}{16}$ yds. long by $4\frac{7}{8}$ yds. wide and $3\frac{1}{3}$ yds. deep? (P.T.)

32. If a wall 660 yds. long, 10 ft. high, and $22\frac{1}{2}$ in. thick, be built by 27 men in $31\frac{1}{4}$ days of 11 hrs. each, in how many days of 12 hrs. each would 60 men build a wall 12 ft. high and $2\frac{1}{4}$ ft. in thickness round a park $4\frac{1}{2}$ miles in circumference? (P.T.)

33. If 24 men can excavate 840 c. yds. in 5 days, working 7 hrs. a day, how many hours a day will 20 men have to work to excavate 750 c. yds. in 4 days? Any 5 of the former can do as much work in 3 days as any 4 of the latter in 4 days. (I.I.)

34. If 120 men working 10 hrs. a day can make an embankment 90 yds. long, 20 ft. wide, and 8 ft. high in 5 days, how many days will it take 500 men working 9 hrs. a day to make an embankment 1200 yds. long, 25 ft. wide, and 9 ft. high? (C.S.)

35. If 3 men working 4 hrs. a day can build a wall 80 ft. long, 6 ft. high, and 2 ft. thick in 5 days, how many men working $3\frac{1}{2}$ hrs. a day will be required to build a wall of similar materials 100 ft. long, 7 ft. high, and 4 ft. thick in 25 days? (C.S.)

36. If 124 men in $5\frac{1}{2}$ days of 11 hrs. each dig a trench $232\frac{1}{2}$ yds. long, 3 ft. 8 in. wide, and 2 ft. 4 in. deep, in how many days of 9 hrs. each will 36 men dig a trench $337\frac{1}{2}$ yds. long, 5 ft. $7\frac{1}{2}$ in. wide, and 3 ft. 6 in. deep, supposing that in the same time each of the second set of men does $\frac{1}{8}$ more work than each of the first set? (C.S.)

37. If 3 men, 4 women, and 5 children can reap a field of 9 ac. 2 rds. in 6 days, working 10 hrs. a day, in what time can 7 men, 6 women, and 11 children reap a field of 14 ac. 1 rd., working 12 hrs. a day, supposing the work done by a man, woman, and child to be in the ratio of 3 : 2 : 1? (C.S.L.)

38. 6125 bricks will just suffice for the erection of a wall of a certain length, breadth, and thickness. How many bricks half the length, half the breadth, and twice the thickness of these will be required in the building of another wall twice the length, twice the thickness, and half the height of the former one?

39. If a company of 160 men, in 6 days of 11 hrs. each, can dig a trench 230 yds. long, $5\frac{1}{2}$ yds. wide, $1\frac{1}{2}$ yd. deep, in how many days of 8 hrs. each would 96 men dig a trench 220 yds. long, $3\frac{1}{2}$ yds. wide, and 1 yd. deep, supposing the hardness of the ground in the former case is to

that of the latter as 5 to 7, and that 4 men of the latter company can do as much work as 5 of the former in the same time? (C.S.)

40. If 3 men and 5 women can together do a piece of work in 18 days, and the work of 2 men is equal to that of 3 women, in how many days could 6 men do the whole of the work by themselves? (C.S.)

41. If 224 equally capable men can do a certain quantity of work in 9 days of 10 hrs. each, how many of them will be required to do half that quantity of similar work in 6 days of 8 hrs. each, assuming that each one of the men does $\frac{1}{5}$ more work per hour when he works 8 hrs. per day than when he works 10 hrs. per day? (C.P.)

42. Five men are employed 7 hrs. a day on a certain work. After they have been engaged 4 days it is found they have done $\frac{1}{2}$ of the whole. Six additional men are then employed, and all are to work $7\frac{1}{2}$ hrs. per day. When will the work be completed, supposing that they all work equally fast? (C.S.)

43. If 8 men and 5 boys complete a work in 8 days of $7\frac{1}{2}$ hrs. each, in how many days of 10 hrs. each could 5 men and 8 boys do $\frac{3}{4}$ of the work (1 man being able to do the work of 2 boys)? (I.I.)

44. If 6 men and 9 boys working 12 hrs. a day make a drain 225 yds. long in 9 days, in how many days can 10 men and 12 boys working $10\frac{1}{2}$ hrs. per day make a drain 175 yds. long, the work of a boy being $\frac{2}{3}$ that of a man? (I.I.)

45. Three men can do as much as 5 boys; the wages of 3 boys are equal to those of 2 men. A work on which 40 boys and 15 men are employed takes 8 wks. and costs £350. How long would it take if 20 boys and 20 men were employed, and how much would it cost? (C.S.L.)

46. If 12 men can do a certain amount of hoeing in 9 days, working 10 hrs. per day, how many men should a farmer set to work to do three times that amount of hoeing in 15 days, working 8 hrs. per day, he being aware that 3 men will join them at the beginning of the fourth day, and 4 more men at the beginning of the tenth day? (C.P.)

47. If type-setters' wages are raised from 35*s.* to 2 guins. per week, and they work on an average 8 hrs. instead of 10 hrs. per day, how much faster must they work in order that the cost of type-setting may be the same as before? (C.P.)

48. The value of a Spaniard's work to that of a Frenchman's being as 3 : 4, and their rate of daily wages being 5*s.* and 7*s.* respectively, find the gain obtained by employing 100 Spaniards for 21 days instead of Frenchmen. (K.S.)

49. If 5 mules and 2 horses can remove 50 tons of hay in a certain time, find how many tons 7 horses and 3 mules will remove in the same time, 4 mules being equal to 3 horses. (I.I.)

50. A trench 920 ft. long, 17 ft. wide, and 10 ft. deep has been dug by 7 men and 2 boys; the work could have been done in the same time by 6 men and 5 boys. What length of a trench 15 ft. wide and 12 ft. deep could be dug by 5 men and 3 boys in half the time? (C.S.)

CHAPTER XX.

PROPORTIONAL PARTS.

Definition.—By division into proportional parts is meant the division of a given number or quantity into parts in a given ratio.

Suppose it is desired to divide 1000 units into 5 parts in the ratio of 2 : 3 : 5 : 6 : 4. This means that for every 2 units the first part has, the second has 3, the third 5, the fourth 6, and the fifth 4. If 20 of these units are taken, 2 can be given to one, 3 to another, 5 to another, 6 to another, and 4 to another; hence, as many 20's as there are, so many 2's are there, so many 3's are there, so many 5's, 6's, and 4's. In 1000 units there are $\frac{1000}{20}$, or 50 groups of 20 each; hence—

The first part has	2×50	or	100	units.
,, second ,,	3×50	,,	150	
,, third ,,	5×50	,,	250	
,, fourth ,,	6×50	,,	300	
,, fifth ,,	4×50	,,	200	

The sum of these units is 1000, showing that the several "answers" are probably correct.

We will now apply the knowledge obtained to some examples.

I. Given the sum and the ratios, to find the parts.

Example.—*Divide* 5768 *in the ratio of* 2, 3, 4, and 5.

(1) Add together the numbers to which the parts are to be proportional.

$$\text{Sum of parts} = 2 + 3 + 4 + 5 = 14$$

(2) Divide the given number 5768 by this sum of parts.

$$\frac{5768}{14} = 412$$

(3) Multiply the quotient obtained in (2) by each of the given numbers in succession, and the products obtained will be the required parts.

$$412 \times 2 = 824$$
$$412 \times 3 = 1236$$
$$412 \times 4 = 1648$$
$$412 \times 5 = 2060$$

Or, better still, the problem may be stated as a proportion; *e.g.*—

Sum of numbers : one of them :: the whole quantity : proportional part of it

14	:	2	::	5768	:	824
14	:	3	::	5768	:	1236

and so on.

Ans. 824, 1236, 1648, 2060.

The same method is applicable to fractions.

Example.—*Divide* £287 *among A, B, and C in the proportions of* $\frac{3}{7}$, $\frac{4}{21}$, $\frac{5}{14}$.

$$\textit{Sum of fractions} = \tfrac{3}{7} + \tfrac{4}{21} + \tfrac{5}{14} = \tfrac{18}{42} + \tfrac{8}{42} + \tfrac{15}{42}$$

Now, insomuch as the common denominator (42) merely gives its *name* to the parts (see p. 60), it may be disregarded, and the *parts* alone considered.

$$\textit{Sum of parts} = 18 + 8 + 15 = 41$$

Then, sum of parts : one of them :: the whole quantity : proportional part of it

then 41	:	18	::	£287	: A's share, or	£126	} *Ans.*
and 41	:	8	::	£287	: B's ,,	£56	
and 41	:	15	::	£287	: C's ,,	£105	

Proof.—(1) The sum of the parts found should equal the given amount, *e.g.*—

$$£126 + £56 + £105 = £287$$

(2) The numbers must be proportional to the given numbers—

$$126 : 56 :: \tfrac{3}{7} : \tfrac{4}{21}, \text{ or } 126 \times \tfrac{4}{21} = 56 \times \tfrac{3}{7} \text{ or } 24 = 24$$
$$56 : 105 :: \tfrac{4}{21} : \tfrac{5}{14}, \text{ or } 56 \times \tfrac{5}{14} = 105 \times \tfrac{4}{21} \text{ or } 20 = 20$$
$$126 : 105 :: \tfrac{3}{7} : \tfrac{5}{14}, \text{ or } 126 \times \tfrac{5}{14} = 105 \times \tfrac{3}{7} \text{ or } 45 = 45$$

To be correct, the parts found must answer *both* these tests.

Example.—*The sum of* $7\frac{1}{2}d.$ *was divided among A, B, and C in such proportion that A received* $1\frac{1}{2}d.$ *more than C, and B* $2\frac{1}{4}d.$ *less than C. Had a sovereign been divided among them in the same proportion, what would each have received?* (C.W.)

C receives $2\frac{1}{4}d.$ more than B, but $1\frac{1}{2}d.$ less than A.

$$\therefore \text{ C's share} = \tfrac{1}{3}(7\tfrac{1}{2} + 2\tfrac{1}{4} - 1\tfrac{1}{2})d. = \tfrac{1}{3} \times 8\tfrac{1}{4}d. = 2\tfrac{3}{4}d.$$
$$\text{and A's ,, } = 2\tfrac{3}{4}d. + 1\tfrac{1}{2}d. = 4\tfrac{1}{4}d.$$
$$\text{and B's ,, } = 2\tfrac{3}{4}d. - 2\tfrac{1}{4}d. = \tfrac{1}{2}d.$$

Then $240d.$ have to be divided in the proportion of $4\frac{1}{4}d.$, $\frac{1}{2}d.$, and $2\frac{3}{4}d.$

$$\text{Sum of fractions} = 4\tfrac{1}{4} + \tfrac{1}{2} + 2\tfrac{3}{4} = 4\tfrac{1}{4} + \tfrac{2}{4} + 2\tfrac{1}{4}$$
$$\text{sum of parts} = 17 + 2 + 11 = 30$$

Arranging in proportion form, we get—

Sum of parts : one of them :: the whole quantity : proportional part of it

30	:	17	::	240*d.*	: A's share, or	11*s.* 4*d.*	} *Ans.*
30	:	2	::	240*d.*	: B's ,,	1*s.* 4*d.*	
30	:	11	::	240*d.*	: C's ,,	7*s.* 4*d.*	

II. Given the ratios, but not the sum, to find the parts.

Example.—*A, B, and C jointly own an estate, their shares being as* $4 : 2\frac{1}{2} : 1\frac{1}{2}$. *A sells* $\frac{1}{8}$ *of his share to C, and C sells* 100 *acres to B, and their shares (B's and C's) are then equal. How many acres was each entitled to at first?* (C.S.)

$$4 + 2\tfrac{1}{2} + 1\tfrac{1}{2} = 8 = \text{sum of parts}$$

Then ratio of shares is $\frac{4}{8}$, $\frac{2\frac{1}{2}}{8}$, $\frac{1\frac{1}{2}}{8}$, or $\frac{8}{16}$, $\frac{5}{16}$, $\frac{3}{16}$ respectively. But A sells $\frac{1}{2}$ of his share to C, and the ratios then become respectively $\frac{4}{16}$, $\frac{5}{16}$, $\frac{7}{16}$.

C sells 100 ac. to B, and the ratios of the shares then become—

$$\tfrac{4}{16},\ \tfrac{5}{16} + 100 \text{ ac.},\ \tfrac{7}{16} - 100 \text{ ac.}$$

$$\text{and } \tfrac{7}{16} - 100 \text{ ac.} = \tfrac{5}{16} + 100 \text{ ac. (by question)}$$

$$\therefore \tfrac{2}{16} = 200 \text{ ac.}$$

$$\text{and } \tfrac{16}{16} \text{ or the whole} = \frac{200 \text{ ac.} \times 16}{2} = 1600 \text{ ac.}$$

and the parts, at first, were in the proportion of 8, 5, and 3.

$$\left.\begin{array}{l} \therefore \text{ A's share} = \tfrac{8}{16} \times 1600 \text{ ac.} = 800 \text{ ac.} \\ \text{and B's} \quad ,, \quad = \tfrac{5}{16} \times 1600 \text{ ac.} = 500 \text{ ac.} \\ \text{and C's} \quad ,, \quad = \tfrac{3}{16} \times 1600 \text{ ac.} = 300 \text{ ac.} \end{array}\right\} \textit{Ans.}$$

III. Mixed Denominations.

Example.—*How many half-sovereigns, half-crowns, and sixpences are together worth £579½, the coins being in the proportion of* $3\frac{1}{2}$, $4\frac{1}{4}$, *and* $5\frac{1}{3}$ *respectively?*

$$3\tfrac{1}{2} + 4\tfrac{1}{4} + 5\tfrac{1}{3} = \tfrac{7}{2} + \tfrac{17}{4} + \tfrac{16}{3} = \tfrac{42}{12} + \tfrac{51}{12} + \tfrac{64}{12}$$

Proportional integers are 42, 51, 64.

$$\left.\begin{array}{rl} \text{Value of these numbers in sixpences,} & 42 \times 20 = 840 \\ & 51 \times 5 = 255 \\ & 64 \times 1 = 64 \end{array}\right\} 1159$$

$$\text{and } £579\tfrac{1}{2} = 23180 \text{ sixpences}$$

$$\therefore \text{ value of } \tfrac{1}{2} \text{ sovereigns in sixpences} = \tfrac{23180}{1159} \times 840 = 16800$$

$$\text{and} \quad ,, \quad \tfrac{1}{2} \text{ crowns} \quad ,, \quad = \tfrac{23180}{1159} \times 255 = 5100$$

$$\text{and} \quad ,, \quad \text{sixpences} \quad ,, \quad = \tfrac{23180}{1159} \times 64 = 1280$$

16800 sixpences = £420 = 840 half-sovereigns
5100 ,, = £127½ = 1020 half-crowns
1280 ,, = £32 = 1280 sixpences

Ans. 840, 1020, and 1280 respectively.

IV. Examples involving Continued Proportion.

Several ratios may be so compounded that the consequent of one becomes the same number as the antecedent of the next; *e.g.* consider the following ratios:—

$$1 : 2, \quad 3 : 4, \quad 5 : 6$$

Now, $1 : 2 = 6 \times 1 : 6 \times 2$; *i.e.* $1 : 2 :: 6 : 12$
and $3 : 4 = 4 \times 3 : 4 \times 4$; *i.e.* $3 : 4 :: 12 : 16$
and $5 : 6 = \tfrac{6}{5} \times 5 : \tfrac{6}{5} \times 6$; *i.e.* $5 : 6 :: 16 : 19\tfrac{1}{5}$

The given ratios have been changed into the ratio of a continued proportion.

Given ratios, 1 : 2, 3 : 4, 5 : 6
Continued proportion, 6 : 12 :: 16 :: $19\tfrac{1}{5}$

To produce this result several ratios have been compounded, and it should be noted that the value of the compound ratio is the same as the ratio of the *first antecedent* to the *last consequent*, e.g. :—

$$\text{If } \overbrace{1 : 2 :: 2 : 3 :: 3 : 4 :: 4 : 5 :: 5 : 6}$$

$$\text{then } 1 \times 2 \times 3 \times 4 \times 5 : 2 \times 3 \times 4 \times 5 \times 6 :: 1 : 6$$

$$\text{and } \frac{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times 5}{\cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times 6} = \frac{1}{6}$$

$$i.e.\ \tfrac{1}{6} = \tfrac{1}{6}$$

This truth enables us to deal with certain problems involving the division of a sum of money in the ratios of a continued proportion. In the formation of the ratios it is desirable, so far as possible, to keep clear of fractions.

Example.—*Divide* £3900 *among A, B, C, D so that A's share may be to B's as* 3 : 4, *B's to C's as* 8 : 9, *and C's to D's as* 9 : 16.

A : B :: 3 : 4 :: 6 : 8 (multiplying by 2)
B : C :: 8 : 9
C : D :: 9 : 16
A : B : C : D = 6 : 8 : 9 : 16

The proportional parts are 6, 8, 9, 16, and their sum = 39.

Then A's share = $£\frac{3900}{39} \times 6 = £600$
B's ,, = $£\frac{3900}{39} \times 8 = £800$
C's ,, = $£\frac{3900}{39} \times 9 = £900$
D's ,, = $£\frac{3900}{39} \times 16 = £1600$

Ans. A, £600; B, £800; C, £900; D, £1600.

Compound Proportional Parts.

Example.—*A man, woman, and boy do a piece of work for which they get* £15 7*s.*; *the man works* 9 *days, the woman* 10, *and the boy* 12, *and the amount of work they do per day is in the proportion of* $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$. *How much money should each get?*

man = $9 \times \frac{1}{2} = \frac{9}{2}$
woman = $10 \times \frac{1}{3} = \frac{10}{3}$
boy = $12 \times \frac{1}{5} = \frac{12}{5}$

$$\text{sum of fractions} = \tfrac{9}{2} + \tfrac{10}{3} + \tfrac{12}{5} = \frac{135 + 100 + 72}{30}$$

Disregarding the denominator (p. 60), we have—

Sum of the parts = 135 + 100 + 72 = 307

$$\therefore \text{man's share} = \frac{£15\ 7s.}{307} \times 135 = £6\ 15s.$$

$$\therefore \text{woman's share} = \frac{£15\ 7s.}{307} \times 100 = £5$$

$$\therefore \text{boy's share} = \frac{£15\ 7s.}{307} \times 72 = £3\ 12s.$$

Ans. Man, £6 15*s.*; woman, £5; boy, £3 12*s.*

EXERCISE XXXVII. 37

Proportional Parts.

1. Divide £144 10*s.* among 3 persons in the proportion of ·3, ·33, and ·3. (C.S.)

2. A man having 3 sons, left £9656 to be divided among them in proportion to their ages at the time of his death. When he died, their ages were 25, 22, and 21 years respectively. What was the share of each? And what difference would it have made to each of the sons if the father had lived a year longer? (B.M.A.)

3. Divide £857 5*s.* 8¼*d.* among A, B, C, and D so that A may have twice as much as B, B three times as much as C, and C four times as much as D. (C.S.)

4. A person left his property to 5 persons in the following proportions: $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, and the residue to Guy's and St. Thomas's Hospitals in the ratio of 2 : 3. Guy's receives £769 4*s.* What sum did he leave? (C.M.)

5. A man, a woman, and a boy do a piece of work for which they get £15 7*s.* The man works 9 days, the woman 10 days, and the boy 12 days, and the amount of work which they do per day is in the proportion of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$. How much money should each get? (P.T.)

6. If the work done by a man, a woman, and a boy be in the proportion of 3, 2, 1; and if 12 men, 15 women, and 15 boys can complete a certain work in 270 days, in what time would 18 men, 12 women, and 12 boys complete it? (P.T.)

7. A, B, and C, working together, do a piece of work for £3 7*s.* 6*d.* A, working alone, could do it in 10 days; B, working alone, could do it in 12 days; and C, working alone, could do it in 15 days. Divide the money between them in proportion to the quantity of work done by each. (S.K.)

8. If 3 men can do as much work per day as 5 women, and a boy half as much as a man, and if, after working a certain number of days, the aggregate wages of 6 men, 12 women, and 8 boys be £22 18*s.* 8*d.*, how ought the money to be divided, in proportion to the actual work done by each? (C.W.)

9. A sum of £4000 was bequeathed to 4 persons, A, B, C, and D; but by an inadvertency in the will, it was stated that they were to receive in the proportions of $\frac{1}{2}$ A, $\frac{1}{3}$ B, $\frac{1}{4}$ C, $\frac{1}{6}$ D of the whole sum respectively. How much should each receive according to the spirit of the will? (P.T.)

10. A person in his will directed that $\frac{1}{2}$ of his property should be given to A, $\frac{1}{4}$ to B, $\frac{1}{5}$ to C, and $\frac{1}{6}$ to D. Show that these directions cannot be carried out; and if his property amounted to £536 18*s.* 1¾*d.*, show how it should be divided so that the shares may have to one another the ratio he intended. (C.W.)

11. A man makes a will by which he leaves a sum of £16500 to his three sons, A, B, and C, in the proportion of 4, 5, and 6; but A dies, and

then the man increases the legacies of B and C by $\frac{3}{5}$ of their former amounts. Find by how much the sum to be disposed of has been increased in the interval. (S.K.)

12. If £100 10*s.* is divided among A, B, and C to-day, so as to give A 10*s.* more than B, and B 10*s.* more than C, and if to-morrow £10 17*s.* 9*d.* is divided among A, B, and C in the same proportion as the first sum, what total sums will A, B, and C have received respectively? (P.T.)

13. A poor-rate of £1178 2*s.* 6*d.* is voted by the guardians in a union consisting of three parishes in which the number of ratepayers is 150, 100, and 75 respectively. Find the portion of the rate which must be contributed by each parish (1) if the average sum to be paid by each ratepayer is the same in each parish; (2) if the average varies as 2 : 3 : 4 respectively. (C.S.)

14. A bar of gold is worth £31 2*s.* Find the value of an equal bar of silver, having given that the weights of equal volumes of gold and silver are as 19 to 10, and gold and silver worth £3 17*s.* 9*d.* and 57*d.* per ounce respectively. (C.P.)

15. Of the population of a certain village, the number of men is to that of women as 13 : 15. Of the men, the married and the unmarried are in the ratio of 6 : 7, and of the women in that of 7 : 5. There are 25 unmarried women. What is the adult population, and how many of the men are married? (C.P.)

16. In three class-rooms of a school are 222 boys; the numbers in the first and second being in the proportion of 3 : 5, and those in the second and third in the proportion of 7 : 11. Determine the number in each. (C.P.)

17. A, B, and C together purchase an estate, their shares being as $1\frac{1}{2} : 2\frac{1}{2} : 4$. C sells one-third of his share to A, who sells 96 ac. to B. If A and C have then equal portions, how many acres had each at first? (K.S.S.)

18. A field of 7 ac. is sown with turnips, beet, and cabbages, the areas of the crops being respectively $1\frac{1}{3}$, $1\frac{1}{2}$, $1\frac{1}{4}$. If the value of an acre of each be also respectively in the same ratios, and an acre of turnips be worth £7, what is the worth of the whole crop? (P.T.)

19. A warehouse of five stories is let in flats. Each flat (except the top one) lets for three-fourths of the rent charged for all the flats above it, and the rent of the whole warehouse is £2401. What is the rent of the middle flat? (S.K.)

20. Each of 3 girls does the same number of sums in the year, and between them they get 1064 correct answers. One girl gets two sums right out of three, another three out of four, and the other four out of five. How many sums did each work correctly in the year? (K.S.)

21. A boy took up five papers in an examination; his marks for these papers were in proportion 6 : 7 : 8 : 9 : 10. Full marks were the same in each paper, and the boy got $\frac{2}{3}$ of the total full marks. Find in how many papers he got more than half marks. (C.J.)

22. In a co-operative factory, how should a sum of £125 3*s.* 5*d.* be divided among 3 workmen, who work 8, 9, and 10 hrs. per day respec-

tively, whose rates of production per hour are also in the same ratios, and who have been employed 10, 11, and 12 days respectively on the work? (K.S.)

23. Divide 7872 into three parts, so that the first shall be four times the second, and the second three times the third. Explain the principle of solving such questions. (K.S.)

24. The number of passengers in a train was 270, and the first, second, and third classes were in the proportion of 4, 7, and 16. The miles travelled by each class were 40, 35, and 25, and the rates were 2*d.*, $1\frac{1}{2}d.$, and 1*d.* per mile. Find the total earnings of the train. (C.S.)

25. First, second, and third class fares are $2\frac{3}{4}d.$, $1\frac{3}{4}d.$, and 1*d.* per mile respectively, and there are 5 first-class passengers to every 3 second and to every 47 third class. What sum did each class of passengers who travelled from A to B contribute to a total receipt of £53974 16*s.* for fares between A and B? (K.S.)

26. Men, women, and children are employed in a factory in the proportion 1 : 2 : 3, and the wages of a man, a woman, and a child are as 6 : 3 : 2. When 50 men are employed, the total weekly wages of all the hands amount to £225. Find the weekly wages paid to a man, a woman, and a child. (C.W.)

27. A man having 3 sons, aged respectively 24, 16, and 8 yrs., left his estate to be divided amongst them in proportion to their ages. Eight years afterwards the second son died and left his share to be divided between his surviving brothers in amounts inversely proportional to their ages. What portion of the original estate had each of the 2 sons after this accession to their property? (O.J.)

28. A gentleman bequeaths $\frac{1}{2}$ of his estate to his son, and $\frac{1}{3}$ of it to 3 friends, A, B, and C, in shares in the ratio of 3 : 2 : 1; he afterwards disposes of £500, and divides the residue between B and C in shares in the ratio of 4 : 1. If C's portion comes to £6500, how much does each of the others receive? (C.S.)

CHAPTER XXI.

PARTNERSHIP.

Definitions.—*Partnership* (or fellowship) is in arithmetic applied to the rule for the division of profit or loss among the owners of a business.

In a business sense a partnership is an association of two or more persons who agree to share profits or losses according to—

(*a*) The capital subscribed by each.
(*b*) The time the capital is employed.

Firm.—Such an association of persons is called a firm.

Capital.—This includes all the money and all the property used in the business of the firm.

The *net capital* is the excess of the *assets* over *liabilities*.

Kinds.—Partnership, in its arithmetical sense, may be of two kinds—Simple or Compound.

Simple Partnership.—The capital contributed by the several partners may be equal or unequal (generally the latter), but the *time* is the same in each case. The risk that each partner runs is therefore proportionate to the amount of capital he advances; hence the profits or losses are shared in that proportion.

Example.—*A, B, C, D form a partnership. A subscribes* £640, *B* £500, *C* £820, *and D* £250. *If the year's profits amount to* £1105, *what should each receive?*

$$\text{Total capital} = £640 + £500 + £820 + £250 = £2210$$

$$\left.\begin{aligned}
\text{Then A's share} &= \tfrac{1105}{2210} \times £640 = £320 \\
\text{B's} \quad ,, \quad &= \tfrac{1105}{2210} \times £500 = £250 \\
\text{C's} \quad ,, \quad &= \tfrac{1105}{2210} \times £820 = £410 \\
\text{D's} \quad ,, \quad &= \tfrac{1105}{2210} \times £250 = £125
\end{aligned}\right\} \textit{Ans.}$$

Rule.—1. Add the contributed amounts together.

2. Divide the total gain (or loss) by this sum.

3. Multiply the quotient by each man's capital. The products will be the respective amounts.

The rule is also used for the division of a bankrupt's assets among his creditors, for the distribution of prize money, and similar problems.

Example.—*A bankrupt's assets are* £402 ; *he owes to creditors* £266, £424, *and* £650 *respectively. Find the share of each.* (C.S.L.)

Total liabilities = £266 + £424 + £650 = £1340

A's share of assets = $\frac{402}{1340} \times$ £266 = £79 16*s.*
B's ,, ,, = $\frac{402}{1340} \times$ £424 = £127 4*s.*
C's ,, ,, = $\frac{402}{1340} \times$ £650 = £195

Ans.

Compound Partnership.—Both capital and time have to be considered, and both may vary in each case. The profits or losses are divided proportionally to the amount contributed and the time for which it is contributed.

If A contributes £1000 for 12 months, and B £700 for 10 months, some common unit must be found for the comparison of these varying quantities. Now, £1000 for 12 months is the same as £12000 for 1 month. Similarly for B. £700 for 10 months is the same as £700 × 10 or £7000 for 1 month.

Example.—*A began business with* £450 *capital, and at the end of four months he took B into partnership with* £300. *Two months later they took in C with* £500 *capital, but three months afterwards C withdrew* £250 *from the concern. At the end of twelve months there was a profit of* £502$\frac{1}{2}$ *to be divided among them. Find each partner's share of that profit.* (P.T.)

Make £1 for 1 month the unit
then £450 for 12 months is the same as 12 × £450 for 1 month

Now, A's capital = £450 × 12 = £5400 for 1 month
B's ,, = £300 × 8 = £2400 ,,
C's ,, = (£500 × 3) + (£250 × 3) = £2250 ,,

Total capital = £10050 ,,

Then A's share = £$\frac{502{\cdot}5}{10050} \times$ £5400 = £270
B's ,, = £$\frac{502{\cdot}5}{10050} \times$ £2400 = £120
C's ,, = £$\frac{502{\cdot}5}{10050} \times$ £2250 = £112$\frac{1}{2}$

Ans.

The method may also be used for problems dealing with the grazing of cattle.

Example.—*Two persons, A and B, hold a ground in common at a rental of* £54. *A puts in* 23 *horses for* 27 *days, and B* 21 *horses for* 39 *days. How much of the rent should be paid by each?* (C.S.)

A, 23 horses for 27 days = 23 × 27 horses for 1 day = 621 units
B, 21 ,, 39 ,, = 21 × 39 ,, ,, = 819 ,,

Hence proportion of rent due from each—

$$A = \frac{621}{621 + 819} \times 54 = \frac{621}{1440} \times 54 = £23\ 5s.\ 9d.$$

$$B = \frac{819}{621 + 819} \times 54 = \frac{819}{1440} \times 54 = £30\ 14s.\ 3d.$$

Ans.

Rule.—1. Multiply each man's capital by the time for which it is used.

2. Add the products.

3. Divide the gain (or loss, or rent) by the sum of these products, and multiply the quotient by each partner's number of units.

Example.—A more complex case.

A and B enter into a speculation. A puts in £50, *and B puts in* £45; *at the end of* 4 *months A withdraws* $\frac{1}{2}$ *of his capital, and at the end of* 6 *months B withdraws* $\frac{1}{3}$ *of his. C then enters with a capital of* £70. *At the end of* 12 *months the profits of the concern are* £254. *How ought this to be divided?* (C.W.)

Take £1 for 1 month as the common unit

A puts in	£50	for 4 months	= £50 × 4 = £200	for 1 month	}	= £400	for 1 month
,,	£25	,, 8 ,,	= £25 × 8 = £200	,,	}		
B ,,	£45	,, 6 ,,	= £45 × 6 = £270	,,	}	= £450	,,
,,	£30	,, 6 ,,	= £30 × 6 = £180	,,	}		
C ,,	£70	,, 6 ,,	= £70 × 6			= £420	,,

Hence profits must be divided in the proportion of 400, 450, and 420.

Sum of shares = 400 + 450 + 420 = 1270

Then A's share = £254 × $\frac{400}{1270}$ = £80
B's ,, = £254 × $\frac{450}{1270}$ = £90
C's ,, = £254 × $\frac{420}{1270}$ = £84

Ans.

THEORY.

1. Define partnership. By what other name is it known? On what principle are profits usually divided in partnerships?

2. How many kinds of partnership are there? Name and define each with examples.

3. Give, with an example, the rule for "simple partnership." To what other uses may this rule be put?

4. How does the element of "time" influence problems in compound partnership?

5. Write down the rule for compound partnership, and illustrate with an example.

EXERCISE XXXVIII. 38

Simple Partnership.

1. Two partners make a profit of £182; one puts £300, the other £400 into the business. How should they divide the profits? (O.J.)

2. A partner in a business is entitled to $\frac{3}{5}$ of the profits. What were the annual average profits when his share for $2\frac{1}{2}$ yrs. was £1010 17*s.* 9*d.*? (K.S.)

3. The sums invested in a business by two partners are £5500 and £3000; the profit for 9 mths. is £1275. What will be the share of each, and what is the rate of gain per cent. per annum? (K.S.)

4. A, B, and C invest £300, £500, and £700 respectively in a business. They share a profit of £450 in this way: one-half in equal shares to each, and the other half in shares proportional to the capital of each. What does each receive? (P.T.)

5. A and B embark £5000 and £6000 as capital in a business. A is the only working partner, and is paid both for superintendence and capital; B is paid for capital only, and out of a profit of £770 he receives in the proportion of 3 : 4. How much is allowed to A for superintendence? (C.P.)

6. Two men own adjoining farms of 360 and 540 ac. respectively. They unite farms and take a partner, who pays them £2000 on the understanding that a third share of the land shall in future belong to each. How is the £2000 to be divided among the original owners? (K.S.)

7. Two partners start in business, one with £2500, and the other with £1500, and borrow £1200 at 4 per cent. At the end of a year they have gained a profit of 10 per cent. on the whole capital. What should each receive in proportion to his capital? (C.W.)

8. A, B, and C are partners in a business, and they contribute to the capital sums of £2603, £2329, and £1781 respectively. Express as a fraction in its lowest terms the part of the capital which each has contributed. If in a certain year the firm has made a profit of £627 8*s.* 1*d.*, find each partner's share. (S.K.)

9. A, B, and C entered into partnership. A had £175, and B had £210; their profits amounted to £422 10*s.*, of which C's share was £172 5*s.* Find (1) what sum C put into the business; (2) what share of the profits was due to A and B respectively. (C.W.S.)

10. A and B enter into partnership on the following terms: (1) B is to put in half as much capital again as A; (2) A, who is the working partner, is to receive 8 per cent. on his capital, and B is to receive 4 per cent. on his capital, out of the profits; (3) the rest of the profits is to be divided between them in the ratio of their respective capitals. At the end of the first year A receives in all £2456, and B £3084. What capital did each put into the business? (O.J.)

11. Three partners in trade contribute respectively the sums of £438, £292, and £730, with the agreement that each was to receive 5 per cent. on their respective investments, and that the remainder of the gains of the firm, if any, was to be divided between them in the proportion of the sums originally advanced. The whole gain of the firm was £200. What was each man's share? (L.M.)

12. Two merchants enter into partnership. A brings £1500 more than B, who as acting partner is to have £150 a year for the management. At the end of 3 yrs. the gross profits, computed at 15 per cent. per annum, are found to be £2925, from which B's salary is to be deducted. It is required to find the stock of each partner and his share of the net profit.

Compound Partnership.

13. There are three partners in a business. A puts in £600 for 3 yrs., B £800 for $2\frac{1}{2}$ yrs., and C £1250 for 2 yrs., and the profits are £708 15*s.* How ought they to be divided? (S.L.C.)

14. A, B, and C rent land. A puts 500 sheep on it for 4 mths., B 600 for 3 mths., and C 1200 for 2 mths. If the whole rent is £279, how much should each pay? (O.S.L.)

15. Three men hire a pasture for £12 12*s*. One puts in 10 oxen for 9 mths., another 12 oxen for 8 mths., the third 14 oxen for 7 mths. What part of the rent should each pay? (K.S.)

16. A and B join in partnership; A puts in the business £300, and B £500; at the end of 6 mths. A puts £400 more, and B takes out £100. How should a gain of £61 15*s*. be divided amongst them at the end of 12 mths.? (P.T.)

17. A and B engage in a business, their capitals being as 7 : 11. At the end of 3 mths. A withdraws $\frac{1}{3}$ of his capital, and 1 mth. afterwards B adds twice what A had withdrawn. How should a profit of £337 7*s*. 6*d*. be divided at the end of the year? (P.T.)

18. A and B enter into partnership for a year. A contributes £500 on January 1, and B on May 1 contributes as much as entitles him to half the gain at the end of the year. What was B's contribution? (I.I.)

19. Two troops of cavalry rent a field in common, for which they are to pay £23 10*s*. One troop puts into the field 27 horses for 29 days, the other 19 horses for 33 days. How much should each pay? (O.S.L.)

20. Three farmers send sheep to grass. A sends 100 sheep for $2\frac{1}{4}$ mths., B sends 150 for $2\frac{1}{2}$ mths., and C sends 236 for $3\frac{3}{4}$ mths. If the whole cost be £132, what will each have to pay? (C.P.)

21. Three persons, A, B, and C, hold a grazing farm in common at a yearly rent of £570. A puts 126 oxen on the farm for 3 mths., B puts 162 for 5 mths., and C puts 72 for 12 mths. How much of the rent should each person pay? (I.I.)

22. On January 1 A begins to trade with a capital of £800. In 3 mths. he takes in B as a partner. What capital must B bring into the concern that the profits may be divided equally at the end of the year? (C.S.)

23. Three graziers, A, B, and C, rent a piece of pasture land for a month. A puts on 27 cattle for 21 days, B 19 for 24 days, and C 23 for 25 days. If at the end of the month the rent and other charges amount to £33 5*s*. 10*d*., how much of this ought to be paid by each? (C.S.L.)

24. A starts a business with a capital of £3000. After 5 mths. he takes B, with a capital of £2000, into partnership, and one month later C, with a capital of £4500. At the end of the year the profits are divided, and B finds that he has to pay 5 guins. as income-tax. Find the total profits of the concern, the tax being at the rate of 6*d*. in the £. (C.M.S.)

25. A tradesman began business on March 4 with a capital of £900; on September 12 he was joined by a partner who put £500 into the business; on December 31 the profits were £204 5*s*. Find the share of each partner. (O.P.)

26. Two people enter into partnership with £3000 and £4500 respectively. At the end of 8 mths. the partner with smaller capital advances £2500 more. Seven months later they close business with a profit of £1040. How should this profit be divided? (K.S.)

27. A and B enter into partnership, their capitals being as 5 : 6. At the end of 2 mths. they withdraw respectively $\frac{1}{3}$ and $\frac{1}{4}$ of their capital, and 4 mths. afterwards B increases his capital by the addition of $\frac{1}{3}$ of its original capital. In what proportion are they to divide the profits, amounting at the end of the year to £379 2*s*. 6*d*.? (C.S.)

28. A commences business with a capital of £4000, and after 4 mths. takes B into partnership with a capital of £300. Two months later they take C into partnership with a capital of £5000. At the end of the year their net profits amount to 16 per cent. on the whole capital invested. What should each receive of the profits? (R.M.A.)

29. A with £2500 capital joins in partnership with B, who has £4000 capital. At the end of a year's trading they have lost £520, and their capitals are diminished accordingly. C now joins them with £4500 capital. At the year's end they had gained £2358. What did each receive? (C.H.L.)

30. A, B, and C are partners in a bank, in which they invest £12842, £8000, and £5000 respectively. C is to have 15 per cent. of the profits for managing the business; A and B are each to have 2 guins. a day allowed for attendance. A attends 6 days, B 4 days. If the profits amount to £7650, how should they be divided? (C.S.)

31. A and B enter into a speculation; A puts in £50, and B puts in £45. At the end of 4 mths. A withdraws $\frac{1}{2}$ his capital, and at the end of 6 mths. B withdraws $\frac{1}{3}$ of his. C then enters with a capital of £70. At the end of 12 mths. the profits of the concern are £254. How ought this to be divided? (C.W.)

32. A began business with £450 capital, and at the end of 4 mths. he took B into partnership with £300. Two months later they took in C with £500 capital; but 3 mths. afterwards C withdrew £250 from the concern. At the end of 12 mths. there was a profit of £502 10*s.* to be divided among them. Find each partner's share of that profit. (P.T.)

33. A starts a business with a capital of £4000. After 3 mths. he takes B with a capital of £7000 into partnership, and 4 mths. later C with a capital of £3000. At the end of the year the profits are divided, and A finds that he has to pay £6 as income-tax. Find the total profits of the concern, the tax being at the rate of 6*d.* in the £. (C.PR.)

34. Two partners, A and B, started a business, B finding half as much again of the capital as A. At the end of 8 mths. B withdrew $\frac{1}{4}$ of his capital, and 2 mths. later on A withdrew $\frac{1}{4}$ of his capital. At the end of the year the profits were found to be £530. To how much of this profit was each man entitled? (C.S.)

35. A and B together rent a field of 27$\frac{1}{2}$ ac. for £38. A puts 30 sheep in for 6 wks., and B 20 sheep for 10 wks. They agree to pay the rent and receive the profits according to the amount of feed taken by each, and they clear £57 by the sale of the hay. What are the rents and profits of A and B respectively? (P.T.)

36. A, B, and C's capitals are respectively as 2, 3, and 4. They make a joint speculation, and the total gain is £7200; A putting in $\frac{1}{2}$ of his capital for 4 mths., B $\frac{1}{3}$ of his for 3 mths., and C $\frac{1}{4}$ of his for 2 mths.; 20 per cent. is realized. Find the respective capitals, investments, and gains. (C.S.)

37. Three graziers, A, B, and C, agree to hold a pasture in common. A puts in 8 oxen for 10 mths.; B, 7 oxen for 12 mths.; C, 20 oxen for 5 mths., at a rent of £66 per annum. How much should they each respectively contribute towards the rent? (L.M.)

38. A and B enter business together; A gives $\frac{3}{5}$ of the capital. At the end of the first year they have made a profit of 3$\frac{1}{2}$ per cent., and at the end of the second a profit of 5 per cent.; at the end of the third year

they are bankrupts, and only pay 10*s.* in the £. The remaining money is £16663 10*s.* How much did each contribute? (C.S.)

39. A, B, and C rent a pasture for a year. A puts in 500 sheep for the first half of the year; B, 750 for the second half; and C, 400 all the year round. The feed is considered to be worth twice as much in the first as in the other half of the year. Find (1) what fractional part of the rent each person ought to pay, and (2) the amount to be paid by each when C's share is £21 more than A's. (C.S.)

40. A and B enter into partnership, each contributing capital, and B having a salary as manager; A supplies £30000 and B £18000. The gross receipts for a year are £11200, of which $\frac{1}{4}$ is paid away for expenses, and 5 per cent. for insurance. A receives £5000 as his share of the profits. What is the amount of B's salary as manager? (S.A.)

41. A, B, and C agree to rent a meadow in common for £69 13*s.* 4*d.*, and to pay towards the rent a sum proportional to the use each has made of it. A puts in 3 cows for 3 mths. and 25 sheep for 4 mths.; B puts in 7 cows for 4 mths. and 15 sheep for 2 mths.; C puts in 5 cows for 3 mths. and 20 sheep for 4 mths. Supposing a cow to eat as much as 4 sheep, what will be each man's share of the rent? (C.S.)

42. A and B embark in partnership for 7 yrs. B is to have $\frac{5}{16}$ of the net profits during the first half of the time, and $\frac{1}{2}$ of them afterwards. After $4\frac{1}{2}$ yrs., owing to a duty being lowered, the annual profits, which had so far remained stationary, are suddenly increased in the proportion of 6 to 5; but at the same time they are reduced by the creation of an income-tax of 7*d.* in the £. By the end of 7 yrs. B's share of the net profits has altogether amounted to £976. What were the annual profits before the duty was lowered? (C.S.)

43. Three merchants entered into partnership on January 1, 1890, and agreed to allow their profits to accumulate for 10 yrs. A subscribed $\frac{1}{3}$ of the capital and £1000 besides; B put in $\frac{1}{4}$, and C $\frac{1}{2}$ of the capital. A withdrew his capital on December 31, 1896, B withdrew his on June 30, 1898. How is a profit of £35500 to be divided between them at the end of 10 yrs.? Also find the capital.

44. Three persons invest sums of £575, £450, and £300 respectively in a venture, on the agreement that the profits shall be divided in such a manner that the rate of interest made by each of them shall be proportional to the sum which he subscribed. In 8 mths. the profits amount to £124 12*s.* 6*d.* How much of this will each receive, and what rate of interest does each make? (C.S.)

45. A, B, and C rent a field for £180. A puts in 12 horses for 5 mths. and 45 sheep for 3 mths.; B puts in 15 oxen for 6 mths. and 54 sheep for 2 mths.; C puts in 6 horses and 47 oxen for 3 mths. Four horses and 3 sheep together eat as much as 5 oxen and 1 horse, and 2 oxen eat as much as 7 sheep. What should A, B, and C pay respectively to the landlord as their shares of the rent?

CHAPTER XXII.

CHAIN RULE.

Definition and Explanation.

Chain rule has been variously defined as—

1. A form of continued proportion.
2. An application of the unitary method.
3. Another form of variation.
4. A complex form of ratios.
5. An operation in simple reduction.
6. Compound barter.

As a rule, a series of quantities is given in equivalent pairs. Each pair is connected with the preceding pair, the second quantity of the one pair being of the same kind (but rarely of the same quantity) as the first quantity of the succeeding pair. *The problem is to find how many of the last quantity are equivalent to a given number of the first.* As a rule, the *unit* of one measure is not given in terms of the other, but so many of one quantity are given as equivalent to a certain number of another quantity; *i.e.* the respective values are connected by certain fractions or ratios, and a *chain* of relative values is thus formed.

Example.—*If* 4 *fowls are worth* 3 *ducks, and* 7 *ducks are worth* 2 *geese, and* 9 *geese are worth* 7 *turkeys, what is the price of a fowl if a turkey costs* 18*s.*? (C.P.)

Now, 9 geese are worth 7 turkeys

$\therefore$ 1 goose is ,, $\frac{7}{9}$ turkey

and 2 geese are ,, $\frac{2 \times 7}{9}$ turkeys

$\therefore$ also 7 ducks are ,, $\frac{2 \times 7}{9}$ turkeys

whence 1 duck is ,, $\frac{2 \times 7}{9 \times 7}$ turkeys

and 3 ducks are ,, $\frac{3 \times 2 \times 7}{9 \times 7}$ turkeys

$\therefore$ also 4 fowls are ,, $\frac{3 \times 2 \times 7}{9 \times 7}$ turkeys

and 1 fowl is ,, $\frac{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{7}}}{\underset{3}{\cancel{9}} \times \underset{1}{\cancel{7}} \times \underset{2}{\cancel{4}}}$ turkey $= \frac{1}{6}$ turkey

$\therefore$ 1 fowl costs $\frac{18}{6}s. = 3s.$

Ans. 3*s.*

Example.—*Given* the following equivalent values :—

4 *houses* = 6 *fields*
3 *fields* = 150 *sheep*
12 *sheep* = 2 *bicycles*
50 *bicycles* = 5 *motor cars*
2 *motor cars* = 8 *horses*

To find how many horses are equal in value to 10 *houses.*

By Rule of Three.

1. If 4 houses are worth 6 fields, what are 10 houses worth?

$$\overbrace{4 : 10}^{\text{houses}} :: \overbrace{6 : x}^{\text{fields}};\ x = \frac{10 \times 6}{4}$$

2. If 3 fields are worth 150 sheep, what are $\frac{10 \times 6}{4}$ fields worth?

$$\overbrace{1 : \frac{10 \times 6}{4}}^{\text{fields}} :: \overbrace{150 : x}^{\text{sheep}};\ x = \frac{150 \times 10 \times 6}{4 \times 3} \text{ sheep}$$

3. If 12 sheep are worth 2 bicycles, what are $\frac{150 \times 10 \times 6}{4 \times 3}$ sheep worth?

$$\overbrace{12 : \frac{150 \times 10 \times 6}{4 \times 3}}^{\text{sheep}} :: \overbrace{2 : x}^{\text{bicycles}};\ x = \frac{150 \times 10 \times 6 \times 2}{12 \times 4 \times 3} \text{ bicycles}$$

4. If 50 bicycles are worth 5 motor cars, what are $\frac{150 \times 10 \times 6 \times 2}{12 \times 4 \times 3}$ bicycles worth?

$$\overbrace{50 : \frac{150 \times 10 \times 6 \times 2}{12 \times 4 \times 3}}^{\text{bicycles}} :: \overbrace{5 : x}^{\text{motors}};\ x = \frac{150 \times 10 \times 6 \times 2 \times 5}{50 \times 12 \times 4 \times 3} \text{ motors}$$

5. If 2 motors are worth 8 horses, what are $\frac{150 \times 10 \times 6 \times 2 \times 5}{50 \times 12 \times 4 \times 3}$ motors worth?

$$\overbrace{2 : \frac{150 \times 10 \times 6 \times 2 \times 5}{50 \times 12 \times 4 \times 3}}^{\text{motors}} :: \overbrace{8 : x}^{\text{horses}}$$

$$\therefore x = \frac{\overset{4}{\cancel{8}} \times 150 \times 10 \times 6 \times \overset{1}{\cancel{2}} \times 5 \times \overset{2}{\cancel{8}}}{\underset{1}{\cancel{2}} \times \underset{1}{\cancel{50}} \times \underset{1}{\cancel{12}} \times \underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}}} = 50 \text{ horses}$$

Ans. 50 horses.

By Chain Rule.

? horses = 10 houses
4 houses = 6 fields
3 fields = 150 sheep
12 sheep = 2 bicycles
50 bicyles = 5 motor cars
2 motor cars = 8 horses

$$\text{then number of horses} = \frac{10 \times 6 \times 150 \times 2 \times 5 \times 8}{4 \times 3 \times 12 \times 50 \times 2} = 50$$

It will be noted that the answer is found by dividing the product of the

numbers in the right-hand column by the product of the numbers in the left-hand column, and that the final statement in the proportion method and the ordinary method of Chain Rule is the same in each case.

Broken Chain.

In some problems the quantities are given the same in kind, but not always of the same denomination. The chain is then said to be "broken," and the connecting links have to be supplied, or the equivalent values expressed in the form of a fraction.

Example.—*If* 6 *lbs. of tea are worth* 12 *cwt. of coal, and* 5 *tons of coal are worth* 80 *lbs. of coffee, and* 8 *oz. of coffee are worth* 7 *oz. of butter, how many pounds of butter are worth* 20 *lbs. of tea?*

There are three missing links, *e.g.*—

20 cwt. of coal = 1 ton of coal
16 oz. of coffee = 1 lb. of coffee
16 oz. of butter = 1 lb. of butter

We must either supply these, or express the values as fractions; *i.e.*—

12 cwt. of coal as $\frac{12}{20}$ ton of coal
9 oz. of coffee as $\frac{9}{16}$ lb. of coffee
7 oz. of butter as $\frac{7}{16}$ lb. of butter

Then ? lbs. of butter = 20 lbs. of tea
6 lbs. of tea = 12 cwt. of coal
20 *cwt. of coal* = 1 *ton of coal*
5 ton of coal = 80 lbs. of coffee
1 *lb. of coffee* = 16 *oz. of coffee*
8 oz. of coffee = 7 oz. of butter
16 *oz. of butter* = 1 *lb. of butter*

$$\therefore \text{ amount of butter} = \frac{\overset{2}{\cancel{20}} \times \overset{1}{\cancel{12}} \times 1 \times \overset{1}{\cancel{80}} \times \overset{2}{\cancel{16}} \times 7 \times 1}{\underset{1}{\cancel{6}} \times \underset{1}{\cancel{20}} \times \underset{1}{\cancel{5}} \times 1 \times \underset{1}{\cancel{8}} \times \underset{1}{\cancel{16}}} = 28 \text{ lbs.} \quad \textit{Ans.}$$

Rule.—The rule can be deduced from a study of the preceding examples:—

1. Express in the form of equations the relations that exist between the several quantities.

2. Place the given equations in consecutive order under one another, beginning with that term which is similar to the one sought.

3. Each quantity should appear on both sides of the equation, so that the *linkage* may be preserved. For this purpose, let the quantity on the right-hand side of one equation be of the same kind and denomination as the next on the left hand of the next equation.

4. Let the *left-hand* quantity of the *first* equation, and the *right-hand* quantity of the last be of the same kind and denomination.

5. The continued product of the numbers on the right-hand side of the equations divided by the continued product of the numbers on the left-hand side will be the quantity required.

THEORY.

1. Define and explain "chain rule," and state what is the problem involved.

2. What is the ordinary form of statement in the method of chain rule, and how is the answer found from that statement?

3. When is the chain said to be "broken" in chain rule, and how would you deal with such a case?

4. Write out the rule for chain rule. Give an illustrative example, and show how the rule applies to each step in its solution.

EXERCISE XXXIX. 39

Chain Rule.

1. If 3 ducks are worth 4 chickens, and 3 geese are worth 10 ducks, find the value of a goose, a pair of chickens being worth 4*s.* 6*d.* (C.J.)

2. If 7 rix dollars are worth 2 ducats, and 9 ducats worth 4 moidores, and 20 moidores worth £27, how many rix dollars are there in £72? (R.M.A.)

3. If 250 rup. have the same value as 84 dols., and 36 dols. have the same value as 7 guins., what is the value of a rupee in £ *s.* *d.*? (O. and C.H.C.)

4. If 23·3 fl. are equal to 49·78 fr., 191·5 fl. to 218·5 m., and 273 m. to 128 roub., find to one decimal place how many francs are equal in value to 233 roub. (K.S.S.)

5. If 26 sheep be worth 5 oxen, 2 oxen be equal in value to 3 horses and 7 horses cost 91 guins., find value of 1 sheep. (C.S.)

6. If 68 oranges be worth 51 buns, and 28 buns be worth 10½ lbs. of sugar, what is the value of 64 oranges when sugar is 2½*d.* per pound? (K.S.)

7. If 20 apples are worth 12 pears, 9 pears worth 4 oranges, 12 oranges worth 240 nuts, find the cost of 25 apples, 5 pears, and 10 oranges, nuts being sold at 4½*d.* per 100.

8. If 2 sheep be worth 185 fr., and 2 calves be together worth one-third as much as an ox; also if 15 sheep be worth 2 oxen, and 55·50 fr. be equal to £2, find how many calves can be bought for £25. (C.S.)

9. If 1⅔ yds. of cloth are worth $\frac{5}{12}$ of a bushel of corn, and 12 yds. of cloth will fetch 4½ dols., what is the value of 5 qrs. of corn, a dollar being equal to 4*s.* 2*d.*? (C.S.)

10. An orchard contained 192 fruit trees. Of these for every 3 apple trees there were 2 pear trees, and for every 3 cherry trees there were

2 walnut trees; there were 2 plum trees to every 11 apple trees, and 7 cherry trees to every 2 plum trees. Find the number of each sort. (C.S.)

11. A can copy 6 pages while B copies 5, B copies 15 while C copies 12, and C can copy 4 while D copies 3. A, who can write 20 pages a day, receives a paper of 240 pages to copy, and after doing a quarter of it calls in B, C, and D to help him. When will the work be finished? (C.S.)

12. The first of 5 clerks can copy 3 lines as soon as the second can copy 4, the second can copy 5 as soon as the third can copy 3, the third can copy 7 as soon as the fourth can copy 8, and the fourth can copy 14 as soon as the fifth can copy 17. How many lines will the fifth copy while the first is copying 490? (C.S.)

13. Assume that 4 English navvies can do as much work in a day as 5 French navvies, and that 4 French navvies can do as much as 7 negroes. It is found that 13 English and 12 French do a piece of work in 3 days. How long will it take 10 negroes? Express your result decimally to 3 significant figures, and use none but arithmetical symbols in your work. (L.M.)

14. If 48 lbs. of tea are worth 55 gals. of ale, and 63 gals. of ale 24 bots. of wine, and 11 bots. of wine 9 prs. of gloves, how many pounds of tea must be given for 20 prs. of gloves? (C.W.)

15. If 8 lbs. of coffee cost as much as 5 lbs. of tea, and 3 lbs. of tea as much as 32 lbs. of sugar, what is the price of each when 1 lb. of tea, of coffee, and of sugar together cost 3*s.* $5\frac{1}{4}$*d.*? (K.S.)

16. Assuming that 5 lbs. of coffee are worth 7 lbs. of chocolate, 10 lbs. of chocolate worth 3 lbs. of tea, 5 lbs. of tea worth 35 lbs. of sugar, 7 lbs. of sugar worth 2 lbs. of cocoa, and 1 cwt. cocoa cost £8 6*s.* 8*d.*, find the price of 1 lb. of coffee. Work by the shortest method you can.

17. If the Hamburg foot = 126·9667 Paris lines, and 10 lines = $\frac{8}{9}$ of an English foot, find in English measure the length of the Hamburg foot. (C.S.)

18. $8\frac{1}{6}$ Cornish acres = 6 Irish acres, 4 Cunningham acres = $6\frac{1}{4}$ Devonshire acres, 196 statute acres = 121 Irish acres, 5 Cornish acres = $7\frac{1}{3}$ Devonshire acres. How many square yards are there in a Cunningham acre? (O. and C.H.C.)

19. Hay is worth $7\frac{1}{2}$*d.* per stone of 14 lbs. Supposing 5 sheep cost as much as 2 calves, and 33 sheep as 4 oxen, what weight of hay must be given with 27 sheep and 3 calves in return for 7 oxen, the price of sheep being £2 9*s.* per head? (C.S.)

CHAPTER XXIII.

RATES AND TAXES.

Rates are sums of money paid by householders to local authorities, such as the County Council, the Town or Borough Council, etc.

The annual value of every house is fixed by an Assessment Committee, and this annual value is called the *ratable value*. Thus, if the rate was 6*s*. in the pound, and a man lived in a house the ratable value of which was £40 a year, his rates would be $40 \times 6s. = 240s. = £12$.

Taxes are sums of money paid directly or indirectly to the National Exchequer. The amount of a tax is fixed by the House of Commons. Thus taxes are uniform throughout the country; rates vary according to localities.

The Income-Tax is the one most often dealt with in problems in arithmetic. In practice the Income-Tax is levied so that the poor man pays nothing and the man of moderate means pays on part of his income only, but in problems in arithmetic, the principle of exemption from income-tax is often disregarded, and, except anything is stated to the contrary, the student must suppose that the income-tax is levied upon the whole of the income uniformly.

Example.—*If the income-tax be at the rate of* 9*d*. *in the pound, what does a man pay on an income of* £500?

Income-tax on £1 = 9*d*.
„ „ £500 = 9*d*. × 500
= £18 15*s*. *Ans*.

Example.—*Find the net income after paying a tax of* 1*s*. 3*d*. *in the* £ *on a gross income of* £350, *assuming that exemption is claimed on* £160.

Amount to be taxed = £350 − £160 = £190.

After paying an income-tax of 1*s*. 3*d*. in the pound, a man has 18*s*. 9*d*. left on each pound.

If the sum left from £1 = 18*s*. 9*d*.
then „ „ £190 = 18*s*. 9*d*. × 190
= £178 2*s*. 6*d*.
Net income = £178 2*s*. 6*d*. + £160
= £338 2*s*. 6*d*. *Ans*.

Example.—*What income will leave* £1000 *after paying* 1*s. in the* £ *income-tax? Give the answer correct to the nearest penny.*

If £1 is the gross income, then the net income is 19*s.*
Required to find the gross income when £1000 is the net income.

19*s.* : £1000 :: £1 : gross income in pounds

$$\therefore \text{ gross income} = £\frac{1000 \times 20}{19}$$

$$= £1052\ 12s.\ 8d. \quad \textit{Ans.}$$

EXERCISE XL. 40

RATES AND TAXES.

To find Income.

1. After paying income-tax at the rate of 7*d.* in the £, a person had £734 13*s.* 6¾*d.* What was his gross income? (P.T.)

2. A person left a sum of money sufficient, after paying 6*d.* in the £ income-tax, to give £2 13*s.* 9*d.* to each of 52 poor people. What was the sum left? (C.W.)

3. After deducting a rate of 8*d.* in the £, each of 49 poor persons receives from a charity a dole of £4 11*s.* 10*d.* What was the whole benefaction? (C.H.L.)

4. A man pays a quarterly income-tax of £1 9*s.* 2*d.*, the tax being 5*d.* in the £. What is his annual income? (P.T.)

5. When an income of £1150 is reduced by taxation to £1063 15*s.*, what is the income of a person who has £518 left after deducting the taxes? (O.J.)

6. If, after paying an income-tax of 8*d.* in the £, a person's net income is £483 6*s.* 8*d.*, what was his gross income? (C.P.)

7. After deducting a tax of 8*d.* in the £ from a man's income it is £333 10*s.*, what is his gross income? (C.S.)

8. If the income-tax be raised from 7*d.* to 9*d.* in the £, a man's net income is reduced by £6 1*s.* 8*d.* What is his gross income? (O.S.)

9. A man pays £116 2*s.* 9*d.* income-tax. If the rate be 7½*d.* in the £, what is the man's income? (C.PR.)

10. A man pays £6 13*s.* 4*d.* income-tax when the tax is at 8*d.* in the £ and £160 of his income free from the tax. What is his income? (C.P.)

To find Total Tax.

11. If I pay £2 7*s.* 8¼*d.* as income-tax on a rental of £71 10*s.* 7½*d.*, what should I pay on a rental of £510 15*s.*? (C.J.)

12. What should a clerk, whose stipend was £326 10*s.* 3¾*d*, pay for income-tax at 1*s.* 4*d.* in the £? (P.T.)

13. A parish is rated at £4368 12*s.*, and upon this amount a rate is levied at the rate of 1*s.* 3*d.* in the £. How much money is thus raised? (P.T.)

14. The rates of a parish amount to 3*s.* 6*d.* in the £; $\frac{1}{3}$ is poor rate, highway rate, the rest is school rate. What will a man pay for school purposes who is rated at £170 per annum? (K.S.)

15. What is the income-tax on £478 10*s.* a year at 5*d.* in the £? Also on what income will a tax of 5*d.* in the £ amount to £28 2*s.* 6*d.*? (C.P.)

To find Rental and Ratable Value.

16. The annual poor rates on a net rental of £365 7*s.* 3*d.* amount to £36 8*s.* 9*d.* What should be the net rental of an estate for which the poor rates amount to £24 5*s.* 10*d.* per annum? (P.T.)

17. A rate of 2*s.* 9*d.* in the £ on a rental of a parish produces £352. What is the total rental? (K.S.)

18. Find the total rental of a town if a tax of 9*d.* in the £ on rents brings in £2163 13*s.* 4*d.* (S.L.C.)

19. Find a man's gross rental if, after paying an income-tax of 6*d.* in the £ on the whole, and 3*s.* 6*d.* in the £ on three-fourths of his rental, his net income is £2700. (P.T.)

20. The inhabited house duty at 9*d.* in the £ on the rent of a house is £3 10*s.* more than the income-tax at 6*d.* in the £. Find the annual rent of the house. (K.S.)

21. A man can afford £125 for rent, rates, and house-tax. If the house-tax is 9*d.* per £ upon the whole rent, and the rates are 3*s.* 10*d.* per £ upon three-fourths of the rent, find, to the nearest shilling, how much rent he can pay. (C.H.L.)

22. The expenses of the poor in a parish amount to £110 7*s.* 8*d.* A rate of 4*s.* 4$\frac{3}{4}$*d.* in the £ is raised, and proves insufficient by 11$\frac{1}{2}$*d.* What does the whole rental of the parish amount to? Work by the rule of three, and prove by practice. (K.S.)

To find Rate in the Pound.

23. The sum of £231 18*s.* is to be raised in a parish, the ratable value of which is £6184. What is the rate in the £? (P.T.)

24. If the net income of an estate, after paying all taxes, be £534 15*s.*, and the gross income be £570 8*s.*, how much in the £ did the taxes amount to? (P.T.)

25. If the tax on £335 7*s.* 6*d.* amounts to £58 13*s.* 9$\frac{3}{4}$*d.*, what is that in the £? (P.T.)

26. If the poor rates on a rental of £525 7*s.* 6*d.* be £61 5*s.* 10$\frac{1}{2}$*d.*, what is that in the £? (C.S.)

27. My rent is £42, and I am rated at £35. My rates for the year are £12 13*s.* 4*d.* How much is that in the £ (to the nearest penny)?

28. If the net income of an estate, after paying all taxes, be £937 10*s.*, and the gross income be £1000, how much in the £ did the taxes amount to? (P.T.)

Miscellaneous.

29. I pay an income-tax of 1*s.* 11*d.* in the £, and a further tax of 5*s.* 7*d.* in the £ on the rent of a house which produces $\frac{1}{7}$ of my gross income. These taxes amount to £71 5*s.* annually. What is the amount of my gross income? (C.S.)

30. When the income-tax is 3*d.* in the £, a person's net income is £2370 per annum. What will it be when the income-tax is raised to 7*d.*? (C.PR.)

31. A, B, and C have incomes amounting in all to £572. Each contributes the same amount annually to a charity. This amount is, per £ of income, $2\frac{1}{2}d.$ for A, $3\frac{1}{2}d.$ for B, $4\frac{1}{2}d.$ for C. Find the amount. (C.W.)

32. A man pays a poor rate which exceeds his income-tax at 7*d.* in the £ by £22 10*s.*, and after paying both, his net income is £486 per annum. Find his gross income. (C.M.)

33. Incomes below £150 being subject to income-tax of 5*d.* in the £, and incomes above £150 to a tax of 7*d.*, find what income above £150 a man must have to be just $7\frac{3}{4}d.$ a year poorer than the man with £149 10*s.* a year. (P.T.)

34. (*a*) Which is the higher tax—9*d.* in the £ or $9\frac{1}{2}d.$ in the guinea? (*b*) Suppose the income-tax to be 10*d.* in the £, and to be levied only on incomes exceeding £100. What is the lowest taxable income, consisting of an exact number of pounds, which is more profitable to its possessor than the highest untaxable income would be? (P.T.)

35. Two men, A and B, with equal incomes, live in different towns and are rated at 2*s.* 6*d.* and 2*s.* 2*d.* in the £ respectively, and the one pays £8 3*s.* 2*d.* more in rates than the other. Find their incomes. Suppose A's income remains the same, but B's is altered so that he has to pay the same amount for rates as A does, although the rate in the £ is unchanged. What would B's income be then? (P.T.)

CHAPTER XXIV.

BANKRUPTCY.

A Bankrupt is a person who is unable to pay his debts, and who has had to surrender his property in order that it may be proportionately divided among his creditors.

Liabilities or *Debts* are what a man owes.

Assets are what a man possesses.

If a man's assets are greater than his liabilities, he is said to be *solvent*; if his assets are less than his liabilities, he is *insolvent*.

A Debtor is a man who owes money; a *Creditor* a man to whom money is owing.

When a man's liabilities exceed his assets, he may call a meeting of his creditors, and an arrangement called a *composition* may be arrived at. Under a composition, the creditors agree to receive whatever the debtor can pay. The arrangement is then a private one, and such a person is insolvent, but not an official bankrupt. If, however, recourse is had to the law, a judge, after examining the case, makes an order declaring the debtor a bankrupt. An Official Receiver is appointed, who realizes (*i.e.* turns into money) all the assets. After deducting the legal expenses, the Receiver determines the ratio which the assets bear to the debts.

If a bankrupt's liabilities were £10,000, and his net assets £2500, the Official Receiver would declare a *dividend* of 5*s.* in the pound, *i.e.* for every pound a creditor should have, he would receive only 5*s.* When the dividends have been paid, the business is said to be *wound up*.

Example.—*A bankrupt pays* 12*s.* 6*d. in the pound. If the total loss sustained by his creditors is* £325, *what are his assets?* (O.S.L.)

In every £1, his assets are 12*s.* 6*d.* and his creditors' losses 7*s.* 6*d.*, *i.e.* for £12½ assets there are £7½ losses.

Hence the following proportion:—

$$\text{As } \underbrace{£7\tfrac{1}{2}}_{\text{Losses}} : £325 :: \underbrace{£12\tfrac{1}{2}}_{\text{Assets}} : \text{assets in £s}$$

$$\therefore \text{assets} = £\frac{325 \times 12\frac{1}{2}}{7\frac{1}{2}} = £541\ 13s.\ 4d. \quad \textit{Ans.}$$

Example.—*An executor, administering an estate, finds it encumbered with debts to the amount of* £4322 10*s.* *over and above its realized value of* £10729 10*s.* *How much in the pound can he offer his creditors by way of composition?* (C.P.)

Total liabilities = £10729 10*s.* + £4322 10*s.* = £15052

$$\text{Composition in the pound} = \frac{\text{assets}}{\text{liabilities}} = £\frac{10729\frac{1}{2}}{15052}$$

= 14*s.* 3·07*d.* *Ans.*

EXERCISE XLI. 41

"BANKRUPT" PROBLEMS.

To find Bankrupt's "Composition" or Dividend.

1. A bankrupt has assets to the amount of £10252 4*s.* 8*d.*, and liabilities to the amount of £13901 6*s.* 8*d.* How much can he pay in the £? (C.J.)

2. A bankrupt's debts amount to £5670, his assets to £3348 16*s.* 10½*d.* How much can he pay in the £? (C.S.)

3. The realized value of an estate is £5364 15*s.* It is encumbered with debts amounting to £3218 17*s.* over and above this value. What composition—*i.e.* how much in the £—can be offered to the creditors? (C.P.)

4. A bankrupt's debts amount to £3427 6*s.* 8*d.*, and his assets are only £2184 18*s.* 6*d.* How much ought a man to receive to whom the bankrupt owes £375 10*s.*? (C.S.L.)

To find Bankrupt's Assets.

5. If a man, owing £4582, finds that he can pay only 17*s.* 6*d.* for each complete pound, what amount of money has he in hand? (C.P.)

6. A bankrupt pays 13*s.* 2*d.* in the £. How much will a creditor receive to whom he owes £563 15*s.*? (C.PR.)

7. A bankrupt pays 8*s.* 10½*d.* in the £. Find by practice how much a creditor will receive to whom the bankrupt owes £1527 13*s.* 4*d.* (C.P.)

8. A bankrupt's liabilities are £6235 10*s.*, and he pays his creditors 5*s.* 6*d.* in the £. Find by practice the amount of his assets. (O.R.)

9. A bankrupt's liabilities are £3758 17*s.* 6*d.* What are his assets if he can pay 13*s.* 7½*d.* in the £? (C.PR.)

10. A bankrupt, whose assets are £3420, can pay a composition of 9*s.* 6*d.* in the £ of his debts. What is the amount of his debts? (P.T.)

11. Find the dividend on £83 9*s.* 4*d.* at 13*s.* 5¾*d.* in the £. (O.S.L.)

12. Find the dividend on £2856 15*s.* at 9*s.* 5*d.* in the £. Deduce the dividend on the same amount at 10*s.* 5*d.* in the £. (C.W.)

To find Bankrupt's Debts.

13. A bankrupt's effects realize £1912 14*s.*, which enables him to pay his creditors 15*s.* 6*d.* in the £. What amount does he owe? (P.T.)

14. An estate, which was wound up in bankruptcy, realized the net sum of £1290 10*s.* 7*d.*, and this amount, on being distributed among the creditors, was sufficient to pay 15*s.* 5*d.* in the £. What was the total amount of the bankrupt's debts? (K.S.)

15. A creditor receives £444 7*s.* 6*d.*, being a dividend at the rate of 4*s.* 11¼*d.* in the £ upon a debt due to him. Find the debt. (K.S.)

16. A bankrupt can pay 6*s.* 8*d.* in the £; if his assets were £500 more he could pay 7*s.* 4*d.* Find his debts and his assets. (P.T. and C.P.)

Miscellaneous.

17. A bankrupt owes £1500 19*s.* 7½*d.*; but if he obtained £420 17*s.* 6*d.* more than he has in hand, he could pay his debts in full, and would have £250 over. How much has he in hand? (C.P.)

18. A bankrupt owes £1537 3*s.* 4*d.*, but his estate is worth only £960 14*s.* 7*d.* What can he pay in the £? and what does a creditor receive to whom he owes £325 10*s.*? (K.S.)

19. A creditor receives on a debt of £296 a dividend of 12*s.* 4*d.* in the £, and he receives a further dividend upon the deficiency of 3*s.* 9*d.* in the £. How much does the creditor receive in all? (P.T.)

CHAPTER XXV.

PROBLEMS ON WORK, PIPES, ETC.

PROBLEMS which involve work done in various times can be solved by finding what fraction of the work is done in a unit of time. Many problems on the filling of cisterns, the emptying of barrels, etc., can be solved by the adoption of some unit.

Choice of Unit.—The greater part of the difficulty in working many problems of this class is the choice of the best unit with which to work. The great thing to endeavour to avoid is the use of too many fractions, and this is often best done by taking as a unit the L.C.M. of the denominators of the fractions concerned.

The method of manipulation will be best shown by the use of examples.

Example.—*A and B do a piece of work in* 12 *days; A alone could do it in* 20 *days. In what time would B do it alone?*

A and B in 1 day do $\frac{1}{12}$.

A in 1 day does $\frac{1}{20}$.

Finding the L.C.M. of $\frac{1}{12}$ and $\frac{1}{20}$, we obtain $\frac{1}{60}$, which we adopt as our unit.

A and B in 1 day do 5 units of work
A „ 1 „ does 3 „
then B „ 1 „ „ 2 „
and B does the whole work in (60 ÷ 2) days = 30 days
Ans. = 30 days.

Example.—*A cistern can be filled by one tap in* 3 *hrs., by another in* 3 *hrs. and* 40 *min., and emptied by a third in* 2 *hrs. and* 20 *min. If they are all opened together, in what time will the cistern be filled?* (C.S.)

Taking the cistern as the unit—

Part filled by 1st tap in 1 hr. $= \frac{1}{3}$

„ „ 2nd „ 1 „ $= \frac{1}{3\frac{2}{3}} = \frac{3}{11}$

„ emptied 3rd „ 1 „ $= \frac{1}{2\frac{1}{3}} = \frac{3}{7}$

When all are turned on, part filled in 1 hr. $\Big\} = \frac{1}{3} + \frac{3}{11} - \frac{3}{7} = \frac{41}{231}$

Number of hours taken to fill cistern $= \frac{1}{\frac{41}{231}} = \frac{231}{41} = 5$ hrs. 38·048 min.

By the method of L.C.M.

The L.C.M. of the denominators of $\frac{1}{3}$, $\frac{3}{11}$, and $\frac{3}{7} = 231$.

Taking $\frac{1}{231}$ of the cistern as a unit—

Then $\frac{77}{231}$, or 77 units, flow through the 1st pipe in 1 hr.
and $\frac{63}{231}$, or 63 ,, ,, ,, 2nd ,, 1 ,,
and $\frac{99}{231}$, or 99 ,, ,, ,, 3rd ,, 1 ,,
When all are turned on, part filled in 1 hr. $\Big\} = 77 + 63 - 99$ units $= 41$ units
Number of hours to fill cistern $= \frac{231}{41} = 5$ hrs. 38·048 min.

Ans. 5 hrs. 38·048 min.

Example.—*A and B undertake to do each* $\frac{1}{2}$ *of a piece of work. A begins at* 9 *a.m., B at* 10.30, *and both stop at* 12, *having then done* $\frac{1}{3}$ *of the work between them. They resume work at* 1 *p.m., and A finishes his share at* 4 *o'clock: when will B have finished?* (C.S.)

A works 6 hrs. and does $\frac{1}{2}$ the piece of work.
If A in 6 hrs. does $\frac{1}{2}$ the work,
then A in 3 hrs. (from 9 a.m. to 12) does $\frac{1}{4}$ of the work;
then B in $1\frac{1}{2}$ hr. (from 10.30 to 12) does $\frac{1}{3} - \frac{1}{4}$, or $\frac{1}{12}$ of the work.
If B does $\frac{1}{12}$ in $1\frac{1}{2}$ hr.,
then he does $\frac{1}{2}$ in $1\frac{1}{2}$ hr. $\times 6 = 9$ hrs.
Deducting the $1\frac{1}{2}$ hr. he works before noon, he has $7\frac{1}{2}$ hrs. to work after 1 p.m. Thus the time he finishes is 8.30 p.m.

Ans. 8.30 p.m.

Example.—*If A, B, and C could reap a field in* 18 *days, B, C, and D in* 20 *days, C, D, and A in* 24 *days, and D, A, and B in* 27 *days, in what time would it be reaped by them altogether?* (R.M.A.)

A, B, and C in 1 day reap $\frac{1}{18}$ of the field
B, C, and D ,, 1 ,, $\frac{1}{20}$,,
A, C, and D ,, 1 ,, $\frac{1}{24}$,,
A, B, and D ,, 1 ,, $\frac{1}{27}$,,

By addition—

3 A's, 3 B's, 3 C's, 3 D's in 1 day reap $\frac{1}{18} + \frac{1}{20} + \frac{1}{24} + \frac{1}{27}$ of the field $\Big\} = \frac{597}{3240}$

then A, B, C, and D in 1 day reap $\frac{1}{3}$ of $\frac{597}{3240} = \frac{199}{3240}$ of the field.

If they reap $\frac{199}{3240}$ of the field in 1 day
then ,, $\frac{1}{3240}$,, ,, $\frac{1}{199}$ day
and ,, $\frac{3240}{3240}$,, ,, $\frac{3240}{199}$ days $= 16{\cdot}28$ days. *Ans.*

EXERCISE XLII. 42

Problems in Work and Pipes.

Work.

1. A can do a piece of work in 6 days, B can do it in 8 days, and C in 12. How long would they take to complete it all working together? (P.T.)

2. A can do a piece of work in 10 days, and B can do it in 8 days. After A has worked 3 days, in how many days will A and B together finish the remainder? (P.T.)

3. Three men working together complete a piece of work which would take them respectively 15, 20, and 25 days to do if working alone. They are paid £4 14*s.* to be divided in proportion to the work done by each. Find their shares. (C.M.)

4. Three men together fill a pit in $2\frac{1}{2}$ hrs., throwing in respectively 6, 8, and 9 equal shovelfuls of earth per minute. In what time could each man have filled up the pit by himself? (K.S.)

5. A can reap in 8 days a field which takes B 9 days to reap. They work together at harvest, and earn £22 10*s.* 6*d.* between them. If they share it in proportion to the work they have respectively done, what should each receive? (C.W.)

6. If a man, a woman, and a boy can do a piece of work in 3, 4, and 6 days respectively, what fraction of the work would remain unfinished after 1 man, 1 woman, and 2 boys had been working at it for a day? (O.J.)

7. If A can do as much work in 5 hrs. as B in 6 hrs., or as C in 9 hrs., how long will it take C to complete a piece of work, one-half of which has been done by A working 12 hrs., and B working 24 hrs.?

8. A can finish a piece of work in 12 days of 8 hrs., B can finish it in 7 days of 9 hrs. Find in how many days they can finish it together, if they work 6 hrs. a day. (C.W.S.)

9. A can do a piece of work in 14 days of 9 hrs., and B can do the same piece in 12 days of 7 hrs. How long will they take to do it, working together for 8 hrs. a day? (S.L.C.)

10. A and B can do a piece of work separately in 12 and 16 days respectively. They work together at it for 3 days, when A leaves, but B continues, and after 2 days is joined by C, and they finish it together in 3 days. In what time would C do it alone? (C.M.)

11. A can do a piece of work in $3\frac{1}{3}$ days, and B in $2\frac{1}{2}$ days. They work together for a day, and then A leaves B to finish the work alone. How long will B take to finish it? (P.T.)

12. A does $\frac{3}{5}$ of a work in 6 hrs., and his place is then taken by B, who does $\frac{3}{4}$ of the remainder in $3\frac{3}{5}$ hrs. How long will C take to finish what is left, if A, B, and C can jointly do the whole in 4 hrs.? (K.S.S.)

13. C does half as much in a day as A and B can do together, and B does half as much again as A. If all three working together can mow 20 ac. of barley in 16 days, how long would each, working by himself, take to mow 5 ac.? (C.J.)

14. A, B, and C together can do a piece of work in 3 days. A could do it by himself in 5 days, or B could do it by himself in 12 days. How many days would C take to do it by himself? (O.S.L.)

15. Two men, A and B, working alone can finish a piece of work in 7 and 8 hrs. respectively. If they work at it alternately for an hour, A beginning, in how many hours will the work be finished? (C.PR.)

16. A, B, and C can build a certain wall respectively in 16, 20, and 24 days. In how long a time would A and B together build a similar wall of double the length if C helped them for $\frac{1}{2}$ of each day? (C.S.)

17. Two men undertake to do a piece of work for $1\frac{1}{3}$ guin. One could do it alone in 7 days, the other in 8 days. With the assistance of a boy they finish it in 3 days. How should the money be divided? (C.M.S.)

18. Two men undertake to do a piece of work for a guinea. One alone could do it in 6 days, the other in 7 days. With the assistance of a boy they can finish it in 3 days. How should the money be divided? (C.PR.)

19. After a certain number of men had been employed on a piece of work 24 days, and had half finished it, 16 men more were set on, and the remaining half was completed in 16 days. How many men were employed at first, and what was the whole expense of the work at 1*s.* 6*d.* a day per man? (C.S.)

20. Thirteen men equal 20 boys and 12 men can just draw a certain load on level ground. Given the help of 5 men, how many boys would be required to draw a load $\frac{3}{7}$ as heavy where the work is harder by $\frac{1}{10}$ for gradient and by $\frac{1}{8}$ for friction?

21. Two men undertake to do a piece of work for £6. One of them could do it by himself in 16 days, while the other would require 20 days. They employ an assistant, and, all three working together, they complete the work in 8 days. How much of the money has each of them earned? (L.M.)

Problems specifying the Amount of Work two or more do jointly.

22. A and B can reap a field in 10 days, A and C in 12 days, B and C in 15 days. They all work together for 4 days, when A leaves; B and C work together for 6 days more, when B leaves. In how many days will C be able to finish the work? (K.S.)

23. A and B can do a piece of work in 12 days, A and C can do the same amount of work in 15 days, and B and C in 18 days. In what time could each do it separately? (C.S.)

24. A and B can do a piece of work together in 4 hrs., A and C together in $3\frac{3}{5}$ hrs., B and C together in $5\frac{1}{7}$ hrs. In what time will each do the work alone? (P.T.)

25. A and B together can do a piece of work in 12 days, which B and C together can do in 16 days. After A has been working at it for

5 days, and B for 7 days, C finishes it in 13 days. In how many days could each do the work by himself? (C.P.)

26. Three persons, A, B, and C, share a cask of beer. B and C would finish it in 20 days, C and A in 24 days, and B and A in 30 days. After they have shared it together for 8 days, C goes away, and A and B share it for 12 days more. If then B goes away, how much longer will it last for A? (C.H.L.)

27. If 4 men, working 7 hours a day, do a piece of work in 5 days and 1 hour, how many hours a day must 6 men work to do twice as much in 6 days and 4 hours, allowing for a half-holiday on one of the 6 days? (C.S.L.)

28. A, B, and C can do together a piece of work in 10 days. After working at it with B and C for 3 days, A leaves off, and B and C finish it in 10 days more. In what time can A do it alone? (C.M.S.)

29. A, B, and C working together can do a piece of work in 6 days. A could do it alone in 24 days. After working together for 2 days, A is taken ill. How long will B and C take to finish it? (R.M.A.)

30. A, B, and C working together do a piece of work for £3 7*s.* 6*d.* A working alone could do it in 10 days, B working alone could do it in 12 days, and C working alone could do it in 15 days. Divide the money between them in proportion to the quantity of work done by each. (S.K.)

31. Two men, A and B, working together can finish a piece of work in 3 hrs. 44 min.; A working alone could finish the work in 7 hrs. If they worked for 1 hr. alternately, A commencing, how long would it be before the work was finished? (L.M.)

32. If 2 men and 3 boys can do a piece of work in 16 days, and 5 men and 6 boys can do the same in 7 days, how long will 4 men and 8 boys take to do it? (O.S.L.)

33. A piece of work can be done by 3 men and 4 boys in 6 days, by 3 men and 1 boy in 8 days, and by 4 women and 8 boys in 5 days. How long would a woman take to complete the work single-handed? (O. and C.H.C.)

34. Four men, 3 women, and 2 boys can reap 7 ac. of corn in 2 days; 2 men and 6 women can reap 11 ac. in 4 days; and 3 men and 5 boys can reap 5 ac. in 2 days; the men working 10 hrs., the women 8 hrs., and the boys 6 hrs. a day. If a man receives 8*d.* an hour, a woman 6*d.*, and a boy 4*d.*, what would 6 men, 5 women, and 4 boys earn in reaping 33 ac., each working the same number of hours a day as before? (O.S.L.)

35. One man, 3 women, and 4 boys can do a piece of work in 96 hrs; 2 men and 8 boys can do it in 80 hrs.; and 2 men and 3 women can do it in 120 hrs. In how many hours can it be done by 5 men and 12 boys? (O.S.L.)

36. If I must hire 2 men and 3 boys for 6 days to do the same piece of work as 11 men and 5 boys could do in $1\frac{1}{2}$ days, compare the work of a boy with that of a man. (P.T.)

37. Three men and 4 boys can level and turf 372 yds. of a cricket-ground in 3 days, and 4 men and 3 boys can complete 640 yds. in 5 days. Compare the rates of working of a man and a boy. (C.PR.)

38. If 7 men and 2 boys working together can do six times as much

work per hour as a boy and a man together, compare the work of a boy with that of a man. (C.PR.)

39. If a man and 3 boys can do a piece of work in 12 days, and 5 men and 10 boys can do the same in 3 days, in what time could a man alone do it? (C.M.)

40. On a piece of work 2 men and 5 boys are employed, who do $\frac{1}{2}$ of it in 6 days. After this, 1 man and 1 boy more are put on, and $\frac{1}{3}$ more is done in 3 days. How many more men must be put on that the whole may be completed in 1 day more? (C.S.)

41. If either 5 men or 9 boys can do a certain piece of work in 19 days, in how many days can 13 men and 7 boys working together do a piece of work twice as great? (S.K.)

Taps.

42. A tub is filled with water by 2 taps running together in 1 min. 40 sec., and it would be filled by one of the taps alone in 3 min. How long would the other tap running alone take to fill the tub? (L.M.)

43. A cistern holding 800 gals. is filled by 3 pipes in 25 min. One pipe discharges 5 gals. more, and the other 3 gals. less, than the third per minute. Find how much flows through each pipe in a minute. (P.T.)

44. A tank which is $\frac{5}{8}$ full is emptied by a pipe which lets out $11\frac{1}{4}$ gals. in 3 min. 18 sec. At the end of $2\frac{2}{13}$ hrs. $\frac{1}{11}$ of the water in the tank has run out. Find the capacity of the tank. (C.W.)

45. Three taps, A, B, and C, can fill a cistern; A by itself in 24 min., B in 10 min., and C in 27 min. They are all three turned on at once, and after running for $4\frac{1}{2}$ min. B and C are turned off. How long will A then take by itself to complete the filling of the cistern? (C.M.)

46. Two pumps, discharging respectively 80 and 100 gals. per minute, are employed to drain a well. It is observed that if the smaller pump only is at work the water sinks 1 in. per minute, and if both are employed from the first, the well is pumped dry in 2 hrs. Find its depth. (C.S.)

Taps and Waste.

47. A cistern can be filled by one pipe in $30\frac{1}{4}$ min., by another in $38\frac{1}{2}$ min., and it can be emptied by a third pipe in $17\frac{7}{8}$ min. If it be empty and they are all opened together, in what time will the cistern be filled? (K.S.)

48. A cistern can be filled by one pipe in 6 hrs., and by another in 5 hrs. There is a waste-pipe also, and if this is open when the second pipe is working, the cistern takes $7\frac{1}{2}$ hrs. to fill. How long will the cistern take to fill if all three pipes are open? (C.PR.)

49. A cistern is filled by 2 pipes, A and B, and emptied by a pipe, C. A can fill it in 20 min., and B in 30 min., while C can empty it in 48 min. If all 3 pipes are open together, in what time will the cistern be half full? (C.W.)

50. Two pumps are employed to empty a full reservoir holding 4800000 gals. One pump throws 150, the other throws 350 gals. per minute. A stream discharges into the reservoir 20000 gals. per hour. The first pump breaks down after working 18 days. How long will the second pump then take to empty the reservoir? (K.S.)

51. A cistern has a tap, A, by which water runs in, and two others, B and C, fixed in the bottom, by which water runs out. When A and B are open, the cistern is filled in 25 min.; when A and C are open, in 20 min.; when A, B, and C are open, in 30 min. Find how long it will take to empty the cistern if B and C are opened and A kept closed. (C.PR.)

52. Two taps can separately fill a cistern, when the waste-pipe is closed, in 10 and 12 min. respectively, and when the waste-pipe is open they can together fill it in 15 min. How long does it take the waste-pipe to empty the cistern when the taps are not running? (C.S.)

53. Two taps could separately fill a bath in 10 min. and in 12 min. respectively, and a waste pipe could empty it in 8 min. The bath being empty, the two taps are turned on, but after 3 min. it is found that the waste pipe has been left open all the time. The waste pipe is now closed; in how many minutes more will the bath be full? (O.S.L.)

54. Water is rising in a cellar whose level floor has an area of 2000 sq. ft., at the rate of 6 in. per hour, and is $1\frac{1}{2}$ ft. deep when a pump which discharges 1800 c. ft. in an hour is set to work. If the pump is alternately worked for 10 min. and stopped for 3 min., how soon will the cellar be dry? (C.S.)

55. Water runs into a cistern by one pipe and from it by two, in each pipe at a uniform rate when the pipe is open. The cistern holds 1000 gals. If all three pipes be open at once, the cistern, supposed full, would empty itself in 200 min. If one escape pipe and the supply pipe only be open, it would, if empty, fill in 200 min.; but if the other escape pipe and the supply pipe be open, in 100 min. How long would it take the two escape pipes to empty the cistern if the supply pipe were closed? (L.M.)

CHAPTER XXVI.

PROBLEMS INVOLVING UNIFORM INCREASE WITH AN ORIGINAL SUPPLY.

On considering such a simple question as, "If 8 cows eat the grass in a field in 2 days, in what time would 16 cows eat it?" we readily see that a very simple case of simple proportion is involved. If, however, the *uniform growth* of grass in the field is to be considered, then a new element arises.

If the grass in the field grows uniformly, and 8 cows eat the grass in 2 days, it does not follow that 16 cows would eat the grass in 1 day, for although a 2 days' growth is double a day's growth, yet an important consideration is omitted, viz. *the original supply of grass in the field.*

Questions of the type now to be considered readily fall into two classes: (*a*) the class in which the *number of agents* is sought, and (*b*) the class in which the *time* of a certain operation is required.

Examples of Class (*a*).

If 3 *sheep take* 30 *days, and* 5 *sheep* 15 *days, to eat the grass of a field, how many sheep will eat the grass in* 12 *days, the grass being supposed to grow uniformly?*

3 sheep eating for 30 days = 90 sheep eating for 1 day
5 ,, ,, 15 ,, = 75 ,, ,, ,, 1 ,,
∴ 90 ,, ,, 1 day consume the original supply + 30 days' growth
and 75 ,, ,, 1 ,, ,, ,, ,, + 15 ,, ,,
∴ 15 ,, ,, 1 ,, ,, 15 days' growth
and 1 ,, ,, 1 ,, ,, 1 ,, ,,

Now, the required number of sheep has 12 days' growth to eat; *i.e.* they have 30 − 12, or 18 days' growth *less* than the first lot of sheep.

Thus there will be 18 less sheep required, for one sheep eats up one day's growth.

∴ The number of sheep (each eating for 1 day) required = 90 − 18 = 72 sheep for 1 day.

Then, if for 1 day there must be 72 sheep
for 12 days ,, ,, $\frac{72}{12}$, or 6 sheep

Ans. 6 sheep.

Examples of Class (*b*).

If 32 *oxen eat up a field in* 42 *days, and* 29 *could eat up the same field in* 49 *days, how many days would* 25 *oxen take, the grass in all three cases being supposed to grow uniformly?*

32 cows eating for 42 days = 1344 cows eating for 1 day
29 ,, ,, 49 ,, = 1421 ,, ,, 1 ,,
∴ a growth of 7 ,, = 77 ,, ,, 1 ,,
and ,, ,, 1 ,, = 11 ,, ,, 1 ,,

Now, 1344 cows eating for 1 day eat the original supply + 42 days' growth.
But 42 days' growth is eaten by 42 × 11, or 462 cows.
∴ (1344 − 462) cows, or 882 cows, eating for one day eat the original supply.
Now, 25 cows must eat in the required number of days the original supply + the growth during that time.

Then 25 × required time = 882 + 11 × required time
i.e. 14 × ,, ,, = 882
and the ,, ,, $= \frac{882}{14} = 63$

Ans. 63 days.

Questions relating to the emptying of a cistern having an original supply as well as a supplying pipe may be solved by a similar method.

Thus we could "clothe" the preceding problem in this form: "A cistern partly full, into which a steady stream of water is flowing, has a number of holes of equal size in the bottom, which can be opened or closed at will. If 32 holes are opened the cistern is emptied in 42 minutes, if 29 are opened it is emptied in 49 minutes. If 25 holes are opened, in how many minutes will the cistern be emptied?"

The *figures* are the same in both problems, the *principle* is the same; the *words* only have been changed.

EXERCISE XLIII.

Problems in Continuous Growth and Original Supply.

1. If 3 cows take 30 days, and 5 cows take 15 days, to eat the grass of a field, how many cows will eat it in 6 days, the grass growing uniformly?

2. If 20 oxen eat the grass of a field in 30 days, and 16 oxen eat the grass of the same field in 45 days, how many oxen would eat the grass in 60 days if it grow uniformly?

3. If 26 cattle take 8 days, and 24 take 10 days, to eat the grass in a field, how many cattle would eat the grass in 16 days, it being presumed that the grass grows uniformly in all cases?

4. If 18 cows eat the grass of a field of 16 ac. in 14 days, and 15 cows eat the grass of 12 ac. in 11 days, how long will it take 12 cows to eat the grass of 8 acres of the field? (II.)

5. If 109 cattle take 8 days, and 64 take 14 days, to eat the grass of a field, how long would 44 take, it being supposed that the grass grows uniformly?

6. If in a meadow of 20 ac. the grass grows at a uniform rate, and 20 oxen would consume the whole of the grass in 30 days, or 4 oxen would consume 5 ac. of it in 45 days, how many days would 7 oxen take to eat up 10 ac.?

7. Suppose that in a meadow of 20 ac. the grass grows at a uniform

rate, and that 133 oxen could consume the whole of the grass in 13 days, or that 28 of the oxen could eat up 5 ac. of it in 16 days. How long would 25 oxen take to eat up 4 ac.?

8. I find that I can engage 15 workmen for 11 wks., or 31 workmen for 5 weeks, at uniform wages, and in either case pay the wages exactly by means of the interest now accumulated on a certain sum of money and that which will arise during the particular period of engagement. How many workmen could I engage for 20 wks. on the same principle?

9. The grass growing uniformly on a meadow would be consumed by 32 horses in 42 days, and by 29 horses in 49 days. How long must 53 horses be allowed to graze on it so that all the grass may be consumed?

10. If 23 oxen consume 8 ac. of pasture in 26 days, and 25 oxen consume 7 ac. of the same in 20 days, the grass growing uniformly, how many oxen would consume 3 ac. in $5\frac{7}{3}$ days?

11. In a field in which grass grows uniformly, suppose that 31 oxen can consume $8\frac{3}{4}$ ac. in three-quarters of the time in which 15 oxen would consume $5\frac{1}{4}$ ac., and that 22 oxen would require 3 days longer to consume $7\frac{1}{2}$ ac. than 20 oxen would require for $6\frac{1}{4}$ ac. How many oxen would eat up the $8\frac{3}{4}$ ac. in 21 days?

12. A steady stream of water flows into a reservoir partly filled with water. The reservoir supplies a certain number of houses, each supply pipe having the same diameter. If 10 houses are supplied, the reservoir is emptied in 20 hrs.; if 8 are supplied, it is emptied in 35 hrs. How many houses must be supplied in order that the reservoir may be just empty at the end of 14 hours?

13. A cistern partly full, into which a steady stream of water is flowing, has a number of equal holes in the bottom which can be opened or closed at will. If 29 are opened, the cistern is emptied in 14 min.; and if 23 are opened, the cistern is emptied in 20 min. In what time would it be emptied if 19 are opened?

14. A mine is flooded, and water continues to pour in at a uniform rate. It can be pumped dry by 104 men in 9 days, or by 48 men in 27 days. If 209 men are put to work the pumps, in what time will it be dry?

15. A cistern contains a certain amount of water, and receives a uniform supply from a pipe, and if it be emptied by 20 taps in 30 min., or by 16 taps in 45 min., how many taps would be required to empty it in 60 min.?

16. If a cistern which contains a certain quantity of water receive a uniform supply from a pipe, and if it be emptied by 8 taps in 16 min., or by 10 taps in 12 min., how many taps would be required to empty it in 8 min.?

17. A steady stream of water flows into a cistern partly full. If 3 holes (all of equal size) are opened, the cistern is emptied in 30 min.; if 5 are opened, it is emptied in 15 min. In how many minutes will the cistern be emptied when 6 holes are opened?

18. A cistern partly full, containing 7000 gals., into which a steady stream is flowing, has a number of equal outlets which may be opened or shut. If 10 are opened, the cistern is emptied in 20 min.; if 12

had been opened, the cistern would have been emptied in 14 min. Show that in order to empty the cistern at least 6 outlets must be opened, and find the amount of water steadily flowing into the cistern.

(O. and C.S.C.)

19. When a leak was first discovered in a vessel the water had only 6 ft. more to rise to sink her. For the first 8 min. 12 of the 18 men on board worked at the pumps, the water gaining 3 ft. against them; then 14 pumped for 6 min., but the water rose 2 ft. more. All hands then took to the pumps till the ship foundered. How long a time elapsed from the moment when the leak was discovered until the vessel sank?

(C.S.)

20. An empty cistern has two supplying pipes, A and B, and two taps, C and D. A would fill the cistern in $42\frac{1}{2}$ min. and B in 46 min., and D can carry off per minute half as much again as C. After A and B running together have supplied a certain quantity, C is allowed to run with them, and it takes 51 min. to empty the cistern; but had D been turned on with C the two would have taken only $5\frac{3}{4}$ min. In what time would the cistern have been emptied if D had been turned on instead of C? How much of the cistern was filled when C was set open?

CHAPTER XXVII.

PROFIT AND LOSS.

Profit and Loss.

These terms are used in connection with the cost price and selling price of goods. If the cost price is greater than the selling price, the difference is called *loss*; but if the selling price is greater than the cost price, the difference is called *profit* or *gain*.

Absolute and Relative Gain or Loss.

Absolute gain or loss differs from relative gain or loss. Consider the case of two articles.

Cost price, £30; selling price, £40; gain, £10
Cost price, £50; selling price, £60; gain, £10

The *absolute* gain (£10) is the same in each case, but the *relative* gain of the first is greater than that of the second; for £10 is $\frac{1}{3}$ of £30, whereas £10 is only $\frac{1}{5}$ of £50. In the first case there is a gain of $\frac{1}{3}$ of the cost price, in the second case only $\frac{1}{5}$ of the cost price. It will thus be seen that *absolute gain deals with the intrinsic or face value of the money gained, while relative gain is estimated with reference to the cost price.* It is important that this distinction should be remembered, as the failure to understand relative gain or loss is a fertile source of error.

The Standard Cost Price.

Where the cost price is not given, it is both convenient and customary to adopt 100 as a standard cost price. The gain or loss required can then be expressed as a ratio of 100; *i.e.* as a gain or loss of so much per cent. More will be said on this point in the chapter on "Percentages."

Method of Work.

Generally profit and loss may be regarded as an application of proportion and the unitary method, but no general rule can be devised that will meet all cases. Or again, many of the questions, as will be seen in the next chapter, require nothing more than the reduction of

certain fractions to other fractions with 100 for a denominator in each case.

The following formulæ may be noted :—

S.P. = C.P. + gain, or C.P. − loss.
profit = S.P. − C.P.
loss = C.P. − S.P.

where S.P. = selling price, and C.P. = cost price.

Example.—*A woman bought a certain number of eggs at the rate of 4 for 3d., and sold them at the rate of 5 for 4d., by which she made a profit of 2s. How many eggs did she buy?*

C.P. 4 for $3d.$ S.P. 5 for $4d.$
$\therefore$ 1 costs $\frac{3}{4}d.$ $\therefore$ 1 sells for $\frac{4}{5}d.$

$$\text{Gain per egg} = \tfrac{4}{5}d. - \tfrac{3}{4}d. = \frac{16-15}{20}d. = \tfrac{1}{20}d.$$

total profit = $2s.$, or $24d.$
$\therefore$ number of eggs = $24 \div \frac{1}{20} = 480$ *Ans.*

Or, a preferable method, which avoids fractions—

4 for $3d.$ = 20 for $15d.$
5 ,, $4d.$ = 20 ,, $16d.$

$\therefore$ she gains $1d.$ on every 20 eggs
$\therefore$,, $24d.$ on $(24 \times 20) = 480$ eggs.

Ans. 480.

THEORY.

1. Define profit and loss.

2. Distinguish between absolute and relative gain or loss, and give examples to illustrate your answer.

3. What do you understand by the standard cost price?

4. Give formulæ to show the relation that exists between the terms *selling price, cost price, profit,* and *loss.*

EXERCISE XLIV. 4 4

Profit and Loss.

To find Gain or Loss.

1. 14400 lbs. of wool were bought at $5\frac{1}{4}d.$ and sold at $6\frac{3}{4}d.$ a pound. What was the gain? (P.T.)

2. Bought 176 yds. 2 qrs. 2 nls. $1\frac{1}{2}$ in. at $18s.$ $9d.$ per English ell (= 5 qrs.). What is the gain or loss in selling at $6d.$ per inch? (K.S.)

3. A person buys 136 yds. of cloth for £150, and retails it at £1 $18s.$ a yard. What does he gain by the transaction? (P.T.)

4. A nugget weighing 3 lbs. 14 oz. avoirdupois was purchased at £39 4*s.* per pound avoirdupois. On separating the quartz from the gold it was found that the gold was to the quartz as 21 to 4. The gold was then sold at £3 17*s.* 4*d.* per ounce Troy. Find the gain or loss.

5. A man bought 120 cows at Rotterdam, paying £9 7*s.* for each; their carriage to London cost $\frac{1}{31}$ of his outlay. He then exchanged them for 105 ponies, which he sold for £13 10*s.* per head. What did he gain on his expenditure? (P.T.)

6. I buy 325 sheep at 27*s.* 6*d.* each, and 48 lambs at 15*s.* each. Just 13 wks. after I sell them at 39*s.* 3*d.* and 22*s.* 9*d.* each respectively. But in the mean time they cost me on the average 6*d.* each per week for keep, and other expenses reached 16*s.* 3*d.* What is my net gain on the transaction? (C.P.)

7. I buy 500 oranges at 2 for $1\frac{1}{2}d.$, and again 500 more at 2 for 3*d.*, after which 60 of the better sort are eaten; I then sell the remainder at $1\frac{1}{4}d.$ each. How much do I gain or lose? (C.P.)

8. A grocer bought 11 cwt. 3 qrs. of cheese for £18 16*s.* He sold a quarter of his stock at £3 per cwt., and the remainder at 30*s.* per cwt. What was his gain upon the whole? (C.W.)

9. A coal-pit puts out 1000 tons a week, which sell for £370, the large coal bringing 10*s.* and the slack 4*s.* a ton. If the one costs the owner 7*s.* and the other 6*s.* in production, find his weekly profit. (C.S.)

10. A man buys 100 bales of cotton for £8 15*s.* a bale, and sells them at 6*d.* per pound. His customer fails and pays 15*s.* in the £. If the bales average 500 lbs. each, what is his gain? (I.I.)

11. A dealer invests £1 in new-laid eggs at 15 for 1*s.*, and £1 in fresh eggs at 18 for 1*s.* Two-thirds of the new-laid eggs are sold for 1*d.* each, and the rest are put with the fresh eggs, and all are sold at $\frac{3}{4}d.$ each. What profit is made? (C.H.L.)

12. Suppose that £1 = 10*s.* = 100*d.* = 1000 farthings, and that everything less than a $\frac{1}{4}d.$ is ignored in all payments. Then if I buy 21000 books at 3 for 1*s.* and retail them one by one at a third of a shilling each, how much money do I lose?

13. Four-fifths of a draper's stock was destroyed by fire, $\frac{7}{8}$ of the remainder was injured by water. He sold the injured goods at half their cost price for £210, and the uninjured goods he sold for £75. What did he lose by the fire? (P.P.)

To find Price.

14. A man buys 1365 yds. of calico for a certain sum, and sells it for £42 13*s.* $1\frac{1}{2}d.$, one-sixth of this amount being profit. What would 100 yds. of flannel cost him if he paid as much for 15 yds. of flannel as he did for 24 yds. of the calico? (R.N.)

15. The expenses of farming are four times as great as the rent. What is the rent of a farm of 400 ac. if the farmer clears 10*s.* per acre, each acre yielding 16 bush. of wheat, and the wheat fetching 35*s.* per quarter? (C.W.)

16. A man buys a flock of 117 sheep for £248 12*s.* 6*d.* He loses 4, and finds 24 others so sickly that he must sell them at once for 10*s.* apiece less than he gave for them. At what price per head must he sell the rest to gain £24 on the transaction? (K.S.)

17. A man buys 20 tons of hay, and by selling it at 1*s.* 10½*d.* per bundle gains £3. Had he sold them at 1*s.* 9*d.* the bundle he would have lost £2. Find the weight of a bundle in stones, and the cost price per ton.

18. A shopkeeper received 5 chests of tea, and found that by selling it at 2*s.* 6*d.* a pound the last chest was clear profit. At what price must he sell to lower his gains to one-half this rate? (C.M.)

19. A tradesman sold 318 cwt. of sugar for £775 2*s.* 6*d.*, and realized a profit of £39 15*s.* by the sale. What did the sugar cost him per cwt.? (C.P.)

To find Number or Amount.

20. A man sells 60 articles at a profit of 2*s.* 6*d.* each. What additional number must he sell at a profit of 3*s.* 4*d.* each, so as to realize an average profit of 3*s.* 1*d.* on each article sold? (P.T.)

21. A man laid out £12 18*s.* 8*d.* in gloves at 2*s.* 8*d.* the pair. Some of them faded, and were unsaleable; the rest he sold for £14 13*s.* 10*d.* at 3*s.* 5*d.* the pair. How many pairs faded? (C.S.L.)

22. A man hires a cricket-field for 90 days at 2 guins. a day. His expenses are 3*s.* 9*d.* per day; he charges 3*d.* admission per person, 6*d.* for horses, 1*s.* for carriages. He admits 10070 persons, 3000 horses, and makes on the whole £50. How many carriages has he admitted? (K.S.)

23. A farmer selling wheat at 34*s.* a quarter would just pay his expenses. He has, however, to sell half his crop at 33*s.* 8*d.*; but waits then, and sells the remainder at 35*s.* 10*d.*, making altogether a profit of £31 17*s.* 6*d.* How many quarters of wheat did he sell? (C.PR.)

24. The cost price of an article is £2 12*s.* 6*d.* per ton; the selling price is £2 18*s.* 9*d.* per ton. How many tons must be sold to gain £9 5*s.* 11¼*d.*? (K.S.)

CHAPTER XXVIII.

PERCENTAGES.

Per cent. (L. *per centum*).—From its derivation it will be seen that the phrase "per cent." means "for a hundred." Business men adopt 100 as a convenient standard of comparison for estimating profit and loss, and the phrase "per cent." is merely a shortened form of *per centum*. The phrase has a wider meaning here than in Interest, where it always stands for £100.

In percentages the 100 may be either abstract or concrete, and it may refer to any denomination. A gain of £10 on £100, of 10 lbs. on 100 lbs., of 10s. on 100s., each is called a gain of 10 %. The symbol % is often used for the term "per cent."

Percentage.—A percentage is that part of a quantity which is estimated in hundredths. It thus means a number which has the same ratio to 100 as the given number bears to some other number. Percentages are nothing more than special cases of Proportion or Proportional parts.

Rate per cent.—A rate of 5 % means 5 parts out of a hundred; hence rate per cent. means nothing more than the number of hundredths expressed by the figure which indicates the rate.

Rate per cent. may thus be expressed as a fraction with 100 *for a denominator, e.g.* 5 % may be written $\frac{5}{100}$. But any fraction may be reduced to another fraction with 100 as a denominator, for the fraction can be written in the form of a complex fraction with unity for a denominator, and then both numerator and denominator can be multiplied by 100; *e.g.*—

$$\frac{3}{4} = \frac{\frac{3}{4}}{1} = \frac{\frac{3}{4} \times 100}{1 \times 100} = \frac{75}{100} = 75 \text{ hundredths}$$

Again, since any ratio may be expressed as a fraction, *any ratio may*

also be expressed as a percentage; e.g. the ratio of 5*s.* 6*d.* to £2 can be expressed by the fraction $\frac{5s.\ 6d.}{£2}$ or $\frac{5\frac{1}{2}}{40}$;

$$\text{and } \frac{5\frac{1}{2}}{40} = \frac{5\frac{1}{2} \times 2\frac{1}{2}}{40 \times 2\frac{1}{2}} = \frac{13\frac{3}{4}}{100} = 13\frac{3}{4} \text{ per cent.}$$

Absolute and Relative Values.—It must always be remembered that a percentage is a *relative* value, and not an absolute one. A knowledge of this fact will help the student to distinguish between such expressions as $\frac{1}{20}$ and $\frac{1}{20}\%$.

$\frac{1}{20}$ is *absolute*, and means $\frac{1}{20}$ of the whole number.

$\frac{1}{20}\%$ is *relative*, and means $\frac{1}{20} \times \frac{1}{100}$ or $\frac{1}{2000}$ of the whole.

A concrete example will fix the distinction, and will show that the two values may differ widely; *e.g.*—

A man (A) loses £1000 out of a capital of £20,000
another „ (B) „ £1000 „ „ £100,000

A's loss per cent. $= \frac{1000}{20000} \times 100 = 5\ \%$
B's „ „ $= \frac{1000}{100000} \times 100 = 1\ \%$

The absolute loss is the same in each case, but the relative loss of A is 5 times that of B.

Method of Work.—The rule for "Percentages," in most cases, is one of the many applications of Proportion found in the higher rules of arithmetic. Most questions can be solved by proportion, or by rules which can be deduced from proportion.

To find Rate per cent.

Example.—*What percentage of* 25 *is* 6?

The problem is to find a number which bears the same relation to 100 that 6 does to 25.

25 : 6 :: 100 : required number

$$\therefore \text{ required number} = \frac{6 \times 100}{25} = 24$$

Hence 6 is 24 % of 25. *Ans.*

Example.—*Find to* 2 *significant figures by how much per cent.* $\frac{22}{7}$ *exceeds* $\frac{355}{113}$? (L.M.)

$$\frac{22}{7} - \frac{355}{113} = \frac{2486 - 2485}{7 \times 113} = \frac{1}{791}$$

then $\frac{355}{113} : 100 :: \frac{1}{791}$: required percentage

$$\text{Percentage} = \frac{100 \times 113}{\frac{355}{71} \times 791} = \frac{20}{4997} = \cdot 04$$

Ans. ·04 %.

Example.—*Find the percentage of mineral matter in a specimen of 6362 oz. of water containing $3\frac{31}{1050}$ oz. of mineral matter.* (P.T.)

$$\text{Percentage of mineral} = \frac{3\frac{31}{1050}}{6362} \times 100 = \frac{\overset{2}{\cancel{3181}}}{\underset{2}{\cancel{6362}}} \times \frac{\overset{10}{\cancel{100}}}{\underset{105}{\cancel{1050}}} = \tfrac{1}{21}\ \%\quad \textit{Ans.}$$

Rate per Unit.—It has already been shown that percentages are nothing but fractions with 100 as a common denominator. If quantities or relations between quantities are expressed as fractions with a common denominator, they are easy of comparison, and this is one of the chief reasons for the use of percentages; but there is no doubt that the process would be clearer and equally useful, in most cases, if *the absolute proportion to unity* were the recognized standard instead of a denomination of 100.

Compound Percentages.—A compound or gross percentage is a percentage upon a percentage, and is merely a question of multiplication; *e.g.* if in an examination 60 % of the candidates pass, and of these 80 % are accepted, the gross percentage, or percentage of acceptances, is $\frac{60}{100} \times \frac{80}{100}$, which equals $\frac{4800}{10000}$, or $\frac{48}{100}$, or 48 %.

A simple example like this shows the superiority of the method of absolute proportion to that of percentages, for the denominators would have been omitted and the statement would have been written down thus: $\cdot 6 \times \cdot 8 = \cdot 48$.

Any number of percentages can be combined.

Applications of the Term "per cent."—The use of the term "per cent." is a wide one, and it may be said to embrace the whole field of the commercial and scientific world. Percentages may involve the element of time or they may not, but the term is more usually used in the latter sense than in the former.

1. *Involving "Time."*—Percentages involving "time" are found in questions on interest, discount, and compound partnership. These rules are dealt with elsewhere.

2. *Without "Time."*—These include questions on profit and loss, commissions on purchases, sales, or collections, insurance, rates and taxes, bankruptcy, dividends, discount or abatements for cash, simple partnership, and certain statistical tables concerning population, birth, death, and marriage rates, education, employments, mixtures, etc.

Problems on Houses and Estates.

Example.—*A man expects to let a house for £80 per annum, and calculates that after deducting an annual sum for repairs, etc., he will get $7\frac{1}{2}$ % on the purchase money. He lets the house for £76 per annum,*

spends yearly £5 *more than he intended on repairs, and gets* 6 % *on the purchase money. What did he give for the house?*

The *estimated* cost of repairs is common to both rents, and therefore can be eliminated.

The *actual* cost of repairs is £5 per annum above the estimate. The *net rent*, therefore, in the second case is £76 − £5 = £71.

Now, £80 gives $\frac{7\frac{1}{2}}{100}$ of the C.P.

and £71 ,, $\frac{6}{100}$,, ,,

Then, by question, $\left(\frac{7\frac{1}{2}}{100} - \frac{6}{100}\right)$ C.P. = (£80 − £71) = £9

i.e. $\frac{1\frac{1}{2}}{100}$ C.P. = £9

$$\therefore \text{C.P.} = \frac{£9 \times 100}{1\frac{1}{2}} = £\frac{\overset{3}{\cancel{9}} \times 100 \times 2}{\underset{1}{\cancel{3}}} = £600 \quad \textit{Ans.}$$

Problems involving Income-tax.—In questions of this kind, as a rule, additions or subtractions have to be made. In such cases it is advisable to work by Fractions.

Example.—*If the income-tax were* $3\frac{1}{2}$ % *on a man's income instead of* 8*d. in the pound, what difference would it make to a man whose annual income is* £3125? (K.S.)

The ratio of income-tax in each case respectively is $\frac{3\frac{1}{2}}{100}$ and $\frac{8}{240}$.

Reducing to a common denominator, we get—

$$\frac{7}{200} \text{ and } \frac{8}{240} = \frac{7 \times 6}{200 \times 6} \text{ and } \frac{8 \times 5}{240 \times 5} = \frac{42}{1200} \text{ and } \frac{40}{1200}$$

$$\therefore \text{difference} = \frac{1}{600} \text{ of £3125}$$

$$= £5\ 4s.\ 2d. \quad \textit{Ans.}$$

To find Selling Price.

Example.—*I sell a book for* 18*s.* 2*d. and gain* 9 %. *At what price must I sell it to gain* 15 %? (C.M.)

Let 100 = C.P.

then 18*s.* 2*d.* = 109 %, and the problem becomes—

If 109 is the percentage value of 18*s.* 2*d.*, what sum will be represented by the percentage value of 115?

i.e. 109 : 115 :: 18*s.* 2*d.* : required amount

$$\therefore \text{required amount} = \frac{115 \times 218}{109 \times 12}s. = 11\frac{5}{6}s. = 19s.\ 2d. \quad \textit{Ans.}$$

Note.—It is usually asserted that problems of this kind are not proportions. It is true there is no proportion between the given rates and the S.P., for—

9 is not to 15 as 18*s.* 2*d.* is to 19*s.* 2*d.*

A proportion, however, does exist between the given rates and the profit on the C.P.; for the C.P. is 16*s.* 8*d.*, and the profit is therefore 1*s.* 6*d.* and 2*s.* 6*d.* respectively, to which the given rates *are* proportional; for—

$$9 : 15 :: 1s.\ 6d. : 2s.\ 6d.$$

A proportion also exists if percentage values, which contain the standard or assumed C.P., are taken, *i.e.* 109 : 115 :: 18*s.* 2*d.* : 19*s.* 2*d.* It is, therefore, not wrong, as is sometimes asserted, to regard problems of this nature as examples in proportion. The error lies in the attempt to form the proportion with the given rates and the S.P., instead of with percentage values (which contain the standard or assumed C.P.) and the S.P.

To find Cost Price.

Example.—*An article is sold at a profit of* 15 %. *The purchaser sells it again for* £9 15*s.* 6*d.* *at a profit of* $27\frac{1}{2}$ % *on what he gave for it. How much did the article originally cost?* (C.W.)

The article is sold for $\frac{115}{100}$ of C.P., and again for $\frac{127\frac{1}{2}}{100}$ of $\frac{115}{100}$ of C.P.

$$\text{then original cost} = £9\tfrac{31}{40} \times \frac{100}{127\frac{1}{2}} \times \frac{100}{115} = £6\tfrac{2}{3}$$

or, by proportion, if we let 100 = C.P.

$$\frac{115 \times 127\frac{1}{2}}{100} : 100 :: £9\tfrac{31}{40} : \text{original cost}$$

Ans. £6 13*s.* 4*d.*

Example.—*Watches are sold at a gain of* 5 %. *If the S.P. of each watch were increased by* 2*s.* *the gain would then be* 10 %. *Find the C.P. of each watch.* (C.M.)

$$\tfrac{10}{100} \text{ of C.P.} - \tfrac{5}{100} \text{ of C.P.} = 2s.$$
$$\therefore \tfrac{5}{100}, \text{ or } \tfrac{1}{20} \text{ of C.P.} = 2s.$$
$$\therefore \text{C.P.} = 2s. \times 20 = 40s., \text{ or } £2 \quad Ans.$$

To find Profit or Loss per cent.

The profit or loss must be estimated as so much per cent. on the **Cost Price,** unless specially directed otherwise.

Example.—*An article is sold for* £50 *at a profit of* 8 %. *Find the C.P.*

$$\text{C.P.} + \tfrac{8}{100} \text{ C.P.} = \text{S.P.} = £50$$
$$\text{and } 100 \text{ C.P.} + 8 \text{ C.P.} = 100 \times £50$$
$$108 \text{ C.P.} = £5000$$
$$\therefore \text{C.P.} = £\tfrac{5000}{108} = £46\ 5s.\ 11\tfrac{1}{9}d. \quad Ans.$$

Note.—Students sometimes find 8 % on the £50, and then subtract their answer from £50. This is wrong, because the profit is not 8 % on £50, since the £50 contains the 8 % profit, but 8 % on the C.P., which has to be found or assumed.

Example.—*A man makes a profit of* $2\frac{1}{2}$ % *by selling an article for* £1 1*s.* $4\frac{1}{4}d.$ *How much per cent. would he make by selling it for* £1 1*s.* 3*d.*? (S.K.)

Let 100 = C.P.

The problem then becomes: If £1 1*s.* $4\frac{1}{4}d.$ represents $102\frac{1}{2}$ %, what percentage value will £1 1*s.* 3*d.* represent?

i.e. £1 1*s.* $4\frac{1}{4}d.$: £1 1*s.* 3*d.* :: $102\frac{1}{2}$ per cent. : required percentage

$$\textit{i.e.}\ \text{required percentage} = \frac{\overset{2}{\cancel{4}} \times \overset{51}{\cancel{255}} \times \cancel{205}}{\underset{205}{\cancel{1025}} \times \cancel{2}} = 102$$

Ans. 2 %.

Note.—This type of problem is often misstated thus—

£1 1*s.* $4\frac{1}{4}d.$: £1 1*s.* 3*d.* :: $2\frac{1}{2}$ % : gain %.

It is another instance, in such cases, of the failure to distinguish between rates and percentage values which contain the cost price.

Example.—*A schoolboy buys oranges at the rate of three for* 2*d., and sells them at the rate of two for* $1\frac{1}{2}d.$ *What does he gain or lose per cent.? If the buying and selling prices were reversed, what would the percentage of profit or loss be?* (C.S.L.)

B.P. = 3 for 2*d.* or 6 for 4*d.*
S.P. = 2 for $1\frac{1}{2}d.$ or 6 for $4\frac{1}{2}d.$ } gain = $\frac{1}{2}d.$

Then 4*d.* : 100*d.* :: $\frac{1}{2}d.$: gain per cent.

$$\therefore \text{gain per cent.} = \frac{100}{2 \times 4} = \frac{100}{8} = 12\tfrac{1}{2}\ \% \text{ gain} \quad \textit{Ans.}$$

Reversed prices—

$4\frac{1}{2}d.$: 100*d.* :: $\frac{1}{2}d.$: loss per cent.

$$\text{loss per cent.} = \frac{100 \times 2}{9 \times 2} = \frac{100}{9} = 11\tfrac{1}{9}\ \% \text{ loss.} \quad \textit{Ans.}$$

Examples on Schools.

A school contains 380 *scholars;* 5 % *are between the ages of* 12 *and* 14, 10 % *between* 10 *and* 12, 35 % *between* 8 *and* 10, *and the remainder under that age. Find the number of each class.* (P.T.)

Number between 12 and 14 = $\frac{5}{100} \times 380 = 19$
,, ,, 10 ,, 12 = $\frac{10}{100} \times 380 = 38$ } Total 190
,, ,, 8 ,, 10 = $\frac{35}{100} \times 380 = 133$

$\therefore$ remainder = 380 − 190 = 190

Ans. 190.

At an examination, in which full marks were 1000, *A got* 20 % *more than B, B* 10 % *more than C, and C* 20 % *less than D. If A got* 660, *what percentage of full marks was obtained by D?* (S.K.)

A got 660

then B got $\frac{\overset{5}{\cancel{100}}}{\underset{6}{\cancel{120}}} \times \overset{110}{\cancel{660}} = 550$

then C's marks = $\frac{100}{110} \times \overset{5}{\cancel{550}} = 500$

$$\text{and D's marks} = \frac{120}{\cancel{100}} \times \overset{5}{\cancel{500}} = 600$$

$$\therefore \text{D's percentage of full marks} = \frac{\overset{60}{\cancel{600}}}{\cancel{1000}_{10}} \times \cancel{100} = 60\,\%$$

Ans. 60 %.

Questions on Population.

Example.—*The population of a country increases* 3 % *annually, but is decreased by emigration annually to the extent of* $\frac{1}{2}$ % *on the whole. What will be the increase per cent. in three years?* (C.S.)

Let population = 100.

First year—

$$\text{net increase} = 3 - \tfrac{1}{2} = 2\tfrac{1}{2}\,\%$$

$$\therefore \text{population} = \frac{102\frac{1}{2}}{100}$$

Second year—

$$\text{population} = \frac{102\frac{1}{2}}{100} + \frac{2\frac{1}{2}}{100}\left(\frac{102\frac{1}{2}}{100}\right)$$

Third year—

$$\begin{aligned}\text{population} &= \frac{102\frac{1}{2}}{100} + \frac{2\frac{1}{2}}{100}\left(\frac{102\frac{1}{2}}{100}\right) + \frac{2\frac{1}{2}}{100}\left\{\frac{102\frac{1}{2}}{100} + \frac{2\frac{1}{2}}{100}\left(\frac{102\frac{1}{2}}{100}\right)\right\}\\ &= \frac{41}{40} + \frac{41}{40^2} + \frac{41}{40^2} + \frac{41}{40^3}\\ &= \frac{41}{40} + 2 \times \frac{41}{40^2} + \frac{41}{40^3}\\ &= \frac{65600 + 3280 + 41}{64000} = \frac{68921}{64000}\\ &= 1\frac{4921}{64000}\end{aligned}$$

$$\therefore \text{increase per cent.} = \frac{4921}{\cancel{64000}_{640}} \times \cancel{100} = 7{\cdot}689\ldots\,\%\quad \textit{Ans.}$$

Miscellaneous Problems.

Example.—*If the price of coal be raised* 10 %, *find by how much per cent. a man must reduce his consumption of that article so as not to increase his expenditure?* (L.M.)

Raised price = $\frac{110}{100}$ of original price.

The consumption then, when reduced, must be the reciprocal of the increased price;

i.e. it must be $\frac{100}{110}$ of original amount.

If 100 be the original amount, then the reduced consumption is—

$$\frac{100}{110} \times 100 = 90\tfrac{10}{11}$$

$$\therefore \text{reduction per cent.} = 100 - 90\tfrac{10}{11} = 9\tfrac{1}{11}\,\%$$

Ans. $9\frac{1}{11}$ %.

Example.—*A reduction of* 10 % *in the price of an article per pound would enable a man to buy* 10 *lbs. more for* 25*s. What is the reduced price?*

After reduction of 10 %,
90 % of 25*s.* buys as much as 25*s.* before the reduction,
∴ 10 % of 25*s.* buys 10 lbs.,
i.e. 2*s.* 6*d.* buys 10 lbs.,
∴ 1 lb. costs 3*d.*

Ans. 3*d.* per lb.

Example.—*A workman finds that he has to spend* 50 % *of his income on food,* 15 % *on clothes, and* 12 % *on rent, and he saves the remainder. He emigrates to a country where food costs half as much, clothes three times as much, and rent two and a half times as much, but wages are* 50 % *higher. Find whether he is better or worse off, the criterion being the length of time his savings would support him in the country where he is settled.* (L.M.)

First Case.		*Second Case.*	
Food = 50		Food = 25	
Clothes = 15	100	Clothes = 45	150
Rent = 12		Rent = 30	
∴ savings = 23		∴ savings = 50	

Expenditure in 1st case = $\frac{77}{100}$, and savings $\frac{23}{100}$ for any given period of time.
Expenditure in 2nd case = $\frac{100}{150}$, and savings $\frac{50}{150}$ for the same period of time.

∴ relative lengths of time are—

$\frac{77}{100} \times \frac{23}{100}$ and $\frac{\frac{2}{100}}{\frac{150}{3}} \times \frac{\frac{50}{150}}{\frac{150}{3}}$

$\frac{1771}{10000}$ and $\frac{2}{9}$

$\frac{15870}{90000}$ and $\frac{20000}{90000}$

The second fraction is the greater, and he would therefore be better off.

Ans.

Example.—*The cost per ton of the two principal ingredients of a manufactured commodity are in the ratio of* 5 : 4; *the quantities used are in the ratio* 4 : 3; *labour and other expenses are* £6 *per ton of the manufactured product, which is sold at a discount of* 8 % *of the nominal selling price. The quantity sold is equal to that of the chief ingredient, the bye-products being valueless. The expected profit is* 25 %. *But if the prices of the ingredients rise* 10 % *and the selling price falls* 10 %, *the profit will be* $3\frac{1}{2}$ %. *What was the nominal selling price per ton?* (S.A.)

Cost per ton is as 5 : 4; quantities used are as 4 : 3
∴ the cost of the ingredients is as 5 : 3, or as £100 : £60
∴ total cost at this price = £100 + £60 + £6 = £160 + £6
∴ price = $\frac{125}{100} \times$ £160 = £200
and price per ton = $\frac{128}{100} \times$ £6 = £7$\frac{1}{2}$
But price obtained = $\frac{90}{100} \times$ £200 = £180
and price per ton obtained = $\frac{90}{100} \times$ £7$\frac{1}{2}$ = £6$\frac{3}{4}$

The cost was $\frac{11}{10}\frac{0}{0} \times £160 = £176 + £6$ wages, etc.

$$\therefore \text{price obtained finally} = \frac{103\frac{1}{2}}{100} \times £176 = £182{\cdot}16$$

$$\text{and price per ton obtained} = \frac{103\frac{1}{2}}{100} \times £6 = £6{\cdot}21$$

Then by question £182·16 and £6·21 = £180 and £6·75
$\therefore$ £182·16 − £180 is (£6·75 − £6·21) per ton
i.e. £2·16 is £·54 per ton
$\therefore$ number of tons sold = 2·16 ÷ ·54 = 4
$\therefore$ price obtained at first = £200 + 4 × £7½ = £230
and nominal selling price = $\frac{1}{9}\frac{0}{2}\frac{0}{0}$ of $£\frac{2}{4}\frac{3}{0}\frac{0}{0}$ per ton
= £62 10*s.* per ton. *Ans.*

THEORY.

1. Explain "percentage" and "profit and loss per cent." (K.S., C.W.)

2. What is meant by "½ per cent."? (P.T.)

3. Show that to double the selling prices converts a loss of 25 per cent. into a gain of 50 per cent. (C.M.)

4. Explain "rate per unit," and show what relation it bears to "rate per cent."

5. Any fraction or ratio may be expressed as a percentage. Demonstrate this by means of an example.

6. Distinguish between absolute and relative values.

7. Give some common applications of the term "per cent.," (*a*) involving time; (*b*) not involving time.

8. Discuss the method and value of "absolute proportion to unity."

9. Give an example of a compound percentage, and state clearly what is meant by the term.

10. An article is sold for *a* shillings at a gain of *b* per cent. At what price must it be sold to gain *c* per cent., where *c* is greater than *b*? Discuss the method of working this type of problem.

EXERCISE XLV. 45

PERCENTAGES.

Problems on finding "per cent."

1. What percentage (*a*) of 5 are 20, 3½, ·02, and 1¼? (P.T.); (*b*) of 1 ton are 1½ cwt., 1 qr. 15 lbs. 12 oz., 2 lbs. 12⅘ oz.? (C.W.)

2. One liquid contains 22½ per cent. of water, another 27 per cent. A glass is filled 5 parts of the one liquid and 7 parts of the other. What percentage of water is in the glass? (C.M.S.)

3. By selling 48 yds. at 1*s.* 8*d.* a yard a draper gains 4*s.* What was the cost price per yard, and what was his gain per cent.? In actual

trade the percentage of gain is usually calculated on the selling price. What would be the percentage gained if so calculated in the above question? (P.T.)

4. A grocer buys 30 lbs. of tea; he sells 16 lbs. at 2*s.* 7½*d.* per pound, and the rest at 2*s.* 9*d.* per pound, and thereby makes a profit of 15 per cent. If he had sold all at 2*s.* 9½*d.* per pound, what rate per cent. profit would have been made? (C.M.)

5. If a man receives on ¼ of his capital 3 per cent., on ¾ 5 per cent., and on the remainder 11 per cent., what percentage does he receive on the whole? (K.S.)

6. A merchant sells an article at an advance of 12 per cent. on the cost price. How much per cent. of the selling price is profit? (R.M.A.)

7. A cyclometer maker, in graduating his instrument, uses the fraction $\frac{1}{11\frac{1}{11}}$ as equivalent to $\frac{437}{4400}$. How much per cent. is his error? Also how far must the cycle run that the reading may be 1 mi. wrong? (S.A.)

8. If £1869 are given at the Mint for 40 lbs. Troy of gold, and 1939 oz. at the mine are worth £7000, find, to 3 significant figures, how much per cent. the value of gold at the Mint is above that at the mine. (S.A.)

9. With one set of weights my apparent weight is 1·1 times what it is with another set. The first set is 3⅛ per cent. too light. How much per cent. is the second set wrong? (S.A.)

10. If the silver price of gold has risen 113 per cent., find to 2 places of decimal fractions how much the gold price of silver has fallen. (S.A.)

Problems on finding Selling Price.

11. A person buys a quantity of wine at 16*s.* per gallon; 20 per cent. of it is lost through natural causes. Find at what price per gallon he must sell it so as to gain 20 per cent. on his outlay. (I.)

12. A shopkeeper buys goods for £7 18*s.* 4*d.* He sells them so as to give 5 per cent. off the selling price for cash payment, and still to make a profit of 10 per cent. For what sum would he sell the goods on credit? (S.K.)

13. If silk is bought at 3 fr. per metre, what is the least price per yard, to the nearest penny, at which it must be sold, so that the gain may be not less than at the rate of 10 per cent.? (Take a sovereign to be worth 25 fr., and the length of a metre to be 39·4 in.) (C.P.)

14. If I should lose 12½ per cent. by selling a horse to-day for 99 guins., at what price must I sell the horse in 6 weeks' time so as to gain 12½ per cent., including profit at this rate on the cost of keeping him for the 6 wks. at 18*s.* 6*d.* a week? (C.M.)

15. A man rents a farm for £150 per annum. He stocks it with 50 head of cattle, which cost him £625. He spends £62 10*s.* in labour during the year, and sells the cattle at the end of it, realizing his rent, charge for labour, and 10 per cent. profit on the cost price of the cattle. For how much per head were the cattle sold? (C.W.)

16. A person gave £75 for 20 casks of oil, each containing 30 gals. He sold 5 casks at 3*s.* per gallon, 1 cask was stove in and the whole of

its contents lost, and 15 gals. were also lost by ordinary leakage. He then sold the remainder at a price per gallon which made his gain amount to 20 per cent. on the whole transaction. What was his selling price per gallon at the second sale? (R.M.A.)

17. If the cost price of an article is reduced by 1·25 fr. per kilo the profit per cent. will be doubled, but if the sale price is reduced at the same time by 6*d.* a pound, the profit per cent. will be unaltered. Determine the sale price of the article in shillings per pound. (1 kilo = 2·2 lbs., £1 = 25 fr.) (O.S.L.)

18. A merchant buys two kinds of wine, the price of one being 5 per cent. cheaper than that of the other. By selling the dearer wine at 39*s.* per dozen he makes a profit of 17 per cent. At what price per dozen must he sell the other so as to make on it a profit of 20 per cent.? (O.S.L.)

19. A reduction of 40 per cent. in the price of eggs would enable a purchaser to obtain 56 more for £1. What is the reduced price? (S.A.)

Problems on finding Cost Price.

20. A cwt. of sugar when sold at a gain of 5 per cent. yields 2*s.* 3½*d.* more than when sold at a loss of 6 per cent. Find the original cost. (C.PR.)

21. A person bought a horse and sold it at a loss of 10 per cent.; if he had received £9 more he would have gained 12½ per cent. What did the horse cost him? (I.I.)

22. If I sell a horse for £62, and a cow for £26, I gain 10 per cent. on the original cost of both; but if I sell the horse for £63, and the cow for its original price, I lose 10 per cent. Find the original cost of each. (C.M.)

23. A horse dealer bought 76 horses, and then sold 20 at a profit of 15 per cent., 40 at a profit of 19 per cent., and the remainder at a profit of 25 per cent., making a total profit of £657. What did he give for each horse? (C.M.)

24. A man buys two horses, A and B; A cost £25 more than B. He sells A at a profit of 15 per cent., and B at a profit of 8 per cent.; his total gain is £37 2*s.* What was the original price of each horse? (C.M.)

25. A consignment of 1600 lbs. of sugar is sold at 4½*d.* per pound to gain 12½ per cent.; a second of 1800 lbs. at 5½*d.* per pound to gain 10 per cent.; a third of 2000 lbs. is sold at 3*d.* per pound, and a gain of 10 per cent. is made upon the whole. Find the original price per pound of the third consignment. (C.W.)

26. For a monthly return ticket 25 per cent. more is paid than for a single ticket. At the end of the month an extension of the time for a week is obtained by paying 5 per cent. on the monthly ticket. The whole sum paid is £2 12*s.* 6*d.* Find the price of the single ticket. (C.W.S.)

27. If the manufacturer makes a profit of 20 per cent., the wholesale dealer a profit of 25 per cent., and the shopkeeper a profit of 40 per cent., what was the cost of the manufacture of an article bought at a shop for 17*s.* 6*d.*? (C.M.)

28. A man bought a horse and carriage which he afterwards sold for £83, gaining thereby 10 per cent. on the horse and 25 per cent. on the carriage. Had he sold them so as to gain 25 per cent. on the horse and 10 per cent. on the carriage, he would have realized £81 10*s.* How much did he pay for each? (C.W.)

Problems on finding Gain or Loss.

29. The nominal weight of a truck of coals is 8 tons 10 cwt., the actual is 8 tons 18 cwt. If 10 per cent. profit be made by selling the coal by the truck at 22*s.* per ton nominal weight, what rate of profit will be made by retailing it at 22*s.* 8*d.* per ton actual weight? (C.S.L.)

30. A dealer has two sorts of tea which he sells at the same price, thereby making a profit of 5 per cent. on one and 20 per cent. on the other. Determine his total profit per cent. if he sells 7 lbs. of the former to every 2 lbs. of the latter. (O.S.L.)

31. A dealer was making a profit of 25 per cent. by selling whisky at 15*s.* per gallon. Owing to a rise in the duty on spirits, he paid 6*d.* per gallon more than before for similar whisky, and sold it at 15*s.* 6*d.* per gallon. What was his profit per cent. after the change? (C.J.)

32. A druggist buys a certain commodity by avoirdupois weight and sells it by Troy weight. The buying price is 3*d.* per ounce, and he sells at 6*d.* What is his gain per cent.? (R.M.A.)

33. Apples are bought at the rate of 7 for 3*d.*; half of them are sold at 5 for 2*d.*, one-fifth at 7 for 4*d.*, and the rest at 7 for 5*d.* What is the smallest number of apples that can be so dealt in, and what is the gain or loss per cent.? (I.I.)

34. Postcards stamped with $\frac{1}{2}d.$ stamps are sold by the Post Office at 10 for 6*d.* if thick, and 10 for $5\frac{1}{2}d.$ if thin. What is the saving per cent. on each kind, on the cost of the cards (independently of postage), by using cards bought at 100 for 3*d.* and affixing $\frac{1}{2}d.$ stamps? (C.P.)

35. What profit per cent. is made by selling an article at a certain price, if by selling it at half that price there would be a loss at the rate of $12\frac{1}{2}$ per cent.? (C.P.)

36. A tradesman paid 8*s.* per bushel for plums, and at the rate of 8*s.* per cwt. for their delivery at his place of business, and professed to sell them at 4*d.* per pound, but gave an average weight of $16\frac{1}{2}$ oz. instead of 1 lb. At what rate per cent., correct to one decimal place, was his profit if the average weight of a bushel of the plums was 54 lbs.? (C.P.)

37. A publisher sells books to a retail dealer at 5*s.* a copy, but allows 25 copies to count as 24. If the retailer sells each of the 25 copies at 6*s.* 9*d.*, what profit per cent. does he make? (S.K.)

38. A man buys 45 bush. of apples at 2*s.* 6*d.* a bushel; $\frac{1}{3}$ of them is spoilt; of the remainder he sells $\frac{1}{3}$ at 2*d.* a pound, and $\frac{2}{3}$ at 4*s.* a bushel (a bushel of apples weighs 40 lbs.). What profit does he make? And what is his profit per cent. of his outlay? (S.K.)

Problems on finding Amount.

39. During a campaign an army lost 17 per cent. of its strength by disease, etc., and then lost 16 per cent. of the remainder in a battle; the number then remaining was 6972. Of how many did the army consist at first? (C.W.)

40. The number of sheep on a farm increases at the rate of 20 per cent. per annum. If at the end of two years there are 720 sheep, how many were there originally? (C.M.)

41. A man buys some fruit at 12*s.* 6*d.* per bushel, and makes a profit

of 20 per cent. by selling it at $3\frac{1}{2}d.$ per pound. Find the weight of a pint of the fruit? (C.H.L.)

42. Seventy-five per cent. of the area of a farm is arable. Of the remainder 85 per cent. is pasture, and the rest is waste; the area of the waste is 3 ac. 0 rd. 20 po. What is the area of the farm? (S.K.)

43. In a forest 5 per cent. of the trees are blown over in a gale, and after 3 per cent. of those remaining are cut down there still stand 55290 trees. How many trees were there in the forest before the gale? (C.S.L.)

44. A reduction of 40 per cent. in the price of half-crown tea enables me to buy for my money 6 lbs. more than I could previously buy for a sovereign. What amount have I? (C.P.)

Problems on Houses and Estates.

45. If the price of land fall 30 per cent., so that ·15 of an estate becomes worth £1400, find the original value of the whole estate. (K.S.)

46. The purchaser of half an acre builds a house upon it, and lays out the ground at a total cost of £2094 5*s.* What rent per annum must he obtain so as to realize 9 per cent. on his whole outlay? (R.M.A.)

47. A man buys 18 houses for £11196; he repairs them, and lets each house for £27 12*s.* 6*d.* per annum, thereby getting a return of $4\frac{1}{4}$ per cent. per annum on his money. What did he spend on repairs? (C.W.)

48. If $\frac{7}{15}$ of an estate be worth 200 guins., and the value of the estate be increased ·075 per cent. by improvements, find the value of $\frac{28}{45}$ of the improved estate. (K.S.)

49. A man built a house and sold it for £800, losing 4 per cent. by the transaction. What did it cost him to build? If he had not sold it, at what rent would he have had to let it so as to make 6 per cent. on his outlay? (C.H.L.)

50. A farmer, who just makes ends meet, sells all his produce and pays annually £120 rent, £50 wages, and £50 for household expenses. The price of farm produce falls $20\frac{5}{7}$ per cent., and wages and household expenses remain as before. By how much per cent. must the rent be reduced if the farmer is still to make ends meet? (K.S.)

51. If a single house costs £1600 for building, $\frac{3}{5}$ as much for furniture and $\frac{1}{20}$ as much for drainage, and if in building five houses at once 7 per cent. is saved of the cost of building, $5\frac{1}{2}$ per cent. of that of furnishing, and 9 per cent. of that of drainage, find the cost of the five. (C.H.L.)

52. If a man leaves at his death an estate whose value is between £25000 and £50000, it has to pay a duty of $4\frac{1}{2}$ per cent.; but if the value be over £50000 and under £75000 it has to pay 5 per cent. Find the difference between the net values of two estates whose gross values are £49900 and £50100 respectively. (C.H.L.)

Problems on Rates and Taxes.

53. A farm of 200 ac. is rated at 2*s.* 6*d.* an acre. After the rent has been diminished by 4 per cent. and the rates have been increased by £5, it is found that the rates amount to $12\frac{1}{2}$ per cent. of the rent. Determine the original rent of the farm. (O.J.)

54. An estate has been let for £480 a year, less landlord's taxes

amounting to 15*d.* in the £. If the rent is reduced to £450 a year, and taxes rise to 16*d.* in the £, by how much per cent. will the landlord's net receipts be reduced? (O.S.L.)

55. The gross income of an estate is £5520. What is the net income after paying income-tax at 6*d.* in the £, and rates to the amount of $18\frac{3}{4}$ per cent. on the gross income? (K.S.)

56. The income-tax was reduced from $7\frac{1}{2}d.$ to 6*d.* in the £, but a man's total income was at the same time reduced by $3\frac{1}{8}$ per cent. Find by what percentage his net income was altered. (K.S.)

57. A man finds that his gross income for 1901–2 is greater by 5 per cent. than his gross income for 1900–1. His net income after paying income-tax has increased by £31. The income-tax is now 1*s.* 2*d.* in the £, and was 1*s.* in the £ previously. What is his net income now? (C.M.)

58. If an income-tax of 1*d.* in the £ produce £3857216 11*s.* $1\frac{3}{4}d.$, and the capital of the country be supposed to produce an income of $2\frac{7}{8}$ per cent., find the capital of the country. (C.M.)

59. The gross rental of an estate is £13500, and deductions arising from rates and taxes at 3*s.* in the £, and interest on a mortgage of £3000 amount to 16 per cent. of this rental. What is the rate of interest paid on the mortgage? (P.E.A.)

60. A landlord's net rental, after paying 5 per cent. for collection, and 8*d.* in the £ income-tax on the remainder, is £580. Find his gross rental. (L.M.)

61. A house in France is bought for 16000 fr. The purchaser has to pay annually a ground rent of 40 fr., and 50 fr. in taxes. What must be the annual rent in order that each year his net profit may be 4 per cent. on the outlay? (C.P.)

Problems on Schools and Examinations.

62. In a school of four sections, the first contained 16 per cent., the second 22 per cent., the third 30 per cent. of the scholars, and the fourth section contained 80 scholars. How many scholars are in the first, second, and third sections respectively? (P.T.)

63. In a school of three departments the number in the boys' department is 31·25 per cent. of the whole, and the number in the girls' 28·75 per cent. of the whole. If there are 20 more boys than girls, find the number in the infants' department. (C.W.)

64. The cost of maintaining a certain school is £110 a year. The Government Grant amounts to £29 10*s.*, and the school fees to $\frac{17}{55}$ of that sum. What percentage of the expenditure is met by (*a*) the grant, (*b*) the fees? Give the answers in decimals. (P.T.)

65. In Ontario 47 out of 129 schools are free, and the aggregate school fees amount to 15 per cent. of the total receipts. The receipts from other sources being taken to be the same in all schools, how much per cent. of the average receipts from fee schools are due to fees? (S.A.)

66. If $\frac{1}{5}$ of $\frac{4}{7}$ of the population be taken as the total number of children who should be *on the registers* of elementary schools, how many children should be found *attending* such schools in a parish containing 9065 persons, allowing 15 per cent. for unavoidable absences? (P.T.)

67. Of the children attending school in Switzerland, 71·7 per cent. speak German only, 22·4 speak French only, 4·5 speak Italian only, and

the remainder consists of 5838 children who speak only Romansch. Find the total number of children attending school. (L.M.)

68. The percentages of actual to possible passes obtained at examination by the scholars of a public elementary school in reading, writing, and arithmetic were 95, 89, 87 respectively; the total number of passes was 813. Find the number examined. (K.S.)

69. At a recent examination every candidate took French or Latin: 75·8 per cent. took French, and 49·4 per cent. took Latin. If the whole number of candidates was 2500, how many took both French and Latin? (C.M.)

70. In a mixed school, 20 per cent. of the scholars are infants under 7, and the number of girls above 7 is $\frac{2}{3}$ the number of boys above 7, and amounts to 64. Find the number of scholars in the school. (P.T.)

71. Supposing an arithmetic paper to be done by 5000 candidates, of whom $\frac{1}{10}$ are girls and the rest boys, and supposing 20 per cent. of the girls and 10 per cent. of the boys to fail, find what percentage of the whole number will succeed in passing. (C.W.S.)

Problems on Proportional Parts.

72. Three persons, A, B, and C, send 55, 70, and 75 sheep respectively to a field of grass, the rent of which is £30. What percentage of the rent should each pay? (P.T.)

73. A sum of £3000 is to be divided among A, B, and C, so that B shall have 100 per cent. more than A, and C 50 per cent. more than B. What will each receive if 10 per cent. of the money be first given to a local charity? (P.T.)

74. A man lends money at 20 per cent., but in consequence of bad debts he only makes a profit on the whole of 8 per cent. What is the proportion of his bad debts to the whole sum of money he lends? (C.H.L.)

75. A man divided his property amongst his three sons in the proportion of $\frac{3}{5}$, $\frac{5}{12}$, and $\frac{3}{8}$. What percentage of the property did they respectively get? (Correct to two places of decimals.) (P.T.)

76. A legacy of £12000 is left to A, B, and C. After they have paid legacy duty of 15 per cent., A receives twice as much as B, and B three times as much as C. What are their respective shares? (S.A.)

Problems on Population.

77. If the annual increase in the population of a state is 25 per thousand, and the present number of inhabitants is 2624000, what will the population be in three years' time? and what was it a year ago? (R.M.A.)

78. The populations of the upper and lower parts of a town were equal, and after the former had fallen 20 per cent., and the latter risen 15 per cent., the total number of inhabitants was 39390. What was the population of each part at first? (P.E.A.)

79. The population of a town has increased 1 per cent. in a given interval; but if it had now been 50 less than it is, there would have been a decrease of 1 per cent. Find the original population. (C.P.)

80. The population of London is 4500000. The annual excess of

births above deaths is 10 per thousand per annum. What will be the population in four years' time if the birth rate and death rate per thousand per annum continue unaltered? (C.W.)

81. In a colonial town 28 per cent. of the inhabitants were born in Europe, and the rest were born in the colony. Of the former, $\frac{3}{4}$ are of British origin, and of the latter, 81 persons are of British origin. What is the total number of inhabitants of the town if 18 per cent. of them are of British origin? (C.S.)

82. If in 1871, when 702201 tons of sugar were consumed in the United Kingdom, the consumption was 49·93 lbs. per head, and the corresponding numbers for 1881 were 989208 tons and 62·77 lbs. per head, find to the nearest whole number the increase per cent. in the population during the interval. (C.H.L.)

Miscellaneous Problems.

83. Show that, if the rats in a parish increase in number by 50 per cent. per annum, they will, after an interval of 8 yrs., be more than 25 times as numerous as they were. (C.P.)

84. Supposing that the duty on spirits was 6*s.* 3*d.* per gallon, and that 10740000 gals. were consumed, find how the revenue would be affected if the duty had increased 30 per cent., whilst the consumption was reduced 3 per cent. (C.S.)

85. A publisher, when he sells a certain book at 13 copies for the price of a dozen, gains £7 1*s.* 8*d.* on 100 copies; but when he sells it at 25 copies for the price of 24, gains $10\frac{70}{383}$ per cent. on the cost. Find the cost and the selling price of each copy. (C.P.)

86. A gentleman receives through his agent an income of £375. His agent deducts $2\frac{1}{2}$ per cent. for agency, and also income-tax calculated at a certain number of pence for each pound, both calculated on the gross income; he remits £353 2*s.* 6*d.* What is the income-tax in the pound? (C.P.)

87. A farmer lost 125 beasts by disease, their average value being £17 10*s.* 8*d.* From the county rate he received an amount equal to 56 per cent. of his loss, and his landlord made him an allowance which covered one-fourth of the remainder. To what extent did the farmer suffer? (O.R.)

88. I buy a horse with ·5 of my money, and sell it again so as to gain 15 per cent. I now invest all but 15*s.* of what I got for the horse in 3 cows, one of which dying, I sell the remainder for £96, thus losing 20 per cent. How much had I before buying the horse? (P.T.)

89. If pure gold is worth £4 5*s.* per ounce, and silver is worth 5*s.* 6*d.* an ounce, what would be the weight of a 15*s.* piece containing 92·5 per cent. *of value* pure gold and the rest silver? What would be the weight of the piece if 92·5 per cent. *of weight* be substituted for 92·5 per cent. *of value*? (P.T.)

90. A fishmonger buys a certain quantity of fish; 21 per cent. becomes worthless before it can be sold, and 14 per cent. has to be sold at $\frac{1}{2}$ of its cost price. At what rate per cent. profit must the rest be sold so that upon the whole 50 per cent. may be gained on the original purchase? (C.S.)

91. A tradesman marks his goods at a price which will give him 25 per cent. profit, and he takes off $12\frac{1}{2}$ per cent. from the marked price

for ready money. If an article marked 5 guins. is sold at the reduced price, find (1) what he gets for it; (2) what he gave for it; and (3) what rate per cent. profit on his outlay he still makes? (C.S.)

92. A gas meter, after being used 96 days, registers 11280 c. ft. as the consumption. The meter is then tested, and found to register 17·5 per cent. in excess of the gas actually consumed. Find the amount that ought to be charged at 3*s.* 9*d.* per 1000 c. ft., and the average cost per night. (C.S.)

93. A city footway $12\frac{1}{2}$ ft. wide requires a quarter of a million of 6-in. tiles to lay it down. What is the length of the footway, and what should a contractor be paid in order to gain 10 per cent. on the job, if the tiles cost 13*s.* $10\frac{1}{4}$*d.*, and the labour, etc., $\frac{4}{15}$ of that amount per square yard? (P.T.)

94. The annual consumption of butcher's meat in the United Kingdom being 33697785 cwt., what amount of saving would be effected to the community by a reduction of $2\frac{1}{4}$*d.* per pound, and what percentage would be saved by such reduction if we take the present average price of meat as $9\frac{1}{4}$*d.* per pound? (P.T.)

95. A square cistern is $2\frac{1}{2}$ ft. broad and 3 ft. deep. It is full of water. All the water is allowed to run over a rectangular water-tight yard 20 ft. by 15 ft. There the water completely freezes, and the ice is found to be $\frac{9}{11}$ of an inch thick. Find by what percentage of its volume the water has increased by freezing. (P.T.)

96. A manufacturer sells matches to a wholesale dealer, the dealer supplies them to a retail shopkeeper, and the shopkeeper sells to the public 12 boxes, each containing 60 matches, for $2\frac{1}{2}$*d.* If the net profits of each vendor are 10 per cent. of his receipts, what is the cost of production to the manufacturer per million matches, inclusive of boxes and all other incidental expenses? (O.J.)

97. A tradesman, who professed to sell his goods at 25 per cent. under cost price, explained that what he called cost price was in reality 30 per cent. above it, and that he expected to clear 5 per cent. profit. Find (1) what percentage he really gained or lost; (2) what was the amount of gain or loss on goods which he sold for £92 12*s.* 6*d.* (C.M.)

98. A miner was receiving 5*s.* a day for $4\frac{1}{2}$ days' work per week. When his wages were increased 40 per cent. he worked only $3\frac{1}{2}$ days per week. If now 10 were taken off this advance of 40 per cent., he would work 4 days per week, and if 25 were taken off, $4\frac{1}{2}$ days. Find the weekly wages received in the four cases. (C.H.L.)

99. A merchant, having bought 400 tons of coal, reckons that by selling them at 17*s.* $6\frac{1}{2}$*d.* per ton he will make $5\frac{1}{4}$ per cent. on his outlay. After selling 300 tons at that rate, he disposes of the remainder at a price which reduces his profit on the whole to 5 per cent. Find (1) what he gave for the 400 tons of coal, and (2) at what price per ton the second lot was sold. (C.S.)

100. The expense attending the production of a book, the retail price of which is 7*s.* 6*d.*, is 2*s.* $4\frac{1}{2}$*d.* per copy. The publisher allows the bookseller 25 per cent. on the retail price, and gives 13 copies to the dozen; 3900 copies are printed and sold. The author is to have half the profits. How much will he receive? (C.S.)

CHAPTER XXIX.

COMMISSION, BROKERAGE, INSURANCE.

Commission.—Commission is a percentage on the price paid for goods. A commission is paid to an *Agent*, who is a person engaged in the purchase or sale of goods for another.

Brokerage.—This is the commission of a special kind of agent called a Broker. As a rule he receives smaller percentages than an agent. The broker's chief business lies in the purchase or sale of Stock, in Bills of Exchange, etc.

Discount.—This is an allowance made by merchants and tradesmen for cash payments. It is different from what is usually understood in arithmetic by "discount," as there is no reference to time. Cash payment excludes the idea of time.

Insurance.—This is a system by which people protect themselves from loss or damage due to various causes. It is a contract of indemnity against loss or damage.

The Insurance Company agrees to pay a certain sum in case of loss or damage to the person insured, who, on his part, agrees to make an annual payment of a certain sum to the company.

Underwriter is a special name given to Marine Insurers.

Policy.—The contract or mutual agreement. It is the legal document which secures the insured person, or policy-holder, from loss.

The Sum insured, or Claim.—The amount payable by the Insurance Company in the event of loss or damage.

Premium.—This is the sum paid for insurance. It may be single or annual, but it is generally the latter. It is a certain percentage on the estimated value. The amount of the annual payment, or the rate of the percentage, depends on various conditions, such as—

1. The nature and value of the property insured.
2. The amount of risk to which it is exposed.
3. The age, health, and occupation (in Life Assurance) of the person insured.

Covered Insurance.—A man may insure not only against loss, but also to cover the value of his premiums and any other expenses he may

have been put to in connection with his insurance. He pays a slightly larger premium, and his policy is then said to be covered.

Method of Work.—The rule is merely an application of proportion, or it may be considered as an application of percentages, which rule itself is one of the many applications of proportion.

Commission.

Example.—*A traveller obtains orders to the value of* £780 15*s.*, *and he receives commission at the rate of* 2½ %. *What does his commission amount to?*

By proportion—

$$£100 : £780\tfrac{3}{4} :: £2\tfrac{1}{2} : x$$

$$x = £\frac{3123 \times 5}{4 \times 2 \times 100} = £\frac{3123}{160} = £19\ 10s.\ 4\tfrac{1}{2}d. \quad Ans.$$

By practice—

	£	s.	d.		
½	780	15	0	= commission at 100 %	
			2		
	1561	10	0	= ,, ,,	200 %
	390	7	6	= ,, ,,	50 %
100	1951	17	6	= ,, ,,	250 %
	19	10	4½	= ,, ,,	2½ %

Ans. £19 10*s.* 4½*d.*

To find Commission. Rule.—Multiply the amount by the rate per cent. and divide by 100.

Since a commission is a percentage, the method of working is the same as in percentages.

Premium.

Example.—*What annual premiums should a person pay to insure his life for* £875, *at* £3 7*s.* 6*d. per cent.?*

$$£100 : £875 :: £3\ 7s.\ 6d. : x$$

$$x = £\frac{875 \times 3\frac{3}{8}}{100} = £29\ 10s.\ 7\tfrac{1}{2}d. \quad Ans.$$

To find Premium. Rule.—The rule is practically the same as that for commission.

Covered Insurance.

Example.—*What sum should be insured at* £3 7*s.* 6*d. per cent. on goods worth* £2319, *in order to recover in case of loss the premium as well as the value of the goods?*

If premium + value = £100, then, since the premium is £$3\frac{3}{8}$, the value will be £$96\frac{5}{8}$.

Hence £$96\frac{5}{8}$: £2319 :: £100 : x

$$x = £\frac{2319 \times 100 \times 8}{\overset{3}{\cancel{768}}} = £2400 \quad \textit{Ans.}$$

To find Covered Insurance. Rule—

1. Subtract the rate per cent. from 100. $(100 - 3\frac{3}{8})$
2. Divide the total value by this amount. $\left(\frac{2319}{96\frac{5}{8}}\right)$
3. Multiply the quotient by 100.

Or, more briefly—

$$\text{Divide the total value by } 1 - \frac{\text{the rate}}{100}$$

Note.—Young students often misstate this type of problem. The usual error is to *add* the rate to £100 instead of *subtracting* it. They forget that the £100 includes the total value + the premium as the amount to be paid on, *i.e.* they make, in this case, per cent. to mean £$103\frac{3}{8}$ instead of £100. The amount actually recovered on £100, after the premium has been allowed for, is £100 − £$3\frac{3}{8}$, or £$96\frac{5}{8}$.

THEORY.

1. Define Discount and Insurance, with examples. (C.M.)

2. Explain the meanings of each of the terms "premium," "policy," "claim," as used in connection with life insurance. (C.M.)

3. Why is the charge per cent. for life assurance different for men of different ages, and why different also frequently for men engaged in different occupations? (P.T.)

4. What is the rule for finding covered insurance, and what common error is often associated with this rule?

EXERCISE XLVI. 46

Commission, Insurance.

1. What is the premium on a policy of insurance for £9626 11*s.* 3*d.* at £2 12*s.* per cent.? (C.S.)

2. A house worth £1125 is insured for $\frac{3}{4}$ of its value at £$2\frac{1}{4}$ per £100. How much is the annual premium? (P.T.)

3. A farmer insured his crops at the rate of 6*d.* per acre, and calculated that the produce of his 55 ac. would be worth £550. The value of the crops fell short of his estimate by 25 per cent. What percentage of their actual value did he pay in insurance? (P.T.)

4. The cost of 14560 qrs. of oats at 27*s.* per quarter, together with the dealer's commission (reckoned on the cost of the oats), amounted to £19901 14*s.* At what rate per cent. was the commission charged by the dealer? (P.P.)

5. The annual premium on a fire insurance is £13 5*s.* 3*d.*, the duty charged on it will be £19 17*s.* 10½*d.* How much per cent. of the premium is the duty? (O.S.L.)

6. A person aged 40 next birthday can assure £5 at death by an annual payment of 3*s.* 3*d.* to the Post Office, and £8 16*s.* by a weekly payment of 2*d.* to an insurance office. Show that the rate of premium charged by the insurance office is $\frac{17}{33}$ higher than that of the Post Office. (C.W.)

7. Make out a bill for the following goods shipped to New York, together with the charges upon them: Irish linen, 475 yds. at 3*s.* 4*d.* per yard; black cloth, 37 yds. at £1 0*s.* 6*d.* per yard. Charges: packing, £1 19*s.* 1*d.*; shipping, £4 7*s.* 8*d.*; insurance and commission on £130 at 2½ per cent. (C.J.)

8. A person has $\frac{1}{7}$ of a ship worth £6600, and insures for 91·25 per cent. of its real value. What amount of damage would he sustain in case of the ship being lost? (P.T.)

9. A man pays for insurance on his life 15 per cent. on his gross income, and after paying income-tax on remainder at 6*d.* in the £ he has £513 16*s.* 6*d.* left. What is his gross income? (P.E.A.)

Covered Insurance.

10. For what sum should goods worth £4384 0*s.* 3*d.* be insured at £2 6*s.* 8*d.* per cent. that the owner may recover in case of loss the value of both goods and premium?

11. For what sum must I insure goods worth £2192 0*s.* 1½*d.* at 2½ per cent., so that in case of loss their value and the premium may be both repaid? (C.S.)

12. A merchant insures a ship and cargo for £15320 at a premium of 4¼ per cent. to cover value of ship and costs of insurance. Find the premium paid, and the value of ship and cargo. (C.W.)

13. A ship value £13775 is insured at 5 per cent. on such a sum that the owner in case of loss may receive both the value of the ship and the amount of the insurance. For what sum must it be insured? (C.W.)

14. A man insures to the value of £445 6*s.* 8*d.* on goods, the rate of insurance being 4¾ per cent. Losing the goods by an accident, he finds that he recovers both their actual value and the premium paid. What was their actual value? (C.S.)

15. For what amount must goods worth £3517 10*s.* be insured at 2½ per cent., so that in case of loss the value of the goods and the premium may be recovered? (I.I.)

16. For what sum must a ship worth £2000 be insured at £3 15*s.* per cent., so that in case of loss her cost and also the premium of insurance may both be recovered? (R.U.I.)

Miscellaneous.

17. What would it cost to convey 144 mi. at the rate of 5*s.* per ton per mile (insurance being demanded at $\frac{1}{8}$ per cent.) a box, weighing itself $3\frac{1}{2}$ lbs., and containing 1050 rings, each weighing 1 dwt. 4 grs., and each being worth £1 2*s.*? (P.T.)

18. A shipowner, insuring a certain steamer for £17000, paid insurance $2\frac{1}{4}$ guins. per cent., policy £$\frac{1}{4}$ per cent., brokerage £$\frac{3}{8}$ per cent., commission £$1\frac{1}{8}$ per cent. She received damage to the extent of £3750. What sum should the shipowner recover after deducting expenses of insurance, and allowing $2\frac{5}{16}$ per cent. discount on the damage? (P.T.)

19. A sub-collector receives 10 per cent. of the collector's commission after 45 per cent. of it has been deducted for office expenses. The collector's commission is $1\frac{1}{2}$ per cent. on a ratable value of £33000. What will the sub-collector receive? (C.W.)

20. I can make a profit of 25 per cent. My agent can make a profit of 30 per cent., but charges me a commission of $4\frac{1}{2}$ per cent. on all returns. How much more profit do I make in selling what costs me £1000 than I obtain if my agent sells it? (C.M.)

21. A man spent $\frac{1}{5}$ of his income on the education of his children, $\frac{1}{10}$ in wages, $\frac{1}{12}$ in rent, paid income-tax of 8*d.* in the £, and rates of 3*s.* 9*d.* in the £ on his rental, paid in insurance £120, and household expenses £8 a week. He recovered from the Inland Revenue the income-tax on £150 and on his insurance, and had then £154 5*s.* remaining. What was his income? (C.W.)

22. A commission agent offers to take either $2\frac{1}{4}$ per cent. on the money passing through his hands, or $\frac{1}{3}$ of the net profits of his sales. The sales amount to £300,000, of which 35 per cent. produce no profit; but the remainder realize a profit of $10\frac{1}{2}$ per cent. What difference will it make to the agent which of his offers his customers accept? (P.T.)

CHAPTER XXX.

SIMPLE INTEREST.

Terms defined.

1. **Interest.**—Interest is a money payment for the use of money which has been lent. It is always a percentage of the sum lent, and it is paid quarterly, half-yearly, or annually.

2. **Principal.**—This is the name given to the loan or sum of money which is lent.

It is sometimes called Capital, but the term is not a good one, as "capital" may include much more than "principal."

3. **Rate.**—The rate is the number of pounds sterling paid for the use of each £100 of the money lent, and consequently it is called the rate per cent.

The usual symbol for rate per cent. is %.

4. **Time.**—The time is the period for which the money is lent. The rate per cent. is usually expressed as so much per annum.

The words *per annum* are sometimes omitted, but, unless otherwise stated, they should always be understood.

5. **Amount.**—The amount for any given time is the principal plus the whole of the interest for that time.

Classes of Problems.—In all problems in simple interest four quantities have to be considered, viz. (1) Interest (or Amount); (2) Principal; (3) Rate; and (4) Time. If any three of these are given the fourth can be found. Hence four cases arise –

1. Where interest or amount has to be found.
2. ,, principal ,, ,,
3. ,, time ,, ,,
4. ,, rate ,, ,,

There is one apparent exception, when the three given quantities are amount, principal, and interest; but since $A = P + I$, two quantities only are really given.

To find Interest.

Let P = principal; I = interest; S.I. = simple interest; t = time in years; r = rate per cent.; A = amount.

Example.—*Find the S.I. on* £587 10*s.* 6*d. for* $7\frac{1}{2}$ *yrs. at* $3\frac{1}{2}$ *% per annum.*

I. *By the Unitary Method.*

The I on £100 for 1 yr. is £$3\frac{1}{2}$

,, £1 ,, ,, £$\frac{3\frac{1}{2}}{100}$

$\therefore$ the I on £587 10*s.* 6*d.* for 1 yr. is $\frac{3\frac{1}{2}}{100} \times$ £587 10*s.* 6*d.*

and the I on £587 10*s.* 6*d.* for $7\frac{1}{2}$ yrs. is $7\frac{1}{2} \times \frac{3\frac{1}{2}}{100} \times$ £587 10*s.* 6*d.*

$$\therefore \text{S.I.} = \frac{\overset{3}{\cancel{15}}}{2} \times \frac{7}{\cancel{200}} \times £\frac{23501}{\cancel{40}\ 8} = £\frac{4\,9\,3\,5\,2\,1}{3\,2\,0\,0} = £154\ 4s.\ 6\tfrac{3}{4}d.$$

Ans. £154 4*s.* $6\frac{3}{4}$*d.*

It will be observed that P, r, and t have been multiplied together, and the product divided by 100.

Rule.—(1) Multiply rate and time together.
(2) Multiply P by the product.
(3) Divide the result by 100.

Expressed by *formula*, the rule becomes $\text{I} = \frac{\text{P} \times r \times t}{100}$

If the time be stated in months, we then have—

$$\text{I} = \frac{\text{P} \times r \times t \text{ (in months)}}{1200}$$

If in days—

$$\text{I} = \frac{\text{P} \times r \times t \text{ (in days)}}{36500}$$

II. *By Formula.*

$$\text{I} = \frac{\text{P} \times r \times t}{100} = £5{\cdot}87525 \times 3{\cdot}5 \times 7{\cdot}5$$
$$= £154{\cdot}2253125 = £154\ 4s.\ 6\tfrac{3}{4}d. \quad \textit{Ans.}$$

III. *By the Contracted Method.*

$$r \times t = 3{\cdot}5 \times 7{\cdot}5 = 26{\cdot}25$$
$$\text{and P} \times r \times t = 587{\cdot}525 \times 26{\cdot}25$$

587·525
52·62
1175 05
352 51
11 75
2 93
154·225

Ans. £154 4*s.* 6*d.*

IV. *By Proportion.*—When P and I alone are referred to, the problem may be solved by simple proportion; *e.g.*—

$$\text{As } £100 : £587\ 10s.\ 6d. :: £26\tfrac{1}{4} : \text{I required}$$

$$\therefore \text{I required} = \frac{£587\ 10s.\ 6d. \times 26\frac{1}{4}}{100} = £154\ 4s.\ 6\tfrac{3}{40}d. \quad \textit{Ans.}$$

To find the Amount.

Example.—*Find the amount of* £3000 *at* 5 % *per annum for* 5 *yrs.*

Note that the answer is to be A, which equals P + I

and since 5 % for 5 yrs. is the same as 25 % for 1 yr.
$\therefore$ the total rate = £5 × 5 = £25

Hence we get the following statement:—

£100 : £3000 :: £25 : I required

$$\text{I required} = £\frac{3000 \times 25}{100} = £750$$

and since A = P + I
$\therefore$ A = £3000 + £750 = £3750

Ans. £3750.

Rule.—(1) Find the total I as above.
(2) Add the I to the P.

If A be given, and either t or r be required, the P must be found, since *A is proportional to neither t nor r.*

Interest and Proportion.—Viewing the "proportion" relations existing between the various terms of an interest sum, we note that—

I is directly proportional to P, r, and t
A „ „ „ P, but *not* to r or t.

The reason is obvious. A is the sum of two quantities, a constant one (P) and a variable one (I). If I varies, then A varies, P remaining unchanged. A may therefore be said to vary directly as P. But there is no constant element in r and t, and therefore A cannot vary as either.

To find Principal.

I. When A is given.

Example.—*What sum of money put out at S.I. for* $3\frac{1}{2}$ *yrs. at* $4\frac{1}{4}$ % *will amount to* £1497 4s. 1*d.*?

$$4\tfrac{1}{4}\ \% \text{ for } 3\tfrac{1}{2} \text{ yrs.} = 4\tfrac{1}{4} \times 3\tfrac{1}{2}\ \% \text{ for 1 yr.}$$

$$= \frac{17 \times 7}{4 \times 2} = 14\tfrac{7}{8}\ \%$$

$$\text{amount} = £100 + £\,14\tfrac{7}{8} = £114\tfrac{7}{8}$$

We know that A is directly proportional to P, the amount of £100 for the

given time at the given rate is therefore found, and the proportion stated thus—

$$\left.\begin{matrix}\text{A of £100 for given}\\ \text{time at given rate}\end{matrix}\right\} : \text{given A} :: £100 : \text{P}$$

$$\text{then } £114\tfrac{7}{8} : £1497\ 4s.\ 1d. :: £100 : \text{P}$$

$$\text{P} = \frac{£1497\ 4s.\ 1d. \times 100}{114\tfrac{7}{8}} = £1303\tfrac{1}{3} \quad \textit{Ans.}$$

By formula, we have—

$$\text{I} = \frac{\text{P} \times r \times t}{100}$$

$$\therefore \text{I} \times 100 = \text{P} \times r \times t$$

$$\text{and P} = \frac{\text{I} \times 100}{r \times t}$$

For problems where it is required to find P, a certain convenient sum of money is *supposed* to be invested for the given time at the given rate. The I and A of this sum are then found, and the problem is solved by the aid of this "supposed" sum. Any sum would do, but £100 is universally chosen as the most convenient.

II. When I is given.

Example.—*What P is required to produce* £200 *in* 4 *yrs. at* 5 % *?*

4 yrs. at 5 % = 20 % for 1 yr.

then as £20 : £200 :: £100 : P

$$\text{and P} = £\frac{200 \times £100}{20} = £1000$$

Ans. £1000.

When both P and t are considered in relation to I, the problem may be considered as one in compound proportion; *e.g.*—

$$\left.\begin{matrix}£5 : £200\\ 4\text{ yrs.} : 1\text{ yr.}\end{matrix}\right\} :: £100 : \text{P}$$

$$\text{where P} = £\frac{100 \times 200 \times 1}{5 \times 4} = £1000$$

Ans. £1000.

Notes.—(1) The I is in direct proportion to the P producing it.

(2) Both P and I must be expressed in the same unit, *i.e.* both must be pounds or shillings, etc.

(3) Errors of interpretation are often made in this kind of problem; P and A are confused, and then, instead of I being found on a supposed principal, it is found on A.

To find Time.

Example.—*In what time will* £500 *amount to* £650 *at* 4 % *per annum?*

Given I = A − P = £650 − £500 = £150

$$\text{and I on £500 for 1 yr.} = \frac{£500 \times 4}{100} = £20$$

$$\therefore £20 : £150 :: 1\text{ yr.} : \text{number of years required}$$

$$\text{number of years required} = \frac{150 \times 1}{20} = 7\tfrac{1}{2}$$

Ans. $7\frac{1}{2}$ yrs.

We first find the given I, and then the I on the given P for 1 yr. Then, as *time is in direct proportion to the I*, we can state as follows :—

One year's I : given I :: 1 yr. : number of years required
i.e. £20 : £150 :: 1 yr. : „ „ „

Note.—When the time required is less than a year, it should be reduced to the fraction of a year.

Time with given Dates.—The day on which the money is paid must be included, but not the day on which it is lent ; *e.g.* if money were lent on March 19th and paid back on December 6th, March 19th would not be included, because the borrower would not have had the use of the money for 1 day until the 20th of March ; *i.e.* in counting, the first day is omitted.

Example.—*Find the S.I. on* £251 5*s. from June* 15*th*, 1902, *to August* 27*th*, 1903, *at* $3\frac{1}{3}$ % *per annum.*

Days from 15 vi. '02 to 27 viii. '03 = 15 + 31 + 27 + 365 = 438 days

$$\text{hence I required} = £251\tfrac{1}{4} \times \frac{3\frac{1}{3}}{100} \times \frac{438}{365}$$

$$= £\frac{1005}{4} \times \frac{10}{3} \times \frac{438}{100} \times \frac{}{365} = £\frac{201}{20} = £10\ 1s.$$

Ans. £10 1*s.*

In the above example the quantities cancel out easily. Had they not done so, it would have been advisable to get 73000 as a divisor, *e.g.* in this case this could have been done by dividing both numerator and denominator by 6.

$$\frac{1005 \times 10 \times 438}{4 \times 3 \times 100 \times 365} = \frac{1005 \times 10 \times 73}{2 \times 100 \times 365} = \frac{1005 \times 10 \times 73}{73000}$$

Time given in Days.—If the time is given in days, we can multiply by the number of days and divide by 365 × 100. This is exactly the same thing as doubling the rate and dividing by 73000, for 73000 = 365 × 100 × 2. This leads us to Hunter's device for dealing with 73000 as a divisor, an artifice which is generally known as—

The 3rd, 10th, and 10th Rule.

Example.—*Find the S.I. on* £2187 10*s. for* 219 *days at* $4\frac{1}{4}$ % *per annum.* (C.S.)

$$\text{I} = £\frac{2187{\cdot}5 \times 4\frac{1}{4} \times 2 \times 219}{100 \times 365 \times 2}$$

$$= £\frac{4072031}{73000}$$

$$= £55\ 15s.\ 8\tfrac{3}{4}d.$$

$$\textit{Ans.} = £55\ 15s.\ 8\tfrac{3}{4}d. - 5 \times \tfrac{1}{4}d.$$

$$= £55\ 15s.\ 7\tfrac{1}{2}d.$$

3	4072031
10	1357344
10	135734
	13573
	55·78682
	20
	15·73640
	12
	8·8368
	4
	3·3472

No decimals need be retained during this part of the work.

Now observe what has been done.

1. We proceed to find I by doubling the rate and by dividing by $365 \times 100 \times 2$ (*i.e.* by 73000). This gives us $\frac{4072031}{73000}$.

2. We then take the numerator (4072031), and add $\frac{1}{3}$ of itself to it; $\frac{1}{10}$ of this third, and again $\frac{1}{10}$ of this latter quotient.

3. Add the quantities thus obtained, and divide the result by 100000; *i.e.* move the decimal point 5 places to the left.

4. Deduct $\frac{1}{4}d.$ for each £10 in the result, and the remainder is the required interest; *e.g.* the result obtained was £55 15*s.* $8\frac{3}{4}d.$ The number of pounds is £55, and so $5 \times \frac{1}{4}d.$ has to be deducted.

Reason for deduction in 4 above.

	73000
$\frac{1}{3}$	$24333\frac{1}{3}$
$\frac{1}{30}$ (*i.e.* $\frac{1}{10}$ of $\frac{1}{3}$)	$2433\frac{1}{3}$
$\frac{1}{300}$ (*i.e.* $\frac{1}{10}$ of $\frac{1}{10}$ of $\frac{1}{3}$)	$243\frac{1}{3}$
	100010

It will be noted that divisor 73000 and dividend 4072031 have been multiplied respectively by $1 + \frac{1}{3} + \frac{1}{30} + \frac{1}{300}$; as a consequence the quotient is not altered in value, but the denominator, 73000, has been changed to 100010. But we divided the new numerator by 100000, and not by 100010. Hence, the divisor being too small by about $\frac{1}{10000}$ of itself, the quotient is too great by about $\frac{1}{10000}$ of itself, which comes out to about $\frac{1}{4}d.$ for every £10. Hence the deduction of the $1\frac{1}{4}d.$ from the result.

This method of working this type of problem is adopted because it is easy to take $\frac{1}{3}$, $\frac{1}{10}$, and $\frac{1}{10}$ of certain quantities, and to divide the result by 100000. If the number of days for which interest is required happens to be an aliquot part of 365, then such problems may be worked by the ordinary rule. For this purpose it should be noted that 73, 146, 219, 292 days become $\frac{73}{365}$, $\frac{146}{365}$, $\frac{219}{365}$, $\frac{292}{365}$ of a year, or $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$ of a year.

To find Rate.

Example.—*At what rate per cent. S.I. will* £225 *amount to* £256 10*s. in* 4 *yrs.?*

$$\text{Total I} = £256\tfrac{1}{2} - £225 = £31\tfrac{1}{2}$$

$$\text{I at 1 \% for 4 yrs. on } £225 = £\tfrac{225}{100} \times 4$$

$$\text{then } £\frac{225 \times 4}{100} : £31\tfrac{1}{2} :: 1\ \% : \text{rate required}$$

$$\therefore \text{rate} = \frac{31\frac{1}{2} \times 1}{\frac{225}{100} \times 4} = \frac{\overset{\cancel{4}}{\cancel{100}} \times \overset{3\cdot5}{\cancel{31\cdot5}}}{\underset{\cancel{9}}{\cancel{225}} \times \cancel{4}} = 3\tfrac{1}{2}\ \%$$

Ans. $3\frac{1}{2}$ %.

Example.—*At what rate per cent.* S.I. *will a sum of money double itself in* 20 *yrs.?*

By formula.

$$\text{Since I} = \frac{\text{P} \times r \times t}{100}$$

$$r = \frac{100 \times \text{I}}{\text{P} \times t} = \frac{100 \times 100}{100 \times 20} = 5\ \% \quad \textit{Ans.}$$

The Case in which I is paid Half-yearly or Quarterly.

1. *Half-yearly.*—Halve the rate and double the time.
2. *Quarterly.*—Divide the rate by 4 and quadruple the time.

} Then proceed as in ordinary cases.

e.g. 8 % for 5 yrs. paid half-yearly is the same as 4 % for 10 yrs.
8 % for 5 yrs. paid quarterly is the same as 2 % for 20 yrs.

Some Other Examples.

Example.—*At what rate per cent. will the I on* £474 15*s.* *become* £80 17*s.* 6*d.* *in* $3\frac{1}{2}$ *yrs.?*

Work by decimals.

	474·75	16·61625)80·87500(4·867 . . .
	3	14 41000
	1424·25	1 11700
$\frac{1}{2}$	237·375	12002
	1661·625	371

and 1661·625 ÷ 100 = 16·61625 ∴ rate = 4·867 . . . = $4\frac{7}{8}$ (nearly)

Ans. $4\frac{7}{8}$ (nearly).

If the student works by decimals, it will be sufficient if the answer be found correct to 3 places. The answer will then rarely be wrong to the amount of $\frac{1}{4}d.$

Example.—*What is the least number of years for which S.I. must be reckoned at* 4 % *on* £105 2*s.* 6*d.*, *so that the I may be an exact number of pounds?*

£105 2*s.* 6*d.* = £105·125

S.I. on £105·125 for 1 yr. at 4 % = $\frac{1}{25}$ × £105·125 = £4·205

Number of years required = smallest number by which £4·205 can be multiplied to make the product a whole number.

Now, 4·205 × 2 = 8·410 (decimal places reduced by 1)
and 8·41 × 100 = 841 (a whole number)
hence number of years = 2 × 100 = 200

Ans. 200.

Example.—*Take another case. Let the sum be* £500 12*s.* 6*d.*, *and the rate* 5 %.

Then £500 12*s.* 6*d.* = £500·625

S.I. on £500·625 for 1 yr. at 5 % = $\frac{1}{20}$ × £500·625 = £25·03125

Now, 25·03125 × 2 = 50·0625 (number of decimal places 1 less)
and 50·0625 × 2 = 100·125 (number of decimal places 1 less)
,, 100·125 × 2 = 200·25 (number of decimal places 1 less)
,, 200·25 × 2 = 400·5 (number of decimal places 1 less)
,, 400·5 × 2 = 801 (a whole number)
hence number of years = 2 × 2 × 2 × 2 × 2 = 32

Ans. 32 yrs.

Some Special Percentages.—Time and space may be economized by a knowledge of these.

5 %.

5 % = $\frac{5}{100} = \frac{1}{20}$, and this means either £5 on £100 per year, or 100*s.* on £100 per year, or 1*s.* on £1 per year, or 1*d.* per £1 per month. At 5 % per annum take 1*s.* for every £1, and 1*d.* for every 20*d.*

Example.—*What is the S.I. on* £320 6*s.* 8*d. for* 1 *yr. at* 5 %?

$$I = 320s. + 4d. = £16\ 0s.\ 4d. \quad Ans.$$

Example.—*Find S.I. on* £1356 13*s.* 4*d. at* 5 % *for* 1 *mth.*

$$I = 1356d. = 113s. = £5\ 13s. \quad Ans.$$

If the answer be required to the nearest 1*d.*, add 1*d.* to the answer, as 13*s.* 4*d.* is more than half of £1. The answer then would be £5 13*s.* 1*d.*

Example.—*Find the S.I. on* £50 10*s. for* 4 *mths. at* 5 %.

$50d. \times 4 = 200d. = 16s.\ 8d.$
In this case we consider the pounds as pence, and multiply by the number of months.

The I of £1 at 5 % for 1 mth. = 1*d.*, for 2 mths. = 2*d.*, for 4 mths. 4*d.*, etc.

$2\frac{1}{2}$ %.

$2\frac{1}{2}\ \% = \frac{5}{200} = \frac{1}{40}$, which is equivalent to 6*d.* per £1 per year, or $\frac{1}{2}d.$ per £1 per month.

6 % for a Given Number of Months.—Multiply the sum by half the number of months and divide the result by £100.

Example.—*Find the S.I. on* £200 *for* 10 *months at* 6 %.

$$I = £\frac{200 \times 5}{100} = £10$$

6 % per annum = 10*s.* % per month, and therefore 10*s.* × 10 or 100*s.*, or £5 % for 10 months.

The 6 % rule can be conveniently applied to other percentages, *e.g.*—
for 4 %, find I at 6 % and deduct $\frac{1}{3}$ of result
for 3 %, ,, ,, ,, $\frac{1}{2}$,,

3 % for Half-year.—Take $1\frac{1}{2}$ times the given sum and divide the result by 100.

Example.—*Find the S.I. on* £300 *for* $\frac{1}{2}$ *yr. at* 3 %.

$$I = £\frac{300 + 150}{100} = £\frac{450}{100} = £4\frac{1}{2} \quad Ans.$$

Many other simple cases will be readily perceived by the student ; *e.g.*—

$1\frac{1}{4}\ \% = \frac{1\frac{1}{4}}{100} = \frac{5}{400} = \frac{1}{80} = \frac{3}{240} = 3d.$ in the £

$3\frac{3}{4}\ \% = \frac{3\frac{3}{4}}{100} = \frac{15}{400} = \frac{3}{80} = \frac{9}{240} = 9d.$,,

$6\frac{1}{4}\ \% = \frac{6\frac{1}{4}}{100} = \frac{25}{400} = \frac{1}{16} = \frac{15}{240} = 15d.$ or 1*s.* 3*d.* in the £

$6\frac{2}{3}\ \% = \frac{6\frac{2}{3}}{100} = \frac{20}{300} = \frac{1}{15} = \frac{16}{240} = 16d.$ or 1*s.* 4*d.* ,,

$7\frac{1}{2}\ \% = \frac{7\frac{1}{2}}{100} = \frac{15}{200} = \frac{3}{40} = \frac{18}{240} = 18d.$ or 1*s.* 6*d.* ,,

$8\frac{1}{3}\ \% = \frac{8\frac{1}{3}}{100} = \frac{25}{300} = \frac{1}{12} = \frac{20}{240} = 20d.$ or 1*s.* 8*d.* ,,

Other cases will suggest themselves, like 10 %, $12\frac{1}{2}$ %, 15 %, 20 %, etc.

THEORY.

1. What is meant by *interest*, and how is it generally calculated? (C.W.)

2. What is meant by an interest of $3\frac{1}{2}$ per cent. per annum? (S.K.)

3. Explain fully the meaning of *principal*, *per cent.*, and *amount*, as applied to simple interest. (P.T.)

4. What in common language is the rate per cent. when it is expressed by the decimal ·045? (R.M.C.)

5. Show the shortest method of calculating mentally the interest on £360 for 146 days at $1\frac{1}{4}$ per cent. per annum. (C.W.)

6. Prove the rule for finding the simple interest of a given sum at a given rate per cent. per annum (£275 for 3 years at 7 per cent. per annum). (K.S.)

7. Work a sum in simple interest by the method of proportion, so as to show the truth of the shortened process which is commonly employed. (K.S.)

8. Give, with an example, the rule for shortening the following process: Finding the interest on money for years and months at $2\frac{1}{2}$ per cent. (C.W.)

9. How would you explain the method of finding the rate per cent. per annum when the principal, simple interest, and time are given (*e.g.* principal, £820; simple interest, £6 16*s.* 8*d.*; time, 4 mths.)? (C.W.)

10. Explain that, in finding the simple interest on any sum of money for any number of months at 5 per cent. per annum, we need only reckon each £1 in the given sum as one penny, and the shillings in proportion, and multiply the result by the number of months. Ex.: £315 15*s.* for 7 months. (C.P.)

11. If in simple interest $£P$ = the principal, n = the number of years, r = rate per cent., $£I$ = simple interest, establish formulæ for calculating either principal, time, rate per cent., or interest, the other three being given. (P.T.)

12. Is the amount of a given sum, at a given rate per cent. of simple interest, proportional to the time? (P.T.)

13. A boy thinks that if £120 is the amount of a sum of money invested at 4 per cent. simple interest for 2 yrs., then £240 must be the amount at 8 per cent. for the same time. Why is this wrong? (P.T.)

14. To find the interest at 3 per cent. per annum on any number of pounds for any number of days, multiply the number of pounds by twice the number of days, deduct $\frac{1}{70}$ of the product, and cut off the last two figures; the result will be the interest in pence. Show that the error in the interest given by this rule for any time less than a year cannot exceed 1*s.* on every £2800. (C.S.)

15. When money commands 5 per cent. interest per annum, prove that 5 per cent., payable half-yearly, is exactly as good as $5\frac{1}{16}$ per cent. payable yearly. (L.M.)

16. The amount is directly proportional to principal, but not to rate per cent. or time. Why not?

17. How do you deal with "time" in problems in interest when dates are given? Give an example.

18. What is the "third, tenth, and tenth rule?" Explain it fully, and give the reason for the rule.

EXERCISE XLVII. 47

Simple Interest.

To find Interest.

Find the simple interest on—

1. £325 10*s.* for 4 yrs. at $5\frac{1}{2}$ per cent.

2. £640 10*s.* 6*d.* for $2\frac{1}{2}$ yrs. at $4\frac{1}{2}$ per cent.

3. £650 for 8 mths. at 4 per cent.

4. Find the simple interest on £11000 for 4·4 yrs. at 3·15 per cent. (O.J.)

5. Find by decimals the interest on £358 6*s.* 3*d.* for $2\frac{3}{4}$ yrs. at $2\frac{1}{2}$ per cent. per annum. (C.S.)

6. A, B, and C are partners. The sums deposited by A and B amount together to £900, and those by B and C amount together to £1600. If the entire capital is £2200, what is the interest each one is entitled to at 4 per cent.? (C.P.)

7. If a bank pays $12\frac{1}{2}$ per cent. dividend, what interest should I get by purchasing £5 shares for £12 5*s.*? (P.T.)

To find Principal.

8. What sum must be lent at simple interest, at $2\frac{1}{2}$ per cent. per half-year, to amount in 4 yrs. to £2360? (P.T.)

9. The interest on a sum of money comes to £202·5 in 12 years at 4·5 per cent. simple interest. Find the thousandth part of the principal. (P.T.)

10. If I borrow £175 10*s.* for 10 mths. when money is worth 5 per cent., how much must I lend in return for 12 mths. when money is worth $3\frac{1}{2}$ per cent.? (P.T.)

11. What sum at $3\frac{1}{2}$ per cent. simple interest will amount between March 24 and August 16 (both days included) to £1318 4*s.*? (P.T.)

12. The simple interest on a certain sum for 9 mths., at 5 per cent. per annum, is £150 less than the simple interest on the same sum for 15 mths. at 4 per cent. per annum. Find the principal. (C.M.S.)

13. A man invested two equal sums, and received interest on them

at the rates of $3\frac{1}{4}$ and $1\frac{3}{4}$ per cent. respectively. At the end of 18 mths. he had received £255 interest. What sums did he invest? (P.T.)

14. A tradesman, who commenced business $5\frac{3}{4}$ yrs. ago, increased his capital at the rate of 15 per cent. per annum simple interest, and it now amounts to £5960. What sum did he start with? (P.T.)

15. A man increased his capital annually $\frac{1}{3}$ part, and at the end of 4 yrs. one year's interest thereon at $4\frac{1}{2}$ per cent. amounted to £135. What capital did he start with? (P.T.)

To find Amount.

16. Find the amount of £435 10*s.* in $5\frac{1}{4}$ yrs. at $3\frac{1}{2}$ per cent. per annum simple interest. (K.S.)

17. What will £434 6*s.* 8*d.* amount to if put out at simple interest at 4 per cent. per annum for $3\frac{3}{4}$ yrs.? From your result deduce the amount at 1 per cent. for $1\frac{7}{8}$ yr. (C.P.)

18. If £7833 6*s.* 8*d.* amount to £8587 5*s.* 10*d.* in $3\frac{1}{2}$ yrs., what will £9400 amount to in $5\frac{1}{4}$ yrs. at the same rate per cent. per annum simple interest? (C P.)

19. A sum of £1250 was borrowed on April 1 and repaid on August 25 of the same year with interest at $3\frac{3}{4}$ per cent. per annum. What sum had to be paid? (C.S.L.)

20. A sum of £2954 3*s.* 4*d.* is deposited at a bank on June 15, the bank allowing interest at the rate of 2 per cent. per annum till August 27, and afterwards at the rate of $1\frac{1}{2}$ per cent. If the deposit is withdrawn on November 8, what amount will the bank pay? (C.H.L.)

To find Rate.

21. The amount of £337 10*s.* in $3\frac{1}{2}$ yrs. is £384 15*s.* What is the rate per cent.? (P.T.)

22. At what rate per cent. per annum simple interest will $3\frac{1}{2}d.$ produce $3\frac{1}{2}s.$ in $33\frac{1}{3}$ yrs.? (P.T.)

23. A sum of money has doubled itself in 16 yrs. at simple interest. What is the rate per cent.? (K,S.)

24. A certain sum of money at simple interest amounts to £414 7*s.* 6*d.* in 3 yrs., and to £440 12*s.* 6*d.* in 5 yrs. Find the sum and the rate of interest. (R.N.)

25. If the simple interest on £3138 3*s.* 6*d.* exceeds that on £2762 13*s.* 6*d.* by £7 0*s.* $9\frac{3}{4}d.$ in 3 mths., what is the rate of interest per cent. per annum? (C.S.)

26. The annual dividend on certain shares is 13 per cent. Find the rate at which a man who pays £15 for a £5 share receives interest on the money he invests in such shares. (K S.S.)

27. £317 16*s.* lent January 1, 1890, becomes £397 5*s.* at the end of 1893, and £476 14*s.* at the end of 1897. What is the rate per cent.? (P.E.A.)

28. The gross rental of an estate is £13500, and deductions arising from rates and taxes at 3*s.* in the £ and interest on a mortgage of £3000 amount to 16 per cent. of this rental. What is the rate of interest paid on the mortgage? (P.E.A.)

To find Time.

29. In what time will £350 amount to £392 at 4 per cent. per annum simple interest? (K.S.)

30. The rate of interest in a savings bank is £2 13*s.* 4*d.* per cent. per annum. A man deposits £18, and on withdrawing the same he finds that his interest is 8*s.* For how long was the money deposited? (C.PR.)

31. In what time will a sum of money be doubled at $1\frac{1}{4}$ per cent. simple interest? (K.S.)

32. A is indebted to B in the sum of £1050, and to C in the sum of £1175. In how many years will the debts be equal, B's debt increasing at 5 per cent., and C's at 4 per cent. per annum simple interest? (C.M.)

33. If £100 be placed at interest at 5 per cent., and the interest be added to the principal every 20 yrs., in how many years will it amount to £1000? (P.T.)

34. What is the least number of years for which simple interest must be reckoned at 4 per cent. on £145 7*s.* 6*d.*, so that the interest may be an exact number of pounds? (P.T.)

Miscellaneous Examples.

35. A tradesman marks his goods with two prices, one for ready money and the other for credit of six months. What fixed proportion ought the two prices to bear to each other, allowing 5 per cent. simple interest? (K.S.)

36. Divide £2075 into two such sums that if the first be put out at simple interest for 5 yrs. at $3\frac{1}{2}$ per cent., and the second for 4 yrs. at 3 per cent., the interest of the first sum shall be double that of the second. (C.S.)

37. A dealer purchases goods on January 1 for £80, and receives £90 for them on March 1. With this he trades again, and receives £180 on July 1. What was his gain per cent. for the half-year? (C.S.)

38. A business was paying regularly 7 per cent. per annum; it was extended by increasing the capital by one-half, and after that it paid regularly 6 per cent. per annum. What was the return per cent. on the new capital? (L.M.)

39. £3000 are lent on condition that £500 be repaid at the end of each successive year, with 1 per cent. interest on that year's debt. How long will any of this debt remain, and what interest will the lender receive during the whole time? (C.P.)

40. I deposit £350 on April 10 in a bank paying interest at 3 per cent. per annum; on May 15 the interest fell to $2\frac{1}{2}$ per cent., on June 30 to 2 per cent., and on October 8 it rose to $2\frac{3}{4}$ per cent. Find, to the nearest penny, the interest due on November 7 in the same year. (C.S.)

41. On January 1, 1870, a contractor borrows a sum of money at 5 per cent. simple interest. At the end of a year the rate of interest is reduced to $4\frac{1}{4}$ per cent. The total amount of interest paid up to the end of 1875 is £1760. What was the sum borrowed? (K.S.)

42. A person obtained a loan at 6 per cent. per annum simple interest, but being unable to pay the debt at the appointed time, the loan was continued for 3 mths. more at 10 per cent. per annum. The interest for the whole time would have been, but for the increased rate, £7 2s. less than what it was. Find the amount of the loan. (I.I.)

43. Exactly three years ago a man borrowed £1875 from a bank at 4 per cent. per annum. At the end of a year he paid the interest for that year and part of the loan, altogether £450. Similarly, he paid £650 at the end of the second year. What sum must he now pay to clear off the debt? (C.P.)

44. A cask of $85\frac{1}{4}$ gals. was bought for £67 4s. and kept 10 yrs., during which a pint evaporated yearly. How much per gallon must then be charged to clear 25 per cent., simple interest at 5 per cent. per annum being what the cost price would have earned if free? (C.M.)

***45.** Find, to the nearest pound, how much money must be invested at 6 per cent. per annum in order that the amount to which it accumulates at the end of 3 yrs. may be the same as the joint amount of £1000 invested at $2\frac{1}{2}$ per cent., and £1000 at $3\frac{1}{2}$ per cent. (L.M.)

46. At the time of his bankruptcy a bankrupt paid his creditors 12s. $2\frac{1}{4}d.$ in the £, and 7 yrs. later he paid them the remainder, increased by simple interest on the remainder for 7 yrs. at 4 per cent. per annum. How much was received by a creditor to whom the bankrupt at the time of his bankruptcy owed £1783 12s.? (C.P.)

47. A sum of money having been lent at $3\frac{3}{4}$ per cent. per annum for 3 yrs., the time is afterwards extended for 6 mths., at 5 per cent. per annum, and the amount to be paid in the end, in consequence of the advanced rate of interest, is increased by £1 12s. Find the sum lent, and the whole amount of the interest paid. (C.S.)

48. A sum of £1200 is invested so as to produce a certain rate of interest, and £1600 is invested so as to produce a rate $1\frac{1}{2}$ per cent. per annum higher than the other. It is found that the yearly income from the latter investment is double of that from the former. Find the yearly income arising from each. (C.S.L.)

* *This example should be treated as an exercise in Simple Interest.*

CHAPTER XXXI.

COMPOUND INTEREST.

Compound Interest.—In S.I.* interest is paid periodically on the principal, and the principal is a constant quantity. In C.I., interest is *not* paid out periodically, but is added to the principal, so that the principal is a variable quantity, the amount at the end of one year becoming the principal of the next; thus S.I. is merely interest on the original principal, whereas *C.I. is interest on a growing principal*, i.e. *it is interest on principal + interest (or amount)*.

The justice of this is obvious. If interest is not paid, it is practically retained as a loan, and, like the principal (which is a loan), it should have interest paid upon it at the agreed rate.

Method of Work.—A problem in C.I. is really a series of problems in S.I., just as a problem in compound proportion is a series of problems in simple proportion.

We may work by either of the following methods:—

1. Method of division or practice.
2. Method of multiplication or indices.

The Method of Division or Practice.

Rule—

1. Take the original P. and find the I. on it at the given rate for 1 yr.
2. Add this I. to the P. The amount is the P. for the second year.
3. Find the I. on this new P., and again add the I. to its P. The amount will be the third year's P.
4. Continue this until the amount for the last period has been found.
5. If A. is required, this will be the answer; but if I. is wanted, subtract the first P. from the last A., and the difference will be the answer required.

Example.—*Find the C.I. on* £500 10*s. at* 5 % *for* $2\frac{1}{2}$ *yrs.*

* S.I. = simple interest.
C.I. = compound interest.

Take aliquot parts of £100 for the rate.

Then 5 % for 1 yr. $= \frac{5}{100}$ or $\frac{1}{20}$ of P.
and 5 % for $\frac{1}{2}$ yr. $= 2\frac{1}{2}$ % for 1 yr. $= \frac{1}{40}$ of P.

	£	
$\frac{1}{20}$ of P.	500·5	= P. for first year
	25·025	= I. ,, ,,
$\frac{1}{20}$	525·525	= P. for second year
	26·27625	= I. ,, ,,
$\frac{1}{40}$	551·80125	= P. for third year
	13·79503125	= I. for $\frac{1}{2}$ yr.
	565·59628125	= A. for $2\frac{1}{2}$ yrs.
	500·5	= P. for first year
	65·09628125	= C.I. for $2\frac{1}{2}$ yrs.

20

1·92562500

12

11·10750000

Ans. £65 1*s.* 11·1075*d.*

Note.—The C.I. may also be obtained by adding together the several interests for the respective periods; *e.g.*—

First period, 1 yr. = £25·025
second ,, 1 ,, = £26·27625
third ,, $\frac{1}{2}$,, = £13·79503125

Total = £65·09628125

The respective interests cannot always be found by the aid of *one* aliquot part only. Sometimes two or more are necessary.

Example.—*Find the A. of* £1600 *in* 3 *yrs. at* $5\frac{1}{4}$ % *C.I.*

$5\frac{1}{4}$ % { $3\frac{1}{2} = \frac{7}{200}$ or $\frac{56}{1600}$; $1\frac{3}{4} = \frac{1}{2}$ of $3\frac{1}{2}$ }

	£	
	1600	= P. for first year
	56	= I. for first ,,
	28	
	1684	= P. for second year
$\frac{56}{1600}$	58·94	= I. for second ,,
$\frac{1}{2}$	29·47	
	1772·41	= P. for third ,,
$\frac{56}{1600}$	62·0343	= I. for third ,,
$\frac{1}{2}$	31·0171	
	1865·4614	= A. for third ,,

20

9·2280

12

2·736

Ans. £1865 9*s.* 2·736*d.*

A little ingenuity may sometimes be required to decompose the rate into its aliquot parts ; *e.g.* $2\frac{3}{4}$ %.

$$2\tfrac{3}{4} = \tfrac{11}{400} = \tfrac{8}{400} + \tfrac{2}{400} + \tfrac{1}{400}$$

The Method of Multiplication.—We will work example (p. 281) by this method.

A. of £1 for 1 yr. at 5 % = £1·05.
£500·5 = P. for first year
1·05 = A. of £1 for 1 yr.

25025
50050

525·525 = P. for second year
1·05 = A. of £1 for 1 yr.

2627625
5255250

551·80125 = P. for third year
1·025 = A. of £1 for ½ yr.

275900625
110360250
551801250

565·59628125 = A. for 2½ yrs.
500·5 = P. for first year

65·09628125 = C.I. for 2½ yrs.
Ans. £65 1*s.* 11·1075*d.*

Rule—

1. Find the A. of £1 for 1 yr. at the given rate.
2. Multiply the P. by this amount, and the product will be the A. for first year, or the P. for the second year.
3. Multiply the new P. by the same multiplier, and the product will be the P. for the third year.
4. Proceed thus until all the periods (or years) are exhausted, and the last product will be the A. required.

Reason for Rule.

£100 amounts to £105 at the end of the year
∴ £1 ,, $£\frac{105}{100}$,, ,, ,,
and the A of *any sum* at 5 % in 1 yr. = $\frac{105}{100}$ of that sum
Similarly, the A. for the second year = $\frac{105}{100}$ of the A. for the first year
∴ the A. of *any sum* at 5 % for 2 yrs. = $\frac{105}{100}$ of $\frac{105}{100}$ of that sum, and so on.
If the time be 3 yrs., and the amount £500, then the A. of £500 for 3 yrs. at 5 % will be—

$\frac{105}{100}$ of $\frac{105}{100}$ of $\frac{105}{100}$ of £500
or by decimals 1·05 × 1·05 × 1·05 × £500
or more briefly in statement $(1{\cdot}05)^3 \times$ £500

The Use of Decimals in C.I.—Vulgar Fractions often give

trouble in the solution of problems in C.I., hence the use of decimals is recommended, and especially so when C.I. or A. is required for more than two periods. Their use is advantageous for the following reasons:—

1. Labour and space are saved, for the multiplication by the rate and the division by 100 can be done simultaneously, and that, too, without writing down the multiplier.
2. The I. is thus obtained in each case in a position convenient for its addition to the P.
3. As a rule, not more than 4 places need be retained in the working, unless special exactitude is required, for, since £·0001 is less than $\frac{1}{10}$ of a farthing, 4 places will give the answer correct to farthings, and 3 to pence.

Many principals, especially those producing recurring decimals, are *not* advantageously treated by decimals.

Note.—A review of the two methods shows the Practice Method, where available, to be the better. It is neater and shorter, many of the usual rates being easy aliquot parts of 100.

The Use of Formulæ.

Let A = amount at the end of the given time;
,, P = principal;
,, R = amount of £1 for 1 yr. at the given rate;
,, n = time in years.

Then A for 1 yr. $= P \times R = PR$
and A for 2 yrs. $= P \times R \times R = PR^2$
and A for 3 yrs. $= P \times R \times R \times R = PR^3$
and A for n yrs. $= P \times R \times R \ldots n$ times $= PR^n$
i.e. $A = PR^n$ . . . . . (i.)

from (i.) we can deduce the following rules:—

$$P = \frac{A}{R^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

$$R^n = \frac{A}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)}$$

$$R = \sqrt[n]{\frac{A}{P}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

The use of formulæ is handy for practice, but it offers little or no mental training.

To find the Amount for a Large Number of Periods.

Example.—*Find the amount at C.I. of* £700 *at* 10 % *for* 8 *yrs.*

$A = PR^n = £700 \times (1{\cdot}1)^8$
Now, $(1{\cdot}1)^2 = 1{\cdot}21$
and $(1{\cdot}1)^4$ or $(1{\cdot}21)^2 = 1{\cdot}4641$
and $(1{\cdot}1)^8$ or $(1{\cdot}4641)^2 = 2{\cdot}14358881$
$\therefore A = £700 \times 2{\cdot}14358881 = £1500{\cdot}512167$
$= £1500\ 10s.\ 3d.$ (nearly)
Ans. £1500 10*s.* 3*d.* (nearly).

Note.—(1) Such methods can be used only for periods which are multiples of 2 or 3 or both; *i.e.* where it is possible to solve by square or cubic measure.

(2) For a large number of periods it is best to use logarithms.

Rule.—*To find the amount for any number of periods.*

1. Find the A. of £1 at the given rate for the first period.
2. Raise this to the power expressed by the number of periods.
3. Multiply the product by the given P.

To find the Amount for Periods less than a Year.—The process of finding the C.I. for 1 yr. at 5 % payable quarterly is the same, in practice, as finding the C.I. for 4 years at $1\frac{1}{4}$ %; *i.e.* the time is multiplied by 4 (4 quarters in a year), and the rate is divided by 4.

Example.—*Find the C.I. of* £55 *for* 1 *yr., payable quarterly at* 5 % *per annum.* (L.M.)

$1\frac{1}{4}$ % = $\frac{1}{80}$ of 100, and $\frac{1}{80}$ = ·0125

Hence A. of £1 for 1 yr. at $1\frac{1}{4}$ % = £1·0125

$(1{\cdot}0125)^4 = 1{\cdot}0509$

and £1·0509 × 55 = £57·799

hence C.I. = £57·799 − £55 = £2·799

= £2 15*s.* 11·76*d.*

Ans. £2 15*s.* 11·76*d.*

To find Principal.

I. Given A., R., and T.

Example.—*What sum will amount to* £565·59628125 *in* $2\frac{1}{2}$ *yrs. at* 5 % *compound interest?*

(1) *Find the amount of* £100 *for the given time at the given rate per cent.* (p. 283).

Amount of £100 for $2\frac{1}{2}$ yrs. @ 5 % = £113·00625

(2) *State as a proportion, thus—*

A. of £100 for given time and rate : A. given :: £100 : P.

£113·00625 : £565·59628125 :: £100 : P.

$$P. = £\frac{565{\cdot}59628125 \times 100}{113{\cdot}00625} = £500{\cdot}5$$

Ans. £500 10*s.*

II. Given I., R., and T.

Example.—*What P. will produce* £1698·5856 *compound interest in* 4 *yrs. at* 4 %?

(1) *Find the C.I. on* £1 *for given time at given rate.*

A. of £1 for 4 yrs. @ 4 % = £$(1{\cdot}04)^4$ = £1·16985856 (p. 284)

I. on £1 „ 4 „ @ 4 % = £·16985856

(2) *State a proportion thus—*

I. on £1 for given time at given rate : given interest :: £1 : P.

£·16985856 : £1698·5856 :: £1 : P.

P. = £10000

Ans. £10000.

To find Rate.—This involves a knowledge of evolution. Rates for certain periods* lend themselves to solution by Arithmetic, but to find any root (the nth root) demands a knowledge of Logarithms.

I. Given P., A., and T.

Example.—*At what rate per cent. will* £500 *amount to* £551·25 *in* 2 *yrs.?*

By formula (p. 284)—

$$R = \sqrt[n]{\frac{A}{P}} = \sqrt[2]{\frac{£551{\cdot}25}{£500}} = \sqrt{£1{\cdot}1025} = £1{\cdot}05$$

Then the A. of £1 for 1 yr. = £1·05
and „ £100 „ = £105
∴ rate = 5 % *Ans.*

II. Given P., I., and T.

1. We may add the I. to the P., get the A., and proceed as above ; or—

2. We may proceed as follows :—

I. on £500 for 2 yrs. = £51·25

$$\therefore \text{I. „ £1 „ „} = £\frac{51{\cdot}25}{500} = £{\cdot}1025$$

Then I. „ „ = ·1025 of P.
and A. „ „ = 1·1025 of P.
∴ 1·1025 = second power of A. at end of first year
and $\sqrt{1{\cdot}1025}$ = 1·05 = A. at end of first year

Then, since the A. of £1 for 1 yr. = £1·05
∴ the A. of £100 for 1 yr. = £105
hence rate = 5 %

Ans. 5 %.

To find Time.

Example.—*In what time will* £500 10*s. amount to* £565 1*s.* 11·1075*d. at* 5 %?

£565 1*s.* 11·1075*d.* = £565·59628125

Find the amount of £500 10*s.* for successive years at the given rate per cent. until an amount approximating to £565·59628125 is obtained.

	£
P. for first year	500·5
I. „ „ „	25·025
P. „ second „	525·525
I. „ „ „	26·27625
P. „ third „	551·80125
I. „ „ „	27·5900625
	579·3913125

The last amount is too large, being greater than the given A. The time, therefore, must lie between 2 and 3 yrs.

* Such periods are some power of 2 or 3, or they may be the products of such powers.

It is now required to find a fraction of a year for which, at the given rate, the I. on £551·80125 is—

£565·59628125 − £551·80125, or £13·79503125

$$\text{Now, } T = \frac{I \times 100}{P \times R} = \frac{£13{\cdot}79503125 \times 100}{£551{\cdot}80125 \times 5} = \tfrac{1}{2} \text{ yr.}$$

$\therefore$ total time = $2\frac{1}{2}$ yrs. *Ans.*

The Time necessary for a Sum of Money to double itself.—Observation shows that the number of years in which any given sum of money will double itself at compound interest can be roughly obtained by dividing $70\frac{1}{2}$ by the given rate, *e.g.*—

(1) At $3\frac{1}{2}$ %, $70\frac{1}{2} \div 3\frac{1}{2} = 20\frac{1}{7}$ yrs.
(2) At 4 %, $70\frac{1}{2} \div 4 = 17\frac{5}{8}$ yrs., and so on.

To find the P., the Difference between S.I. and C.I. being given.

Example.—*The C.I. on a certain sum for* 3 *yrs. at* 5 % *per annum is* £7 12*s.* 6*d. more than the S.I. Find the sum.*

A. of £1 in 3 yrs. at 5 % C.I. = $(1{\cdot}05)^3 = £1{\cdot}157625$
$\therefore$ C.I. on £1 for 3 yrs. at 5 % = £·157625
and S.I. on £1 for 3 yrs. at 5 % = 3 × £·05 = £·15
hence difference = £·157625 − £·15 = £·007625
$\therefore$ on a P. of £1 the difference in the I. is £·007625

If, on a principal of £1, the difference in the interest is £·007625, what is the principal when the difference in interest is £7 12*s.* 6*d.*?

£·007625 : £7·625 :: £1 : required principal

$$\text{required principal} = £\frac{7{\cdot}625}{{\cdot}007625} = £1000$$

Ans. £1000.

THEORY.

1. Define compound interest, and show how it differs from simple interest.

2. What advantages, if any, arise in the solution of problems in compound interest by the use of decimals?

3. Write out the rule for finding the compound interest of a given sum for a given time at a given rate per cent., (*a*) by the method of division; (*b*) by the method of multiplication.

4. Write down the formulæ for finding the amount, the principal, the rate per cent., and the time in years, and show how each is derived.

5. What is the rule for finding the amount at compound interest for a large number of periods? What limit is there to the arithmetic rule?

6. The compound interest of a given sum for a given time and at a given rate is greater in proportion as the length of each period is less. Demonstrate this.

7. Write out the rule in compound interest for (*a*) finding the principal; (*b*) for finding the rate; (*c*) and for finding the time.

8. Give a rough rule for finding the time necessary for a sum of money to double itself at compound interest, and illustrate your answer with an example.

EXERCISE XLVIII. 48

Compound Interest.

To find Interest or Amount.

Find the *compound interest* on—

1. £3275 at $2\frac{1}{2}$ per cent. for 3 yrs. (C.P.)

2. £3456 at 4 per cent. for 3 yrs. (K.S.)

3. £1024 at 5 per cent. for 3 yrs. (O.R.)

4. £50000 at $2\frac{1}{2}$ per cent. for 3 yrs. (O.S.L.)

Find the *amount* of—

5. £1302 1*s.* 8*d.* in 3 yrs. at 6 per cent. (C.PR.)

6. £65 2*s.* 1*d.* in 4 yrs. at 4 per cent. (I.I.)

Find the *exact* compound interest on—

7. £3976 for 2 yrs. at 4 per cent. (K.S.)

8. £1250 for 4 yrs. at 3 per cent. (C.M.)

Find *to the nearest penny the compound interest* on—

9. £625 4*s.* for 3 yrs. at 4 per cent. (K.S.)

10. £7692 17*s.* 6*d.* for 3 yrs. at $2\frac{1}{2}$ per cent. (C.H.L.)

Find *to the nearest penny the amount* of—

11. £1425 for 3 yrs. at 4 per cent. (C.P.)

12. £4750 for 3 yrs. at $3\frac{1}{2}$ per cent. (C.PR.)

Find the *compound interest, neglecting fractions of a penny*, on—

13. £$2378\frac{1}{2}$ in $2\frac{1}{2}$ yrs. at $4\frac{1}{2}$ per cent. (C.P.)

14. £3165 in 3 yrs. at $4\frac{1}{2}$ per cent. (C.S.)

Find the *amount, neglecting fractions of a penny, of*—

15. £4175 for 3 yrs. at $2\frac{1}{2}$ per cent. (C.S.)

16. £4875 for 3 yrs. at $3\frac{1}{2}$ per cent. (P.T.)

Find *compound interest, payable half-yearly*, on—

17. £100000 in $2\frac{1}{2}$ yrs. at $3\frac{1}{2}$ per cent. (C.PR.)

18. £410 in $2\frac{1}{2}$ yrs. at $4\frac{1}{2}$ per cent. (I.I.)

Find the *amount*, interest being *payable half-yearly*, on—

19. £6510 8*s.* 4*d.* in $1\frac{1}{2}$ yr. at 8 per cent. (K.S.)

20. £512 in $1\frac{1}{2}$ yr. at $2\frac{1}{2}$ per cent. (S.K.)

21. What sum put out for $1\frac{1}{2}$ yr. at 4 per cent. compound interest, payable half-yearly, will amount to £6632 11*s.*? (K.S.S.)

22. Find the amount of £1000 at the end of 3 yrs. at compound interest at the rate of 4 per cent. per annum, payable quarterly. Deduce the amount at the end of 6 yrs. (C.H.L.)

23. Find the compound interest (correct to the nearest farthing) of £1000 for 1 yr., interest being payable quarterly, at the rate of 4 per cent. per annum. (C.S.L.)

24. By how much does the compound interest on £78125 for 2 yrs. at 4 per cent. per annum, payable half-yearly, exceed the compound interest on the same sum at the same rate per cent. per annum, payable yearly? (P.P.)

25. Find the exact difference between the compound interest of £500, reckoned half-yearly, at £1 19*s.* 3*d.* per cent. per half-year, and the interest on the same sum at 4 per cent. per annum at the end of 1 yr. (C.W.)

To find the Difference between Simple and Compound Interest.

26. By what amount does the compound interest on £195 6*s.* 3*d.* for 3 yrs. at 4 per cent. per annum exceed the simple interest on the same sum for 2 yrs. at $6\frac{1}{2}$ per cent. per annum? (C.P.)

27. What sum of money put out at $4\frac{1}{2}$ per cent. simple interest for 2 yrs. will increase to £5450? If the interest had been compound, what would have been the ultimate amount? (C.P.)

28. Show that the amount of £700 at 10 per cent. per annum compound interest will in 8 yrs. exceed the amount of £1000 at 5 per cent. per annum simple interest. (C.M.)

29. Distinguish between simple and compound interest. What difference will it make at the end of 3 yrs. to an investor of £150 at $4\frac{1}{2}$ per cent. per annum, whether he has put out his money on the former or the latter kind of interest? (C.W.)

30. What is the compound interest to the nearest penny on £83 14*s.* 7*d.* in 7 yrs. at the rate of 3 per cent. per annum? How much does this compound interest exceed the simple interest in the same time on this principal at the same rate per cent.? (R.M.A.)

31. Find the amount of £512 in $1\frac{1}{2}$ yr. at $2\frac{1}{2}$ per cent. per annum—first, at simple interest; secondly, at compound interest, supposing that the interest due is added at the end of each half-year. (Fractions of a penny may be neglected.) (S.K.)

32. The difference between the simple and compound interest on a certain sum of money for 3 yrs. at 5 per cent. is £13 6*s.* $10\frac{1}{2}$*d.* What is that sum of money? (C.S.)

To find Principal.

33. A certain sum of money placed out at simple interest for 3 yrs. at $5\frac{1}{2}$ per cent. per annum brings in £18 8*s.* 9*d.* more in the 3 yrs. than if it were placed out at compound interest for the same time at 5 per cent. per annum. Find the sum. (C.S.)

34. The compound interest on a certain sum for 2 yrs. at 4 per cent. per annum is £7 10*s.* more than the simple interest for the same time at the same rate. What is the sum? (C.PR.)

35. What sum will amount to £1 16*s.* 9*d.* in 2 yrs. at 5 per cent. per annum compound interest? (K.S.)

36. The compound interest on a sum of money for 2 yrs. at 5 per cent. exceeds the simple interest by 1 guin. Find the sum of money. (C.M.)

37. What sum will amount to £1591 13*s.* 2·16*d.* in 3 yrs. at compound interest, the interest for the first, second, and third years being 3, 2, and 1 per cent. respectively? (C.S.L.)

38. What sum of money at $5\frac{61}{240}$ per cent. per annum simple interest will produce in 15 yrs. the same amount of interest that £500 will produce in 3 yrs. at 5 per cent. per annum compound interest? (K.S.)

39. A and B invest equal sums; A invests at 5 per cent. compound interest, payable yearly, and B invests at $5\frac{1}{2}$ per cent. simple interest. At the end of 3 yrs. the total received by B is £3 13*s.* 9*d.* more than that received by A. What was the sum invested? (C.S.)

40. What sum of money would amount to £42998·1696 in 8 yrs. at 20 per cent. per annum compound interest? (K.S.)

To find Rate.

41. At what rate per cent. per annum compound interest will £166 13*s.* 4*d.* amount to £176 16*s.* 4*d.* in 2 yrs.? (C.W.)

42. At what rate per cent. compound interest does a sum of money become fourfold in 2 yrs.? (K.S.)

43. A sum of money put out at compound interest amounts in 2 yrs. to £1760 8*s.* 4*d.*, and in 3 yrs. to £1830 16*s.* 8*d.* Find the rate of interest and the original sum. (C.S.L.)

44. The amounts at compound interest of a certain sum at the ends of two successive years are £9 3*s.* 9*d.* and £9 12*s.* $11\frac{1}{4}d.$ What is the rate of interest per cent.? And if these years are the second and third, what was the sum lent? (C.W.)

45. A sum of money is invested, and the interest at the end of the first year amounts to £70 8*s.* 4*d.* This is added to the principal, and the interest on the total at the end of the next year at the same rate amounts to £73 4*s.* 8*d.* Find the rate of interest and the original capital. (P.T.)

46. The interest charged for the loan of a certain sum is £10 for 1 yr., and £20 8*s.* for 2 yrs. Determine whether the interest is simple or compound, and also the rate per cent. (P.E.A.)

47. (*a*) Distinguish between simple and compound interest. (*b*) Interest being payable yearly, what is the rate per cent. if £1 amounts in a year to £1 5*s.*? And in the course of which year will a debt double itself at this rate of compound interest? (C.M.)

48. A sum is invested at compound interest payable annually. The interest in two successive years was £112 10*s.* and £115 6*s.* 3*d.* Find the rate of interest and the value of the principal at the beginning of the first of these years. (C.W.)

49. A's capital exceeds B's by $\frac{41}{400}$ of B's capital. B puts his out at compound interest for 3 yrs. at 5 per cent. At what rate per cent. of simple interest must A put his out in order that at the end of 3 yrs. the two amounts may be the same? (C.S.)

To find Time.

50. Find in what time £160 will amount to £173 17*s.* $9\frac{9}{25}d.$ at $4\frac{1}{4}$ per cent. per annum compound interest. (I.I.)

51. In what time will £4000 amount to £8574 7*s.* 1·2576*d.* at 10 per cent. per annum compound interest? (K.S.)

52. Find what is the first year in which a sum of money will become more than double in amount if put out at compound interest at the rate of 10 per cent. per annum. (C.S.)

53. What is the least number of complete years in which a sum of money, laid out at compound interest at the rate of 10 per cent. per annum, will amount to a sum more than twice the principal? Find, correct to the nearest penny, what £279 will amount to in that time. (C.S.L.)

Miscellaneous Problems.

54. A man saves £50 each year for 5 yrs., and invests it at the end of the year at $2\frac{1}{2}$ per cent. compound interest. Find the amount of his savings to the nearest penny at the end of the fourth year. (S.A.)

55. The compound interest on a certain sum for 2 yrs. is £563 4*s.*, and the simple interest is £546 2*s.* 8*d.* What is the sum, and what is the rate per cent.? (S.A.)

56. Interest being due yearly, the difference between the total simple interest and the total compound interest on a certain principal lent in each case for 2 yrs. at $3\frac{1}{2}$ per cent. is £12 5*s.* Find this principal. (K.S.)

57. A man saves £100 at the end of each year, and allows his savings to accumulate at 4 per cent. per annum compound interest. How much is he worth at the end of 8 yrs.? (I.I.)

58. If I borrow £2000 at $7\frac{1}{2}$ per cent. compound interest, and pay off £800 at the end of the first year, and the same sum at the end of the second year, what will the remainder of my debt amount to at the end of $2\frac{1}{2}$ yrs. from the time of borrowing? (C.S.)

59. If £3000 amounts to £5333 6*s.* 8*d.* in 10 yrs. at compound interest, what will £3000 amount to in half that time? (C.W.)

60. Find the compound interest on £2150 for 3 yrs. at 5 per cent. A company obtain a loan from their bankers of £2150 at 5 per cent.; they do the same a year afterwards, and again a year after that. What will their whole debt be at the end of the third year, allowing compound interest on the first two loans? (C.S.)

61. A man buys a house on the conditions that he shall pay £500 now, £500 one year hence, and £500 two years hence. What should be (to the nearest shilling) the cash value of the house, compound interest being reckoned at $3\frac{1}{2}$ per cent.? (L.M.)

62. A sum of £6375 is to be divided between A and B, who are respectively 18 and 19 yrs. old, in such a way that, if their portions be invested at 4 per cent. per annum compound interest, they shall receive equal amounts on reaching 21 yrs. of age. How must the sum be now divided, and how much will each receive when 21 yrs. old? (I.I.)

63. Calculate the compound interest for 3 yrs. on £37500 at the rate of 3 per cent. per annum. Assuming that money can be invested at such a rate that it will double itself at compound interest in 24 yrs., find to the nearest pound how much £1000 will amount to in 12 yrs. (C.S.L.)

64. A dealer bought some pictures, and 2 yrs. afterwards sold them for £357 5*s.* less than they cost, and thus received 16 per cent. less than the purchase money would have amounted to at 4 per cent. per annum compound interest. What sum did he give for the pictures? (C.P.)

65. Instead of a payment of £8815 12*s.* 7*d.* at the end of 3 yrs. from the present time, it has been agreed that five equal sums shall be paid, one such sum at the end of each of five successive years from the present time. What should the annual payment be, if money is worth 4 per cent. per annum and compound interest is reckoned? (C.P.)

66. A tenant is offered the rest of a lease of a house for the 5 yrs. it has to run at the rent of £16 a year, on paying down a premium of £150; but, as he can only pay down £50, he proposes to pay down that sum and an increased rent for the 5 yrs. What should the rent be to the nearest shilling, calculating compound interest at 5 per cent. on the unpaid premium? (C.P.)

67. A sum of £5000 is lent on the condition that, if it is not repaid within a year, compound interest will be charged. The option is given of 5 per cent. interest throughout, or $2\frac{1}{2}$ per cent. for the first year, 5 per cent. for the second year, $7\frac{1}{2}$ per cent. for the third year, and so on, the interest increasing by $2\frac{1}{2}$ per cent. every year. Show that, if the loan is repaid within 3 yrs., the latter system will be in favour of the borrower; and find the difference on the two systems if it be not paid off till the end of the fourth year. (C.S.)

68. A testator bequeathed the sum of £110223 in trust for his three infant daughters, A, B, and C, whose ages at his death were respectively 11, 13, and 15 yrs., and he directed his trustees to divide the money into three parts, so that at 3 per cent. compound interest, payable yearly, his daughters on respectively attaining their majority should become entitled to equal sums of money. Find the sums to be set aside for A, B, and C respectively to the nearest pound. (C.S.)

69. A capitalist, having a sum of money to invest, places half in a bank which offers depositors $4\frac{1}{2}$ per cent. compound interest, added yearly to the deposits, and the other half in a bank which also allows compound interest, where the rate is $\frac{1}{2}$ per cent. less, but the additions of interest are made every 6 mths. At the end of 2 yrs. he finds there is a difference of £249 16*s.* $3\frac{1}{4}$*d.* in the two accounts. Find the sum originally invested. (C.S.)

CHAPTER XXXII.

DISCOUNT.

Definition.—Discount is an abatement made for the prepayment of money; *i.e.* it is an allowance made for the payment of money before it is legally due.

Supposing interest to be at the rate of 5 %, £105 due at the end of a year is now worth only £100, because £100 put out at interest for that time and at that rate would amount to £105. The *present worth*, therefore, of the £105 is £100, and the *true discount* is £5.

True Discount.—This is also known as Real, Equitable, Mathematical, or Arithmetical Discount, and *it is the difference between the sum due at some future time, and the present worth of that sum at an agreed rate of interest.*

Present Worth.—The present worth of a certain sum due at the end of a given time *is that sum which, with its interest at the given rate, would amount exactly to the sum due.* True discount is thus the interest on the present worth (see p. 296).

The nomenclature of Discount is somewhat unfortunate, as it is likely to be confusing after the use of similar terms in Interest. What is called *Principal* in Interest is known as the *Present Worth* in Discount, while *Amount* in Interest becomes *Principal* in Discount.

These distinctions must be thoroughly grasped by the student, or he may fail to understand such a definition of Discount as the following:—

"The difference between the Principal and the Amount of the Principal after a certain time and at a certain rate per cent. is called the Discount of the Amount for that time and at that rate."

These distinctions, however, are not universally observed. By some the Present Worth is called the Principal, and the given sum the Amount.

To find Discount.

Example.—*Find the true discount on* £510, *due* 6 *months hence at* 4 %.

Interest on £100 for 6 months at 4 % = £2
∴ amount of £100 at end of 6 months at 4 % = £102

Then the present worth of £102 due at the end of 6 months at 4 % = £100, and by the definition given above, the true discount on £102 due at the end of

6 months at 4 % = £2. Then if the true discount on £102 is £2, what is the true discount on £510?

$$£102 : £510 :: £2 : D^*$$

$$\therefore D = £\frac{510 \times 2}{102} = £10$$

Ans. £10.

Rule.—(1) Let £100 represent a present worth.
(2) Find the amount of £100 for the given time at the given rate per cent.
(3) Then the amount − the present worth = the discount.
(4) State the proportion as follows:—

$$\text{As} \left\{\begin{matrix}\text{the amount of £100}\\ \text{for the given time}\\ \text{at the given rate}\\ \text{per cent.}\end{matrix}\right\} : \left\{\begin{matrix}\text{the given}\\ \text{sum}\end{matrix}\right\} :: \left\{\begin{matrix}\text{the discount}\\ \text{on that}\\ \text{amount}\end{matrix}\right\} : \left\{\begin{matrix}\text{the discount}\\ \text{upon that}\\ \text{sum}\end{matrix}\right\}$$

By formula—

$$D = \frac{P \times I}{100 + I}, \text{ or } D = \frac{A \times I}{100 + I}$$

Note.—P = principal, in the sense in which it is used in discount; A = amount in the sense in which it is used in interest; ∴ P = A in these senses (p. 293); and $I = r \times t$; hence we have—

$$\frac{P \times I}{100 + I} = \frac{P \times t \times r}{100 + (t \times r)}, \text{ a form used on p. 295}$$

A study of the above statements will show that if the discount and two other elements be known, the fourth or remaining term can be found. It is important, however, to note that, *although the discount is proportional to the amount or principal, it is not proportional to either the time or the rate; e.g.* if the discount on a certain sum of money for a certain time at 5 % is £20, the discount on the same sum for the same time at 10 % is not £40, but a little less. The following examples will show the truth of this:—

(1) The discount on £420 for 1 yr. at 5% is £20, the proportional statement being—

$$£105 : £420 :: £5 : £20$$

(2) The discount on £420 for 1 yr. at 10% is £38$\frac{2}{11}$, the statement being—

$$£110 : £420 :: £10 : £38\tfrac{2}{11}$$

To find Present Worth.

I. The P.W. may be found by subtracting the D from the given principal (or amount).

II. It may be found directly, by finding what sum will amount to the given sum in the given time and at the given

* Abbreviations used in this chapter: D (discount); P.W. (present worth); P.V. (present value); I (interest); P (principal); A (amount); t (time); r (rate per cent.).

rate; *i.e.* we may use the same method as that for finding the principal in interest, when the rate, time, and amount are given (p. 271).

Example.—*What is the P.W. of* £1456 10*s. due* 3 *yrs. hence at* $4\frac{1}{2}$ %? (L.M.)

Interest on £100 for 3 yrs. at $4\frac{1}{2}\% = £13\frac{1}{2}$
amount of £100 at end of 3 yrs. at $4\frac{1}{2}\% = £113\frac{1}{2}$

Then if the present value of $£113\frac{1}{2}$ is £100, what is the present value of £1456 10*s.*?

$$\left.\begin{matrix}\text{Amount or}\\ \text{principal}\end{matrix}\right\} : \left\{\begin{matrix}\text{amount or}\\ \text{principal}\end{matrix}\right\} :: \text{present value} : \text{present value}$$

$$£113\tfrac{1}{2} : £1456\tfrac{1}{2} :: £100 : \text{P.V.}$$

$$\text{P.V.} = \frac{£1456\frac{1}{2} \times 100}{113\frac{1}{2}} = £1283\ 5s.\ 2\tfrac{86}{227}d. \quad \textit{Ans.}$$

Example.—*If the P.V. of* £120 is £100, *for the same time and at the same rate find the P.V. of a bill for* £600.

$$\begin{array}{lll} £100 & \text{is the P.V. of} & £120 \\ £\frac{100}{120} & \quad ,, \quad ,, & £1 \\ \text{and } \dfrac{£600 \times 100}{120} & \quad ,, \quad ,, & £600 \end{array}$$

$$\therefore \text{P.V.} = £\frac{\overset{5}{\cancel{600}} \times 100}{\underset{1}{\cancel{120}}} = £500$$

Ans. £500.

To find Discount at Different Rates.—Let the two rates be respectively 5 % and 10 %. Then, by formulæ (p. 294)—

$$D_5 = \frac{P \times 5 \times t}{100 + (5 \times t)}, \text{ and } D_{10} = \frac{P \times 10 \times t}{100 + (10 \times t)}$$

Hence we get the following proportion:—

$$\frac{P \times 5 \times t}{100 + (5 \times t)} : \frac{P \times 10 \times t}{100 + (10 \times t)} :: D_5 : D_{10}$$

It will be observed that the P's and the *t's* (numerators only) can be cancelled, but not the *t's* in the denominators; and so *to get the discount at one rate from that at another, the principal need not be known*, but the *time* must be known. In working such problems, therefore, it will be necessary to place the respective *rates only* in the numerators.

Example.—*The D on a certain sum of money for* 4 *months at* 6 % *is* £50. *What will it be for the same time at* 4 %?

$$\frac{6}{100 + (6 \times \frac{1}{3})} : \frac{4}{100 + (4 \times \frac{1}{3})} :: £50 : \text{D required}$$

$$\frac{6}{102} : \frac{12}{304} :: £50 : \text{D required}$$

$$\text{Discount} = £\frac{\overset{2}{\cancel{12}} \times 50 \times \overset{51}{\cancel{102}}}{\cancel{6} \times \underset{\underset{76}{\cancel{152}}}{\cancel{304}}} = £\tfrac{2550}{76} = £33\tfrac{21}{38}$$

Ans. £33 11*s.* $0\frac{12}{19}$*d.*

Banker's Discount.—This is also known as commercial discount, mercantile discount, practical discount, bank discount, and false discount. True discount has only a theoretical value. A banker does not charge true discount, but deducts the simple interest from the face value of the bill. *Banker's discount is thus the simple interest on the face value of the bill.*

The banker charges interest on the interest as well as on the present worth; *i.e.* he charges interest on the future value, which is larger than the interest on the present value.

As a rule, mercantile bills run for periods less than a year, and simple interest is then allowed. Strictly, if bills run for more than a year, compound interest should be charged; but in arithmetical problems this is not always adhered to. If compound interest is required, it is generally stated in the problem.

Let us suppose a case. A man wishes to cash a bill, and so goes to a banker or bill discounter, who pays the face value of the bill less the discount. Now, this discount is not true discount, but banker's discount or simple interest on the sum, and the banker gains by the process.

Suppose the bill to be one for £5000, that it has to be paid by B in a year, and that the rate of interest is 5 %. A wants money, so he goes to a banker, who pays him £4750, deducting £250 as discount. Now, note that the banker has deducted simple interest on £5000, and pays £4750 for a bill, the present value of which is £4761·9; so that he takes as his discount £250, instead of the true discount, viz. £5000 – £4761·9, or £238·1. He thus gains by the process £250 – £238·1, or £11·9. The banker's gain is, then, the difference between the simple interest and the true discount. It may also be described as the simple interest on the true discount, for the banker calculates simple interest upon the *face value*, that is, upon the *present value plus the true discount*, instead of calculating upon the present value only.

The Difference between Banker's and True Discount.—

1. To show that T.D.* is the I. on P.V.

P.V. + T.D. = sum due, or face value (F.V.)
But P.V. + I. on P.V. = sum due
∴ P.V. + T.D. = P.V. + I. on P.V.
∴ T.D. = I. on P.V.

2. To show that B.D. – T.D. = I. on T.D.

Sum due (F.V.) = P.V. + T.D.
(i.) But S.I. = S.I. on P.V. + S.I. on T.D.
(ii.) and T.D. = S.I. on P.V.
∴ S.I. – T.D. = S.I. on T.D. (subtracting (ii.) from (i.))
i.e. B.D. – T.D. = S.I. on T.D.

3. To show that T.D. is always less than S.I.

(*a*) T.D. is I. on P.V. (proved above),
and P.V. is always < sum due, or F.V.
∴ T.D. on F.V., *i.e.* I. on P.V., < I. on F.V.
i.e. T.D. < S.I.

* Abbreviations: T.D. (true discount); B.D. (banker's discount); S.I. (simple interest); F.V. (face value).

Days of Grace.—A bill of exchange contains the date on which payment is legally due, but three days' grace is allowed. If the last of the three days, however, should fall on a Sunday, the bill must be paid on the Saturday. If a bill is payable "on demand," the 3 days' grace is not allowed.

Example.—*A bill for* £6132 10*s.*, *drawn on March* 14*th and payable* 9 *months after date, was discounted by a banker on May* 12*th at* $3\frac{1}{4}$ %. *How much did the banker give for the bill?* (C.S.L.)

Days.	
May	19
June	30
July	31
Aug.	31
Sept.	30
Oct.	31
Nov.	30
Dec.	14
Days' grace	3
	219

The bill is due Dec. 17th. Hence D is to be allowed from May 12th to Dec. 17th, *i.e.* for 219 days. *Then the D on this bill will be the S.I. on* £6132$\frac{1}{2}$ *at* $3\frac{1}{4}$ % *for* 219 *days.*

$$\text{£100} : \text{£}6132\tfrac{1}{2} :: \frac{\overset{3}{\cancel{219}}}{\underset{5}{\cancel{365}}} \times \tfrac{13}{4} : \text{I}$$

$$\text{I} = \frac{\text{£}\overset{2453}{\cancel{12265}}}{\underset{4}{\cancel{2 \times 100 \times 20}}} \times 39 = \text{£}\frac{95667}{800}$$

$$= \text{£}119\ 11s.\ 8\tfrac{1}{10}d.$$

$$\therefore \text{ price paid by banker} = \text{£}6132\tfrac{1}{2} - \text{£}119\ 11s.\ 8\tfrac{1}{10}d.$$
$$= \text{£}6012\ 18s.\ 3\tfrac{9}{10}d. \quad \textit{Ans.}$$

If the banker had paid T.D., his cash payment would have been £6015 4*s.* 0·847...*d.* The student might verify this.

Tradesman's Discount.—This is a mere reduction in price, and, strictly speaking, it is not discount. A man has to charge a higher price for credit than for cash, and so he generally marks his goods at the credit price. The ready-money buyers simply gets this credit allowance removed, so that discount in this sense means simply the difference between the credit price and the cash price.

To find Principal.

1. Given *t*, *r*, and the difference between I. and T.D.

Example.—*The difference between the S.I. and the T.D. on a certain sum of money due* 2 *yrs. hence at* 5 % *is* £$4\frac{6}{11}$. *Find the sum.*

Suppose the amount of the bill to be £100.

Then S.I. $= 2 \times \text{£}5 = \text{£}10$

and T.D. $= \text{£}\frac{100}{110} \times 10 = \text{£}\frac{100}{11} = \text{£}9\frac{1}{11}$

$\therefore$ difference between S.I. and T.D. $= \text{£}10 - \text{£}9\frac{1}{11} = \text{£}\frac{10}{11}$

$\therefore$ sum required is found thus—

$$\text{£}\tfrac{10}{11} : \text{£}4\tfrac{6}{11} :: \text{£}100 : \text{P}$$

$$\therefore \text{P} = \text{£}\frac{4\frac{6}{11} \times 100}{\frac{10}{11}} = \text{£}\frac{\overset{5}{\cancel{50}} \times 100}{\underset{1}{\cancel{10}}} = \text{£}500$$

Ans. £500.

2. Given r different.

Example.—*The difference between the present worth of a sum at* 4 % *for* 1 *yr. and the same sum for the same time at* 5 % *is* £1 10*s*. *Find the sum.*

Since P.V. at 4 % of £104 = £100

$\therefore$ P.V. of sum due 1 yr. hence at 4 % $= \frac{100}{104} = \frac{25}{26}$ of sum due

and P.V. ,, ,, ,, 5 % $= \frac{100}{105} = \frac{20}{21}$ of sum due

By question $(\frac{25}{26} - \frac{20}{21})$ of sum due = £1$\frac{1}{2}$

i.e. $\left(\frac{525 - 520}{26 \times 21}\right)$ of sum due = £$\frac{3}{2}$

i.e. $\frac{5}{26 \times 21}$ of sum due = £$\frac{3}{2}$

$$\therefore \text{ sum due} = £\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}} \times \frac{\overset{13}{\cancel{26}} \times 21}{5} = £\frac{819}{5} = £163\frac{4}{5}$$

Ans. £163 16*s*.

3. Given t, r, and T.D.

Example.—*On what sum of money is the T.D. for* $1\frac{2}{3}$ *yr. at* 3 % £150?

By the Unitary Method—

£5 is the T.D. for 20 months at 3 % on £105

and £1 ,, ,, ,, ,, £21

$\therefore$ £150 ,, ,, ,, ,, £21 × 150

Ans. £3150.

To find Time.

1. Given T.D., P, and r.

Example.—*The T.D. on a bill for* £3150 *is* £150, *interest being charged at the rate of* 3 %. *Find the time.*

P.V. = £3150 − £150 = £3000

$$\text{and } t = \frac{100 \times \text{I}}{Pr} = \frac{\overset{50}{\cancel{100}} \times \overset{150}{\cancel{150}}}{\underset{30}{\cancel{3000}} \times \underset{1}{\cancel{3}}} = \frac{50}{10} = 1\frac{2}{3} \text{ yrs.}$$

Ans. $1\frac{2}{3}$ yrs.

If T.D., P, and r are known, the P.V. is known, and this may be treated as the P in an interest question. The problem thus becomes, "In what time will the S.I. on £3000 at 3 % amount to £150?"

2. Two different principals produce the same discount; the time is known in one case: required to find it in the other.

Example.—*If* £30 *is the T.D. on* £1030 *for* 9 *mths., for how long is* £30 *the T.D. on* £1530 *at the same rate?*

£30 is the T.D. on £1030
∴ £1000 is the true P.W. of £1030
∴ £30 is the I. on £1000 for 9 mths. at a certain rate

$$i.e.\ \frac{1000 \times \frac{3}{4} \times r}{100} = 30$$

$$\text{or } 3000 \times \text{rate} = 400 \times 30$$

$$\therefore \text{rate} = \frac{\overset{4}{\cancel{400}} \times \overset{30}{\cancel{30}}}{\underset{100}{\cancel{3000}}} = 4\ \%$$

Again, £30 is the T.D. on £1530
∴ £30 is the I. on £1500 at 4 % for an unknown time

$$\text{then } \frac{\overset{15}{\cancel{1500}} \times t \times 4}{\underset{100}{\cancel{100}}} = 30$$

$$i.e.\ 60 \times t = 30$$

$$t = \tfrac{30}{60} = \tfrac{1}{2} \text{ yr. or } 6 \text{ mths.}$$

Ans. 6 mths.

It will be noticed that all the conditions in the first case are known except the rate. This is found. It is then used in the second case, the time thus being the only unknown element. Three terms being given (P, D, *r*), the fourth (*t*) is easily found.

3. To find time, when B.D. and T.D. for different sums are equal.

Example.—*The B.D. on* £1000 *at* 5 % *is the same as the T.D. on* £1100 *for the same time at the same rate. Find the time.*

By question, I. on £1000 = T.D. on £1100
= I. on (true P.V. of £1100)
∴ £1000 = true P.V. of £1100
and ∴ £100 = T.D. on £1100
∴ £100 = I. on £1000 at 5 % for a certain time

$$100 = \frac{\overset{10}{\cancel{1000}} \times t \times 5}{\underset{100}{\cancel{100}}} = 50 \times t$$

$$\therefore t = 100 \div 50 = 2$$

Ans. 2 yrs.

To find Rate.—The method is similar to that for finding time.

1. Given D, P, and *t*.

Example.—*Find the rate of interest, if the true discount on* £339 3*s.* 4*d.*, *due* 3 *mths. hence, is* £5 16*s.* 8*d.* (C.H.L.)

T.D. = £5 16*s.* 8*d.* = £$5\frac{5}{6}$
∴ P.W. = £$339\frac{1}{6}$ − £$5\frac{5}{6}$ = £$333\frac{1}{3}$

The problem then becomes: At what rate per cent. will £$333\frac{1}{3}$ yield £$5\frac{5}{6}$ interest in 3 mths.?

By formula—

$$r = \frac{100 \times \text{I}}{\text{P} \times t} = \frac{100 \times 5\frac{5}{6}}{333\frac{1}{3} \times \frac{1}{4}} = 7$$

Ans. 7 %.

2. Given T.D., B.D., and t, to find both sum and r.

Example.—*The true discount on a sum of money for* 6 *mths. is* £12, *and the banker's discount for the same time at the same rate is* £12 10*s*. *What is the sum and what is the rate?*

(*a*) To find rate.

$$\text{B.D.} - \text{T.D.} = £12\tfrac{1}{2} - £12 = £\tfrac{1}{2}$$

Now, B.D. − T.D. = I. on T.D.

$$\therefore £\tfrac{1}{2} = \text{I. on } £12 \text{ for 6 mths.}$$

$$i.e.\ £\tfrac{1}{2} = \frac{£12 \times \frac{1}{2} \times r}{100}$$

$$\text{and } £\tfrac{1}{2} \times 100 = £12 \times \tfrac{1}{2} \times r$$

$$\therefore \frac{\frac{1}{2} \times 100}{12 \times \frac{1}{2}} = r$$

$$8\tfrac{1}{3} = r$$

Ans. $8\frac{1}{3}$ %.

(*b*) To find sum.

B.D. = I on P for given time and at given rate

$$\therefore £12\tfrac{1}{2} = \frac{\text{P} \times \frac{1}{2} \times 8\frac{1}{3}}{100}$$

$$\text{and } £\tfrac{25}{2} = \frac{\text{P} \times 25}{600}$$

$$\text{and } \frac{25 \times 600}{2} = \text{P} \times 25$$

$$\text{and } \frac{25 \times 600}{2 \times 25} = \text{P} = £300.$$

Ans. £300.

To find P.W. or D. at Compound Interest.

It has already been pointed out that if bills run for more than a year, compound interest should be charged. Assurance companies calculate deferred payments in this way. The method is, however, similar to that for simple interest, for the relations do not change. We still have—

P.W. + I. on P.W. = sum due

or A. of P.W. in given time = sum due

1. To find P.W. or D.

Example.—*Find the P.W. and T.D. on* £121550 12*s*. 6*d*. *due* 4 *yrs. hence at* 5 % *compound interest.*

£5 = $\frac{1}{20}$	1·00
	·05
$\frac{1}{20}$	1·05
	·0525
$\frac{1}{20}$	1·1025
	·055125
$\frac{1}{20}$	1·157625
	·05788125
	1·21550625

Amt. of £1 in 4 yrs. at 5 %

£1·21550625 : £121550·625 :: £1 : P.W.

$$\text{P.W.} = £\frac{121550{\cdot}625}{1{\cdot}21550625} = £100000$$

and D. = £121550·625 − £100000

= £21550·625 = £21550 12*s*. 6*d*.

Ans. (*a*) £100000.

(*b*) £21550 12*s*. 6*d*.

Or we might have proceeded by *formula*.

$$A = P \times \left(1 \times \frac{r}{100}\right)^n \text{ (p. 284), where } P = \text{P.V.}$$

$$121550{\cdot}625 = P \times (1 + \tfrac{5}{100})^4 = P \times (\tfrac{105}{100})^4$$

$$\therefore P = 121550{\cdot}625 \div (\tfrac{105}{100})^4$$

$$= £\frac{121550{\cdot}625}{1{\cdot}21550625} = £100000$$

Ans. £100000.

Note.—Easy examples only can be worked in this way. For a greater number of years logarithms would have to be used.

Example.—*A man is to receive* £300 *a year for* 3 *years, and the first payment is to be made a year hence, the second at the end of the next year, and so on. He elects to receive an immediate equivalent payment, the rate being* 5 % *at C.I. What sum will he receive, the answer being given to the nearest* £?

Required P.V. = { sum of P.V.'s of £300 due at the end of 1, 2, and 3 yrs. respectively.

$$= £300 \div (1{\cdot}05) + £300 \div (1{\cdot}05)^2 + £300 \div (1{\cdot}05)^3$$

$$= £300 \times \{(\tfrac{100}{105}) + (\tfrac{100}{105})^2 + (\tfrac{100}{105})^3\}$$

$$= £300 \times (\cdot9523 + \cdot9070 + \cdot8639)$$

$$= £300 \times 2{\cdot}7232 = £816{\cdot}96$$

$$= £817 \text{ nearly.}$$

Ans. £817.

THEORY.

1. Define present worth, discount, and amount, giving simple examples to illustrate your definitions. (C.S.)

2. Distinguish clearly between *interest* and *discount*. What is meant by (1) *days of grace*, (2) *mercantile* discount? Are there any exceptions to the days of grace?

3. Define true discount, with examples. Which is greater, the discount on a sum due in a certain time, or the simple interest on the sum for that time, the rate per cent. being the same? (K.S.)

4. Distinguish between true discount and ordinary trade or banker's discount. (C.W.)

5. Explain the rule for finding the present value of a given sum, due after a given time, at a given rate per cent. simple interest. (C.W.)

6. Show that banker's discount equals true discount plus interest on the true discount. (C.W.)

7. "Discount equals given sum less present worth; present worth equals given sum less discount." Explain and give an illustration of this statement. (C.W.)

8. Explain what is meant by saying that discount is interest on the present worth. (C.PR.)

9. Show that the discount of any sum is the present worth of the interest on that sum for the same time and at the same rate. (C.S.)

10. Would the "true" discount on a certain sum for a certain time be more or less than double the discount on the same sum for half the time, the rate of interest (supposed simple) being the same? (L.M.)

11. Give reasons for the rule for finding the discount of a given sum

due a given time hence at a given rate per cent. simple interest (*e.g.* the discount of £299 15*s.* due 3 yrs. hence at 3 per cent. per annum simple interest). (K.S.)

12. Prove that the difference between the interest and the discount on any sum of money for a given time is equal to the interest on the discount for that time, simple interest being reckoned at the same rate throughout. (C.S.L.)

13. Show that the simple interest (at 3 per cent.) of a sum of money for 3 yrs. : the discount (3 per cent. simple interest) of the same sum payable at the end of 3 yrs. :: 109 : 100. (O.B.)

14. To which of the following terms is discount proportional: (*a*) principal, (*b*) time, (*c*) rate? Give your reasons in each case.

15. To find the discount at one rate from that at another rate, the principal need not be known, but the time must be known. Prove this statement.

16. Of the two methods for finding the present worth of a given sum, which do you prefer, and why?

17. Show how to find the principal in discount, (*a*) when the rate per cent., the time, and the difference between interest and true discount are given; (*b*) when the rates are different; (*c*) when the time, rate, and discount are given.

18. Prove, generally, that with interest at x per cent. the commercial present worth is less than the true present worth by the fraction $\left(\frac{x}{100}\right)^2$ of the latter. (L.M.)

EXERCISE XLIX. 49

DISCOUNT.

To find Discount.

Find the true discount on—

1. £1336 11*s.* 3*d.* for $3\frac{1}{2}$ yrs. at 5 per cent. per annum.

2. £210 12*s.* 1*d.* for $3\frac{1}{2}$ yrs. at $4\frac{1}{4}$ per cent. per annum.

3. £168 7*s.* 6*d.* for $\frac{1}{5}$ yr. at $5\frac{1}{8}$ per cent. per annum.

4. £555 for 17 mths. at $4\frac{1}{2}$ per cent. per annum.

5. £98 19*s.* 3*d.* for $1\frac{3}{4}$ yrs. at 5 per cent. per annum.

6. Find the discount on a bill for £2828, due in 2 yrs. 146 days, at 5 per cent. per annum. (K.S.)

7. Find the true discount on £929 19*s.* $10\frac{1}{2}$*d.* due 6 mths. hence at 3 per cent. per annum. (C.M.)

8. Determine the discount on a bill of £97 17*s.* 8*d.* fully due on November 7, and discounted on September 13, at $3\frac{3}{4}$ per cent. per annum. (O.S.L.)

9. Find the true discount on a bill for £1319 10*s.*, due on June 19, at 6 mths., and discounted January 12, 1901, at $4\frac{1}{2}$ per cent. (C.W.)

10. A bill for £900, drawn January 30, 1881, for 6 mths. is discounted March 9 at 5 per cent. What does the banker gain by the difference between true and commercial discount? (I.I.)

11. What is the true discount on a bill for £1622 5*s.* drawn on February 4 at 10 mths., and discounted July 14 at $7\frac{1}{2}$ per cent. per annum, 3 days of grace being allowed? (C.M.S.)

12. Find the discount on a bill for £$237\frac{1}{2}$, due 4 mths. hence, at 4 per cent. simple interest. What would be the amount of the error in this case if interest were taken instead of discount, and in whose favour would the error be? (P.T.)

13. Find the difference between the simple interest on £461 5*s.* for 2 yrs. at 4 per cent. and the discount on the same sum due after 2 yrs. at 4 per cent. (C.W.)

14. The *commercial* discount on a sum due in 7 mths. at 3 per cent. per annum is £11 17*s.* 5*d.* What is the *true* discount? (C.P.)

15. Find the true discount on £50 3*s.*, due 6 mths. hence, at 4 per cent. (omitting days of grace). How much more than this would banker's discount be? (C.P.)

To find Present Value.

Find the present worth or present value of—

16. £284 due in 2 yrs. at $3\frac{1}{4}$ per cent. per annum.

17. £151 17*s.* 6*d.* due in 4 yrs. at $5\frac{3}{8}$ per cent. per annum.

18. £656 due in 6 mths. at 5 per cent. per annum.

19. £121 5*s.* 6*d.* due in 3 mths. at $4\frac{1}{4}$ per cent. per annum.

20. £675 2*s.* 6*d.* due in 3 yrs. at $4\frac{1}{4}$ per cent. per annum.

21. What is the present value of £3420, due 4 yrs. hence, reckoning interest at $3\frac{1}{3}$ per cent. per annum? (C.P.)

22. If the present worth of £218, due 2 yrs. hence, be £200, what is the present worth of £1000 due 6 yrs. hence at the same rate? (C.S.L.)

23. A person entitled to £2877 10*s.* $1\frac{1}{2}d.$ in $2\frac{1}{2}$ yrs.' time, discounts his expectation at $7\frac{1}{2}$ per cent. simple interest. What is the estimated present value? (C.S.)

24. What is the true discount on £512 10*s.*, due 2 yrs. 8 mths. hence, at $3\frac{1}{2}$ per cent. per annum simple interest? Express your answer as the decimal of the present value of the sum. (C.M.)

25. Find the present value of a legacy of £2100, payable on the death of a certain person whose probable duration of life is estimated at 10 yrs. (Money at 4 per cent. simple interest.) (C.P.)

26. Find the present worth of £1868, payable as follows: £406 in 3 mths., £618 in 6 mths., and the remainder in 11 mths., interest being at the rate of 6 per cent. per annum. (C.M.)

27. If the price of goods be £171 2*s.* $4\frac{1}{2}d.$, credit for 1 yr. being allowed, what is the true cash price, reckoning $4\frac{1}{2}$ per cent. simple interest per annum? (C.S.)

28. Find the value on March 1 of the expectation of a sum of £3724 10*s.* 5*d.*, due November 30 the same year, the rate of interest being 5 per cent. per annum. (C.S.)

29. Two bills for £456 5*s.* and £301 2*s.* 6*d.* are due on July 5 and 30 respectively. Find their joint values on June 10, reckoning bankers' discount at 5 per cent. (C.W.)

30. If a sum of £1000 becomes due 3 mths. hence, what is its present value as commonly calculated, and what as correctly calculated, interest being reckoned at 5 per cent.? (S.K.)

To find Principal, Sum, or Amount.

31. The true discount on a bill due 8 mths. hence at 2½ per cent. is £176 14*s.* 8*d.* Find the amount of the bill. (K.S.)

32. The commercial present worth of a sum due 219 days hence at 5 per cent. is £499 11*s.* What is the sum, and what is its true present worth? (I.I.)

33. A man discounts a bill due 5 mths. hence, and finds that his banker gives him £1 less than the theoretical present value of the bill. What was the amount for which the bill was drawn, the rate of discount being 2 per cent.? (C.H.L.)

34. The sum of the true discounts at 6 per cent. per annum and at 4 per cent. per annum on a bill having 6 mths. to run is £1 less than the true discount at 5 per cent. per annum on a bill of twice the amount, which also has 6 mths. to run. Find the amount of the first bill. (I.I.)

35. A person holding a bill due 30 days hence discounts it without the usual 3 days of grace at 4½ per cent. per annum (true discount), and invests the proceeds at 5 per cent. interest per annum; he thereby gains 11*s.* 3*d.* in the 30 days. Find the amount for which the bill was drawn. (C.S.)

36. I am allowed 1¼ per cent. discount on an amount charged to me for goods received, and give my acceptance at 5 mths. for the *net* sum. By selling the goods immediately for a bill of £162 12*s.* 2*d.*, payable at 7 mths., my percentage of gain is 11½ per cent. What was the amount originally charged to me, interest being calculated at 5 per cent.? (P.T.)

37. A bill falls due in 6 mths. The creditor agrees to accept immediate payment of half, and to defer payment of the other half for 1 yr. He finds that by this arrangement, reckoning interest at 6 per cent. per annum, he gains 9*s.* What is the amount of the bill? (C.S.)

38. The discount on a certain sum of money due at the end of a certain time is £1870, and the interest on the same sum of money for the same time is £2452 16*s.* 4*d.* What is the sum? (O. and C.S.C.)

39. A owes B a sum of money, and offers to pay him immediately, subject to discount reckoned at 4½ per cent., instead of at the end of 2 yrs. when the debt is due. B can place out the money at 5 per cent. interest, and by that means will gain £36. What was the sum due? (C.W.)

To find Time.

40. The true discount on a bill of £2440 7*s.*, due at a given time at 4¾ per cent., is £145 7*s.* When will the bill become due? (C.W.)

41. If the interest on £253 2*s.* 6*d.* at 5 per cent. be equal to the discount on £257 6*s.* 10½*d.* for the same time at the same rate, when is the latter sum due? (C.PR.)

42. Received to-day (September 29) £456 10*s.* for a debt of

£490 14*s.* 9*d.*, discount 5 per cent. per annum. When should the debt have been payable without discount? (P.T.)

43. The present worth of a sum of money due in 5 yrs. at $3\frac{1}{4}$ per cent. per annum simple interest is $3\frac{1}{3}$ per cent. less than the present worth of an equal sum at $4\frac{1}{2}$ per cent. per annum simple interest. When is the latter sum due? (C.P.)

44. The present values on May 1, of two bills drawn for the same amount were in the ratio of 4638 : 4639. Interest was reckoned in the case of the first bill at the rate of $4\frac{1}{4}$ per cent. per annum, and in the case of the second at the rate of 4 per cent. per annum. If the first was fully due on September 22 of the same year, when did the second become due? (C.S.)

45. The discount upon £220, due at the end of a given number of years, is £44. Find the interest upon the same sum (£220) for the same number of years and at the same rate of interest. Also determine the rate per cent. and the number of years, given that these are indicated by whole numbers greater than unity. (C.S.)

To find Rate.

46. The present worth of £1321 15*s.* due in $2\frac{1}{2}$ yrs. is £1244. Find the rate per cent. per annum simple interest. (K.S.)

47. Find the rate of interest if the true discount on £339 3*s.* 4*d.* due 3 mths. hence is £5 16*s.* 8*d.* (C.H.L.)

48. A bank discounting a bill due 10 mths. hence deducts 4 per cent. from the total amount of the bill as discount. What rate of interest per annum does the bank get for its money? (C.M.)

49. The interest on a certain sum of money for 2 yrs. is £71 16*s.* $7\frac{1}{2}$*d.*, and the discount on the same sum for the same time is £63 17*s.*, simple interest being reckoned in both cases. Find the rate per cent. per annum, and the sum. (C.S.)

50. If the true discount on £2696 13*s.* 4*d.* due 3 mths. hence is £30, what is the rate per cent. simple interest? At what rate per cent. would the banker's discount on the same sum for the same time be £30 6*s.* 9*d.*? (C.S.)

51. The interest on a certain sum for 10 yrs. is greater than the discount on the same sum due in 10 yrs. by one quarter of the discount. Find the rate per cent. per annum simple interest, which is the same in both cases. (C.P.)

52. Find the true discount and the banker's discount on £1062 10*s.* due 16 mths. hence at $3\frac{3}{4}$ per cent. What is the rate per cent. for the aforesaid time when the banker's discount exceeds the true by $\frac{1}{15}$ of the latter? (L.M.)

53. Two tradesmen, A and B, have the same cash price for the same quantity of a certain commodity, but their credit prices are £26 5*s.* and £25 6*s.* 3*d.* respectively. The rate of mercantile discount for cash payment allowed by A. is half as large again as that allowed by B. What rate of discount is allowed by A? (C.P.)

Discount at Compound Interest.

54. Find the present value of £91 17*s.* 6*d.* due in 2 yrs. at 5 per cent. compound interest. (O.S.L.)

55. Find the present value of £84 10*s.* due 2 yrs. hence, compound interest being reckoned at 4 per cent. per annum. (S.K.)

56. Find the true present value of a debt of £3000 due 3 yrs. hence, reckoning compound interest at 5 per cent., and omitting fractions of a penny in the result. (C.H.L.)

57. Find the difference between the amount of £1205 put out for 2 yrs. at 5 per cent. compound interest, and the present value of the same sum due 2 yrs. hence at 5 per cent. compound interest. (C.PR.)

58. If the compound interest on a certain sum of money for 2 yrs. at 4 per cent. is £45 18*s.*, what is the true discount on the same sum of money due 2 yrs. hence at simple interest at 4 per cent.? (C.S.)

59. Of what debt due 2 yrs. hence at 4 per cent. compound interest is £1397 11*s.* 3*d.* the present value? (Get your results right to the nearest penny.) (C.S.L.)

60. Find the compound interest for 3 yrs. at 4 per cent. on the present worth of £573 15*s.*, due 5 yrs. hence at 7 per cent. simple interest. (C.M.)

61. The present value of a sum of money due 3 yrs. hence is £150, and its value in 2 yrs.' time will be £160 3*s.* 4*d.* Find the rate of compound interest. (L.M.)

62. Find (*a*) the simple interest on £468 11*s.* 10½*d.* for 4 yrs. at 4 per cent.; (*b*) the present worth of £86 16*s.* 5¼*d.* due 3 yrs. hence, compound interest being reckoned at 5 per cent. (C.H.L.)

Miscellaneous Problems.

63. A man bought a field, and gave a bill of £482 due in 2 mths. for it, and sold it at once for a bill of £509 5*s.* due in 5 mths. Required his gain per cent., reckoning interest at 2½ per cent. per annum on the bills. (P.T.)

64. The real cost of an article is 56 per cent. of the price at which it is marked for sale. It is, however, sold at a trade discount of 12½ per cent. How much does the seller gain per cent.? (L.M.)

65. A dealer sells 25 tons of coals at 19*s.* 2*d.* per ton, and allows 5 per cent. discount for cash payment. If he had not allowed the discount his profit would have been 15 per cent. of his outlay. Find what his profit per cent. is at the reduced price, and his whole gain on the quantity sold. (C.S.)

66. A cargo was bought for £10282, payable 12 mths. hence, and was immediately sold again for £10712, payable 8 mths. hence. What was the gain per cent. on the value of the purchase-money, the rate of discount being 6 per cent. per annum? (C.M.)

67. A lends B £240 on condition that at the end of 2 yrs. B will repay him £264 principal and interest. B finds that he can pay the money back at the end of a year, and A agrees to let him do so at a (true) discount of 4 per cent. per annum. Find to the nearest penny the

sum that A receives, and the interest per cent. that A makes on his £240 for that one year. (S.K.)

68. If a sum of £2320 10*s.*, due a year hence, is to be paid off at once, and 5 per cent. discount is to be allowed, what difference will it make in the present payment if 5 per cent. of the whole is deducted instead of the true discount? (C.S.)

69. A discount house takes money on deposit and uses the money in discounting bills. It pays 4 per cent. per annum interest on the deposits, and makes $4\frac{1}{4}$ upon them. If the deposits average £10000000, what annual profit is made? (C.PR.)

70. A gives B a bill for £1035 3*s.* due 8 mths. hence in exchange for a bill for £1075 18*s.* due 16 mths. hence. If the true present values correspond to a rate of interest of $4\frac{1}{2}$ per cent. per annum, determine which of the parties to the transaction is the gainer, and by how much? (O. and C.H.C.)

71. The ready-money price of an article at a tradesman's, who allows mercantile discount in the ordinary way, for ready money is £8 11*s.*, and the credit price is £9. What ought the credit price to be in order that, while charging the same ready-money price, he may allow *twice* the rate of discount? (P.T.)

72. Find the present value of a bill for £6433 3*s.* 4*d.* due 3 mths. hence, the rate of interest being $3\frac{1}{3}$ per cent. per annum. A person holding such a bill disposes of it through an agent at the reduced rate of £100 for £$106\frac{1}{3}$ of the nominal value, and the agent charges a commission of $\frac{1}{6}$ per cent. on the purchase-money. What does the person receive? (C.S.)

73. A master sends his clerk to pay bills to the amount of £177 7*s.* 6*d.*, which are not due until a year after. He gives him the money for the purpose, deducting $7\frac{1}{2}$ per cent. true discount. The clerk gets tradesman's discount to the extent of 10 per cent. on £89 19*s.* 2*d.*, and 5 per cent. on the rest of the bills. What profit is made by the clerk? (C.H.L.)

CHAPTER XXXIII.

STOCKS AND SHARES.

Stock.—Stock is the name given to money lent to a Government, a corporation, a county council, or a trading company. Government Stock is money borrowed by a Government to meet some exceptional expenditure, like a war, or an increase of armaments. The money is borrowed at a fixed rate per cent., and the Stock is named from this rate, *e.g.* the Two and a half Per Cents.

From the buyer's or lender's point of view, the money is said to be *subscribed;* from the Government's or company's point of view, it is said to be *issued; i.e.* the Government issues stock, and the public subscribes it.

Shares.—Companies generally raise their capital in shares. For instance, a company may desire to raise £100,000 capital, and it may do this by offering shares at £1, £10, or any other price, to the public.

But the issue price of shares is rarely paid up on application. The portions comprising the total are paid at specified times in specified amounts fixed by the company. For instance, a stock may be issued at 92 and offer 3 % interest, and the money may be subscribed thus—

10 % on application; *i.e.* $\frac{1}{10}$ of the sum will be forwarded when the shares are applied for.
20 % on allotment; *i.e.* $\frac{1}{5}$ of the sum will be paid when the company has decided how many shares are allotted to the applicant.
35 % at some subsequent time; *e.g.* one month after allotment.
35 % on another date, say one month later.

Kinds of Shares.—Shares differ mainly in the way in which the profits or dividends are divided.

1. **Preference Shares** are issued at a fixed rate of dividend, usually about 5 % or 6 %. In the division of profits these shares have the preference; *i.e.* their dividends are paid before those of any other shareholders.

2. **Debentures** are sometimes granted by public companies to raise

more working capital. The distinctive feature is that the whole undertaking (plant, etc.) of the company is mortgaged for the money thus borrowed. Debentures are thus the safest of shares, and as a result, they receive, as a rule, less interest than the other shares. Like preference shares, the dividends on them are paid before the dividends on ordinary or deferred shares.

3. **Ordinary Shares.**—After the two previous classes of shares have been paid, most of the remaining profit is divided among the holders of ordinary shares. A portion of the profits is usually carried over to a reserve fund, to meet the possibility of bad times.

4. **Deferred Shares.**—Holders of this stock get the remnants, if there are any. The payment of their dividends is deferred until all the other shareholders have been paid at a rate previously agreed upon.

Stock and Cash.—The student must distinguish carefully between these two terms.

Stock is the *nominal* value on paper; *i.e.* the *face* value.

Cash is the *real* value in money; the *market* value.

For instance, suppose £1000 is cash to be invested; when it *is* invested it is stock. When it is being sold it is stock; it is stock to the buyer, and becomes cash to the seller only when the sale is completed. In other words, *Stock is not money, but value for money*, and on this value the holder of the stock may claim a dividend.

Value.—To avoid error, the various senses in which this term is used must be clearly understood.

1. **Par Value.**—Face value; the value expressed on the face of the share certificate.

2. **Real Value.**—This varies, and depends on the market price of the day. Stock is said to be at a premium when its real value is *above* its par value; and at a discount when its real value is *below* its par value.

3. **Nominal Value.**—This is invariable. It is the amount of money originally paid for the shares on their first issue.

Suppose the 3 % Tasmanians to be issued at 92, and that £1000 worth of stock be bought. The *nominal value* of the stock is £920; the *par value* £1000; and if the stock be now at 94, the *real value* is £940.

Dividends.—A dividend is the amount to be divided, and it signifies a share, *pro rata* to the amount subscribed, of the total interest or profits on the total amount of stock. Dividends may thus be called interest or profit on stock or shares. It may be noted that—

1. Interest or dividend is paid regularly on the stock—
   (*a*) Either at a fixed rate, as in Preference and Debenture Shares, and in Government or Municipal Stock;
   (*b*) or, at a varying rate, dependent on circumstances (state of trade, etc.), as in the case of ordinary and deferred shares.

2. Dividend is usually calculated on each £100 stock; *i.e.* on the par or face value. It is not calculated on the market value or selling price of the stock.

3. The rate of dividend, if fixed, is invariable, so long as a Government or company, etc., is solvent. The holder of £100 worth of 3 % Stock will receive £3 per £100 stock, whatever the market price of the stock may be at the time. Hence, if a man buys at a discount, he gets a better return for his investment; and conversely if he buys at a premium.

4. When a company declares a dividend—which is usually paid half-yearly—at so much per cent., interest is paid only on the amount of the stock or shares paid up, and not on their full nominal value. For instance, if a man is allotted £1000 worth of stock, and the company has called up only £700, his dividend will be paid on the £700, and not on the £1000.

5. The words "per cent." may prove misleading, as it means so much on the *face value*; *e.g.* a man invests in the 4 per cents. at 20 % premium; *i.e.* he buys £100 stock for £120 cash. His real percentage then is not 4, but $3\frac{1}{3}$, for he is receiving £4 only for the loan of £120.

Buying and Selling.—Suppose a man to hold £1000 stock in the $2\frac{1}{2}$ per cents.

1. He holds bonds or certificates from the Government acknowledging this debt.

2. As a rule, he cannot sell back his stock to the Government, but he can sell it to other people.

Governments often reserve the right of redeeming stock at a specified time.

3. His certificate entitles him to receive interest. The Government undertakes to pay each half-year £$12\frac{1}{2}$; *i.e.* the interest on £1000 for half a year at $2\frac{1}{2}$ %.

If a shareholder wishes to sell, we note that—

1. He may sell at a premium (profit), or at par (neither profit nor loss), or at a discount (loss).

2. The price of stock is constantly varying. Among other things, it depends on—

(*a*) The rate of interest.
(*b*) The credit of the Government, etc.; *i.e.* upon its ability and readiness to pay.
(*c*) The number of persons desirous of becoming investors. Stock is like any other marketable commodity—an increased demand sends up the price.

3. Stocks are usually quoted at two prices on the Stock Exchange. Thus "Consols $104\frac{3}{8} - \frac{3}{4}$" would mean that they could be bought at $104\frac{3}{4}(+\frac{1}{8})$, and sold at $104\frac{3}{8}(-\frac{1}{8})$. The price for the day will fluctuate between these limits most probably.

4. The buying and selling will be done by the aid of certain men,

usually members of the Stock Exchange, called *stockbrokers* and *stock-jobbers*. The *broker* buys and sells for the public and charges a commission for his trouble, which commission is usually termed *brokerage*. He is the agent between the public and the jobber. The usual charges of such men are—

(*a*) 2*s.* 6*d.* or $\frac{1}{8}$ % for Consols and Foreign Government Stocks.
(*b*) $\frac{1}{4}$ % on Railway Stock.
(*c*) $\frac{1}{2}$ % on shares below the value of £100.
In arithmetic problems $\frac{1}{2}$ % is usually given.

The *jobber* is a man who deals in stocks and shares. He lives on the difference between the buying and selling price of stocks. In other words, like a tradesman or merchant, he buys and sells a certain article (stocks), and lives on the profits. He is not an agent like the broker, but a *stock merchant*.

The Funds.—Money which is lent to the Government is said to be in the Funds. The money is lent at a fixed rate of interest, and the investment, being both safe and marketable, carries a low rate of interest. The Funds are Government Stock, and this stock is sold in the form of certificates, which entitle the holder to receive so much interest each year, so that each purchaser of the stock may be said to have purchased an "annuity."

When "Annuities" were first issued they were for *limited* periods, and a tax was earmarked for their redemption. Subsequently, however, annuities were issued for *unlimited* periods, and the payment of them was made from the annual revenue. A fixed redemption fund was no longer set aside. The Funds, therefore, may now be said to mean a sufficient sum of money to redeem these annuities, and they thus represent the credit of the country.

Consols.—This word is an abbreviation of the phrase "Consolidated Annuities." They were so called because they were formed by the consolidation into one stock of various debts, contracted by the Government at different times and at different rates. They represent the major portion of the public debt of this country. They yield $2\frac{1}{2}$ % interest, and dividends are paid quarterly. They are also known as the Funds, and constitute what is called the *Funded Debt*, which practically thus consists of unredeemable loans.

The Unfunded Debt.—This debt is composed of redeemable loans, the common form being that of Exchequer Bills. As a rule the money is borrowed for a few months only. Interest is paid at the specified rate for the time of the bills in *days*, as the life of a bill is a short one. All persons who have a claim upon the country for money thus invested are known as fundholders, whether they hold consols or bills.

Bonds.—These are the documents or printed certificates which the Government issues in acknowledgment of the fundholders' claims. They are also called *Debentures*. If the bond contains a promise to pay back the principal, it is called a *terminable bond;* if it contains no such promise, it is called a *perpetual bond*. *Coupons* or *warrants* are attached to the bond, and one is detached and forwarded to the Government at each period for receiving dividends.

Mortgage.—Sometimes money is borrowed by means of bonds. Houses, lands, or other property may be pledged as security, and a fixed rate of interest may be promised; the bond then becomes a mortgage.

Some worked examples in stocks now follow.

I. To find Income derived from Investment in Stock.

(1) When stock is given.

Example.—*I own* £4400 *in the* 4 *per cents. What is my income?*

If the income on £100 *stock* is £4

then „ „ £4400 „ £4 × 44 = £176

Ans. £176.

Note that the £4400 is *in* the 4 per cents.; *i.e.* it is stock.

(2) When money is given.

Example.—*I invest* £9800 *in the* $3\frac{1}{4}$ *per cents. at* 98. *What is my income?*

Now, £98 (money) would buy £100 (stock), which would yield £$3\frac{1}{4}$ (dividend); *i.e.* £98 (money) yields £$3\frac{1}{4}$ (dividend). The problem thus becomes: If £98 (money) yields £$3\frac{1}{4}$ (income), what income will £9800 (money) yield?

£98 : £9800 :: £$3\frac{1}{4}$: required income

$$\therefore \text{ income} = \frac{£3\frac{1}{4} \times 9800}{98} = £325$$

Ans. £325.

Note that £9800 is *to be* invested; *i.e.* it is money, not stock.

(3) Given the difference between the two incomes and the rates, to find the incomes.

Example.—*The difference between two incomes, the smaller of which is in the* 3 *per cents. and the larger in the* 5 *per cents., is* £500, *and they are in the ratio of* $\frac{3}{98} : \frac{1}{21}$. *Find the incomes.*

Smaller : larger :: $\frac{3}{98} : \frac{1}{21}$

:: $\frac{3}{14} : \frac{1}{3}$

:: 9 : 14 (difference = 5)

:: 900 : 1400 (difference = 500)

Ans. £900 and £1400.

The Use of Formulæ.—Many problems may be solved by the aid of formulæ, which are mere shorthand expressions for the respective rules.

If V = market value
„ M = purchase money
„ S = amount of stock
„ R = rate per cent.
and „ I = income

then (1) $I = \frac{M \times R}{V}$

(2) $S = \frac{M \times 100}{V}$ or $\frac{100 \times I}{R}$

(3) $M = \frac{S \times V}{100}$

(4) $M = \frac{I \times V}{R}$

(5) $R = \frac{I \times V}{M}$

(6) $V = \frac{M \times R}{I}$

II. To find Amount invested or realized.

1. Stock, Quotation, and Rate known.

Example.—*What sum will purchase* £4000 *in the* 4 *per cents. at* 92?

£100 stock : £4000 stock :: £92 sterling : amount required

$$\text{amount required} = £\frac{4000 \times 92}{100} = £3680 \quad \textit{Ans.}$$

By formula $M = \frac{S \times V}{100} = \frac{£4000 \times 92}{100} = £3680$

Ans. £3680.

2. Annual Income, Quotation, and Rate known.

Example.—*What was the amount paid to obtain an income of* £200 *per annum from* 4 % *stock at* 92?

An income of £4 costs £92 cash

„ „ £1 „ £$\frac{92}{4}$ „

„ „ £200 „ £$\frac{92}{4} \times 200 = £4600$ *Ans.*

3. Where Brokerage is included.

Example.—*What is the cost of* £1600 3 % *stock at* $95\frac{1}{2}$, *brokerage being* $\frac{1}{8}$?

Every £100 stock costs $(£95\frac{1}{2} + \frac{1}{8})$, or $£95\frac{5}{8}$, because the buyer pays $£95\frac{1}{2}$ for the stock, and $£\frac{1}{8}$ to the broker for his commission.

Then £100 stock : £1600 stock :: $£95\frac{5}{8}$: required cost.

$$\text{Required cost} = \frac{£1600 \times 95\frac{5}{8}}{100} = \frac{£1600 \times 765}{100 \times 8} = £1530 \quad \textit{Ans.}$$

4. When Income-tax is involved.

Example.—*A person having invested a sum of money in the* 3 *per cents. receives annually therefrom* £233, *after deducting an income-tax of* 7*d. in the* £. *What can the stock be sold for when the consols are at* $94\frac{1}{8}$? (L.M.)

(*a*) *To find amount of stock.*

Income-tax of $\frac{7}{240}$ is deducted from income, and the net income is then £233.

$\therefore$ gross income : net income :: £1 : (£1 − $£\frac{7}{240}$)

:: £1 : $£\frac{233}{240}$

:: £240 : £233

The gross income then is £240

$$\text{and amount of stock} = \frac{£240 \times 100}{3} = £8000$$

(*b*) *To find cash value.*

£100 stock sells for $£94\frac{1}{8}$

£1 „ „ $£\frac{94\frac{1}{8}}{100}$

$\therefore$ £8000 „ „ $\frac{£94\frac{1}{8} \times 8000}{100} = £7530$

Ans. £7530.

Sale and Transfer.—If A sells stock to B, A's name is taken off the register kept by the company or government for that particular stock, and B's name is put on, and this process is called a Transfer. The register for Government Stock is kept by the Bank of England. A may sell at a gain or at a loss, as the transaction is purely one between him and B, but there is no alteration of the *amount* of the stock on the register. The same amount will stand to B's name that formerly stood to A's, although B may have given more or less than A for the stock.

1. Sale, but no Transfer to New Stock.

Example.—£6300 *stock was bought at* £90 *and sold at* £137. *Find gain.*

Gain on £100 stock = £(137 − 90) = £47

„ £6300 „ = £47 × 63 = £2961

Ans. £2961.

2. Sale and Transfer to New Stock.

Example.—*I sell out* £30000 *from the* 3 *per cents. when they are at* 98, *and invest my money in a* 5 % *stock at* 105. *What amount of stock have I, and what change is there in my income?*

Income in first case = £3 × 300 = £900

$$\text{Amount of stock bought at 105} = £\frac{\overset{14}{\cancel{98}} \times \overset{2000}{\cancel{30000}}}{\underset{7}{\cancel{105}}} = £28000$$

$$\text{Income in second case} = \frac{\overset{100}{\cancel{300}}}{\underset{21}{\cancel{£30000}}} \times \frac{\overset{14}{\cancel{98}} \times \cancel{5}}{\underset{7}{\cancel{105}} \times \cancel{100}} = £1400$$

∴ change in income = £1400 − £900 = £500 gain

Ans. (1) £28000; (2) £500.

To find Rate.

1. To find Real Percentage, quotation known.

Example.—*The* 5 *per cents. are at* £120. *What is that per cent.?*

Unitary Method.

£120 cash yields £5 income

£1 „ „ $£\frac{5}{120}$ „

£100 „ „ $£\frac{\overset{5}{\cancel{5}}}{\underset{6}{\cancel{120}}} \times \cancel{100}$ income

$= \frac{25}{6} = 4\frac{1}{6}\ \%$

Proportion.

£120 : £100 :: £5 : rate

$$\text{Rate} = \frac{\overset{5}{\cancel{100}} \times 5}{\underset{6}{\cancel{120}}} = \frac{25}{6} = 4\tfrac{1}{6}\ \%$$

$4\frac{1}{6}\ \%$ *Ans.*

2. Income and Amount of Stock known.

Example.—*From* £4000 *stock an annual income of* £160 *was derived. What was the rate of interest?*

On £4000 stock £160 was paid annually
„ £100 „ $£\frac{160}{40}$ „ „
$\therefore$ rate $= \frac{160}{40} = 4\ \%$. *Ans.*

3. Value, Income, and Quotation known.

Example.—*What was the rate of interest paid on a stock quoted at* £104, *when an investment of* £2080 *cash produced an annual income of* £160?

£2080 cash produced £160 income
£1 „ „ $£\frac{160}{2080}$ „
£104 „ „ $£\frac{160 \times 104}{2080}$ „

$$\therefore \text{ rate} = \frac{\overset{8}{\cancel{160}} \times 104}{\underset{20}{\cancel{2080}}} = 8\ \%. \quad \textit{Ans.}$$

To find Price or Quotation.

Example.—*At what price was* 4 % *stock quoted when* £4600 *cash could produce an income of* £200 *a year?*

An income of £200 was produced by £4600 cash
„ £1 „ „ $£\frac{4600}{200}$ „
„ £4 „ „ $£\frac{4 \times 4600}{200}$ „

$$\therefore \text{ Price} = £\frac{4 \times \overset{23}{\cancel{4600}}}{\underset{1}{\cancel{200}}} = £92 \quad \textit{Ans.}$$

Example.—*A person invests* £5000 *in the* 3 *per cents. at a certain price. The price of the stock rises* $2\frac{1}{2}$ %, *and he sells out and purchases* 4 *per cents. at* 104, *and thus raises his income by* £25. *Find the price at which he originally bought.*

His improved capital is $\frac{£102\frac{1}{2}}{100} \times 5000 = £5125$

His *second* income is therefore—

$£\frac{104}{26} : £5125 :: \frac{1}{4} : \text{income}$

whence income $= £\frac{5125}{26}$

His *first* income then is $£\frac{5125}{26} - £25 = £\frac{5125 - 650}{26} = £\frac{4475}{26}$

then $£\frac{4475}{26}$ income : £3 :: £5000 : price required

$$\text{price required} = \frac{£3 \times \overset{200}{\cancel{5000}} \times 26}{\underset{179}{\cancel{4475}}} = £\frac{15600}{179} = £87{\cdot}15 \quad \textit{Ans.}$$

To find the Comparative Value of Different Stocks.

Example.— *Which is the best investment, the* 3 *per cents. at* 93, *the* $3\frac{1}{2}$ *per cents. at* 98, *the* 4 *per cents. at* 104, *or the* $2\frac{1}{2}$ *per cents. at* $82\frac{1}{2}$?

The interest on £1 in these 4 cases is represented respectively by the 4 fractions—

$$\frac{3}{93}, \quad \frac{3\frac{1}{2}}{98}, \quad \frac{4}{104}, \quad \frac{2\frac{1}{2}}{82\frac{1}{2}}$$

Reducing the numerator to unity, we get—

$$\frac{1}{31}, \quad \frac{1}{28}, \quad \frac{1}{26}, \quad \frac{1}{33}$$

By inspection the largest is $\frac{1}{26}$; therefore the 4 per cents. at 104 is the best investment.

If the question had required that the investments should be arranged in the order of their value, then the answer would have been—

$\frac{1}{26}$	$\frac{1}{28}$	$\frac{1}{31}$	$\frac{1}{33}$
Four per cents.	Three and a half per cents.	Three per cents.	Two and a half per cents.

THEORY.

1. Define stock, funds, consols, par. (P.T.)

2. Distinguish between stock and the money value of stock, illustrating by an example. What is meant by buying stock at a premium? Explain the term "brokerage."

3. What is meant when stock is said to be, (1) at 5 per cent. premium, (2) at 5 per cent. discount, (3) at par? (C.M.)

4. What is meant by saying that "London and North-Western stock is at $127\frac{1}{2}$"? (C.S.)

5. Translate into simple language, "$2\frac{3}{4}$ Consols to-day are quoted at $110\frac{1}{2}$." (C.M.)

6. Explain the following extract from the *Times*: "Consols, which left off last evening at $94\frac{1}{8}$ to $\frac{1}{4}$, opened at 94 to $\frac{1}{8}$, and remained without variation till the close of business." (P.T.)

7. Explain the following extract from the Stock and Share List:—

Making up prices.		Closing prices.		Rise.	Fall.
115	Cape of Good Hope $3\frac{1}{2}$ p. c. ...	M$15\frac{1}{2}$	116	$\frac{1}{2}$	
110	New S. Wales $3\frac{1}{2}$ p. c. 1924 ...	110	$110\frac{1}{2}$		
$123\frac{1}{2}$	Great Northern Pref. Ord. ...	$123\frac{1}{4}$	$123\frac{3}{4}$		
59	Do. Def. Ord. ...	$58\frac{7}{8}$	$59\frac{1}{8}$		$\frac{1}{4}$

Why are New Consols called "Goschens"? (O.J.)

8. Define shares. Say how many kinds there are, and describe each kind.

9. What are dividends? How are they usually calculated?

10. Define stock-jobber and stock-broker. What relation exists commercially between them?

11. Define bonds, debentures, coupons, warrants, and mortgages.

12. Describe the usual process, with an example, in the sale and transfer of stock.

13. What is the method of issuing stock, and how is the money usually subscribed in such cases?

14. Distinguish clearly, with examples, between par value, real value, and nominal value of stock.

15. How does interest or rate affect dividends?

EXERCISE L. 50

Stocks and Shares.

Find the value of the following stock:—

1. £200 in 3 per cents. at 105.
2. £5500 in $2\frac{3}{4}$ per cents. at 88.
3. £10000 in 5 per cents. at 125.
4. £15000 in $6\frac{1}{4}$ per cents. at 150.

How much stock can I buy with—

5. £650 in 4 per cents. at 120.
6. £2500 in 5 per cents. at 120.
7. £6600 in $4\frac{1}{2}$ per cents. at 121.
8. £7600 in $2\frac{1}{4}$ per cents. at 95.

To find Income.

9. What annual income would be derived from £750 5 per cent. debenture stock?

10. What is the annual income from £14500 in the $2\frac{3}{4}$ per cent. Consols?

11. What is the half-yearly dividend upon £20000 of $2\frac{3}{4}$ per cent. stock?

12. What would be the half-yearly dividend from an investment of £3300 in the $3\frac{1}{4}$ per cents. made when the stock was standing at 91? (K.S.)

13. A man invests £10000 in $2\frac{3}{4}$ per cent. Consols at $111\frac{3}{4}$, inclusive of commission. Find within a penny (1) the amount of Consols he will obtain; (2) the sum he will receive each quarter for interest. (C.S.L.)

14. What is the annual income derivable from a sum of £7850 invested in stock standing at $98\frac{1}{8}$, and yielding the same rate of interest at which £7864 12*s.* 6*d.* will amount in 5 yrs. to £9175 7*s.* 11*d.* (P.T.)

15. On a certain day the prices of $2\frac{1}{2}$ and $2\frac{3}{4}$ per cent. stocks are respectively $94\frac{1}{8}$ and 96. What difference will it make to an investor's annual income whether he spend £1600 in buying the former or the latter stock? (The calculation to be correct to the nearest penny, and no charge to be made for brokerage.) (C.S.L.)

16. A man has £1000 left to him. He invests half of it in railway shares of £100 each, which are at 125 and pay 4 per cent. interest on the stock, and the other half in a business which pays 4·45 per cent. on the money invested. Neglecting brokerage, find the exact income from the whole investment. (P.T.)

17. A and B each invest £1200 in the $2\frac{1}{2}$ per cents., A when they are at 90, B when they are at 96. Find the ratio of A's resulting income to B's resulting income. Find also the difference between these incomes. (P.T.)

18. A person who has £1010 stock in a railway company guaranteed to pay 5 per cent. per annum, receives for his half-yearly dividend the guaranteed interest and a further bonus of 5*s.* 3*d.* per cent. for the half-year. Find the amount of his dividend when an income-tax of 5*d.* in the £ has been deducted from it. (C.S.)

With Brokerage.

19. Calculate the total income produced by investing £8000 in equal amounts of each of the three following stocks: (*a*) $2\frac{3}{4}$ per cent. Consols at $101\frac{3}{4}$; (*b*) a 4 per cent. Railway Preference stock at $134\frac{1}{2}$; (*c*) an Ordinary Railway stock, paying $6\frac{1}{2}$ per cent. dividends, at $161\frac{1}{2}$; the expenses of purchase amounting to $\frac{1}{8}$ per cent. on (*a*), 1 per cent. on (*b*), and $1\frac{1}{8}$ per cent. on (*c*). (C.H.L.)

20. How much 3 per cent. stock must be sold out at $89\frac{1}{8}$ in order that the owner's income may be increased £12 by investing the proceeds in 4 per cent. stock at $91\frac{7}{8}$? The brokerage on each sale is $\frac{1}{8}$ per cent. (C.S.)

21. A person holding £5250 stock in the 3 per cents. and £4750 in $6\frac{1}{2}$ per cent. Bank stock sells out, the price of the former being $91\frac{7}{8}$, and of the latter $185\frac{3}{8}$. With part of the proceeds he buys ninety 8 per cent. £100 railway bonds at $142\frac{3}{8}$ (the usual brokerage, $\frac{1}{8}$ per cent., being paid in each transaction), and invests the balance in a mortgage at 4 per cent. Find the difference in his income. (R.M.A.)

22. A man calls in a sum of £10000 lent on mortgage at 4 per cent., and invests the money in £3000 London and North-Western Railway 4 per cent. debenture stock at 133 (including brokerage), and £3000 Great Western Railway 5 per cent. stock at 167 (including brokerage). The balance of the £10000 is placed on deposit at $2\frac{1}{2}$ per cent. What is the change in the income derived from the £10000? (C.PR.)

23. A person sells an estate worth £1200 per annum for 25 years' purchase, and after deducting $1\frac{1}{2}$ per cent. for expenses invests the remainder in North-Eastern Consols at 155. Allowing 2*s.* 6*d.* per cent. for brokerage, find the amount of stock he will receive. If he gets 7 per cent. on his investment, what will be the difference of his income, supposing the management of his estate to have cost him 10 per cent. of the rental? (R.M.A.)

24. I wish to transfer £48000 in 4 per cent. preference stock at 96 to 5 per cent. debentures at 105. It is calculated that the result of each sale of £16000 will be to lower the price of the stock which I am selling by 4 per cent. (*i.e.* the first £16000 will be sold at 96, the next at 92, etc.), and that each purchase of £16000 will raise the price of that which I am buying by 5 per cent. Find my change of income, assuming that the charges for brokerage amount altogether to £52.

To find Change of Income on Transfer.

25. I sell out £3200 in $2\frac{1}{2}$ per cent. stock at $108\frac{3}{4}$, and re-invest half in $2\frac{1}{4}$ per cent. stock at 87, and the other half in 4 per cent. stock at 116. By how much is my income increased? (P.T.)

26. I sell out £300 stock in 3 per cents. at $107\frac{1}{2}$, and buy £10 shares in a gas company at £10 15*s.* each. When the shares fall to £7 13*s.*, I sell out and buy 4 per cent. stock at $114\frac{3}{4}$. How much will my income be diminished? (K.S.)

27. A man invested £990 in Central Argentine railway stock at 198, and when they had fallen to 95, he purchased five £100 shares. If he waits until they rise again to 148, and then sells all his stock, how much will he have gained or lost? (O.R.)

28. A person sells out 4 per cent. stock at 126, and invests the proceeds in equal parts in $2\frac{3}{4}$ per cent. stock at 98, and 5 per cent. stock at 120. How much per cent. does he increase his income by the change? (The answer is to be correct to two places of decimals.) (L.M.)

29. A person sells his property for £27118, the rental of which is £900; he invests £4818 in 3 per cents. at $100\frac{3}{8}$, £2450 in Greek 5 per cents. at 50, and £19850 in New Zealand 4 per cents. at $99\frac{1}{4}$. Find the increase of his income. (R.M.A.)

30. The 3 per cents. are at $91\frac{1}{2}$, and the $3\frac{1}{2}$ per cents. at $96\frac{1}{2}$. A person has a sum of money to invest which will give him £100 more of the former stock than of the latter. Find the difference of the income he could obtain by investing in the two stocks. (C.PR.)

31. A man sold 150 £100 shares of railway stock which were paying 5 per cent. at 105. With the proceeds he purchased 4 per cents. at 90 and re-sold them at 96. He then re-invested in the railway stock, which was still at 105 and paying 5 per cent. What was the change in his income? (R.M.A.)

32. If a person sells for £18 each, fifty £10 shares in a company which pays a dividend of 8 per cent., and then invests the proceeds of this sale in the purchase of £5 shares in another company at £3 10*s.* each, find what difference is made in his income if the second company pays a dividend of $3\frac{1}{2}$ per cent. (K.S.)

33. What is the price of 4 per cent. stock if an investment in it is as profitable as one in $3\frac{1}{2}$ per cent. stock at 98? Find the loss of income when £5200 of the above 4 per cent. stock is sold at the ascertained price, and the money re-invested in $2\frac{3}{4}$ per cent. stock at 96. (Neglect brokerage.) (C.W.)

34. (Neglect brokerage.) The 3 per cents. being at par, and the $2\frac{1}{2}$ per cents. at 90, a man holding £1050 stock in the 3 per cents. and £1500 stock in the $2\frac{1}{2}$ per cents., transfers his capital from these securities to a company paying 4 per cent. per annum on shares of £50 each, now quoted at 60. What is his annual income, and by how much does it differ from his previous income? (C.W.)

To find the Money invested.

35. A person has equal sums invested in three undertakings, which yield respectively 3 per cent., 4 per cent., and 5 per cent., and his total income is £240. What is his capital? (K.S.)

36. If by delaying an investment in $2\frac{1}{2}$ per cent. stock while the price falls from 90 to $89\frac{3}{8}$, the annual income to be received is increased by £6 5*s.*, what is the sum invested? (C.S.)

37. Money is invested in $3\frac{1}{2}$ per cents. at 98, and 4 per cents. at 108. If one of these investments gives £50 a year more interest than the other on the same principal, what is the capital invested? (C.S.)

38. A and B have equal sums of money. A invests his equally in the purchase of $2\frac{3}{4}$ per cents. at 110, and $2\frac{1}{2}$ per cents. at $106\frac{1}{4}$; B in the purchase of equal amounts of the same stocks. The difference in their incomes is 3*s.* What amount of money did each possess? (P.)

39. A, who has invested £3450 in 4 per cent. stock at 115, finds that he has £15 less income than B, who has invested a certain sum in 3 per cent. stock at par. How much did B invest? (C.M.)

40. A man invests a certain sum of money in the 4 per cents. at 89. At the end of a year he takes his dividend, sells out at 81, and invests the proceeds of the sale as well as the dividend in a $4\frac{1}{2}$ per cent. stock at 102. He finds that his income is £12 10*s.* a year less than it would have been had he left his money as originally invested. What was the original sum invested? (O.J.)

41. A man buys railway debenture shares, paying $4\frac{1}{2}$ per cent., and he buys at such a price that he gets $3\frac{1}{3}$ per cent. on his money. At what price does he buy? What sum must be invested in $2\frac{3}{4}$ per cent. Consols at $96\frac{1}{4}$ in order to derive an income of £166 13*s.* 4*d.*? (Neglect brokerage.) (K.S.)

42. I invest a certain sum, half in a 4 per cent. stock at par, and half in a 5 per cent. stock at 120. If I had invested one quarter of the sum in the first stock, and the remainder in the second stock, I should have had £5 more annual income than in the former case. What sum had I to invest? (C.S.L.)

43. A man invested £5173 11*s.* 3*d.* partly in Consols at $101\frac{1}{8}$, and the remainder in railway stock at $146\frac{1}{2}$. When the Consols were at $102\frac{1}{4}$, and the railway stock at $143\frac{1}{8}$, he sold out, and received altogether the same sum he had originally invested. How much money did he invest in Consols? (Neglect brokerage.) (C.S.L.)

44. A man invests a sum of money in a stock at 91, and an equal sum in another stock at 117. After some time the prices of the stocks have exchanged values, when the man sells, and finds that he has made a profit of £52. What sum did he invest? (C.W.)

With Brokerage.

45. What sum of money must I invest in the Indian $3\frac{1}{2}$ per cents. at 109 to produce sufficient income for an annual prize of 1 guin., the cost of brokerage on amounts under £50 being 1*s.*? (C.P.)

46. Having invested a sum of money in 3 per cent. Consols at $100\frac{3}{8}$, I held the stock until I had received a half-year's dividend. I then sold it at $99\frac{5}{8}$, and lent the proceeds of the sale, together with the dividend, at $3\frac{3}{4}$ per cent. per annum. If brokerage amounting to $\frac{1}{8}$ per cent. be charged upon sales and upon purchases, what sum did I originally invest, my income being increased by £9 19*s.* 6*d.*? (C.S.)

To find the Amount of Stock held, purchased, or sold.

47. My income is derived from £10000 stock in the $2\frac{1}{2}$ per cents., which are at 80. How much must I sell out in order that, after re-investment of the proceeds in 4 per cents. at 120, my income may be £12 10*s.* greater than formerly? (C.S.)

48. A speculator bought Spanish bonds (yielding no interest) at 18, 18 Honduras £100 bonds paying 10 per cent. interest at 35, and 17 Turkish £100 bonds paying 5 per cent. at 45. At the end of two years he sells the Honduras stock at 15, and the following year he sold the Spanish at 25, and the Turkish at 46. Including the interest, he gained £1146. What number of Spanish bonds did he buy? (C.S.)

49. A man invested money in the 3 per cents. at $101\frac{3}{8}$, and on their falling to $100\frac{3}{8}$ he sold out, and thereby lost £24. How much stock did he hold, and how much did he pay for it? (P.T.)

50. A man holding $3\frac{1}{2}$ per cent. stock sells out when the stock is at 90, and invests the proceeds in 4 per cent. stock at 85. If his annual income is thereby increased by £25, how much of the first stock did the man hold, no account being taken of brokerage? (O.J.)

51. A person has a certain amount of 4 per cent. stock. He sells it at $117\frac{1}{3}$, and invests half the proceeds in 2 per cent. stock at 96, and the rest in 3 per cent. stock at 99. He then finds that his annual income is reduced by £20. What amount of the original stock had he? (L.M.)

52. How much stock in the $2\frac{3}{4}$ per cent. Consols does a man hold if, by selling out at 99, and investing the proceeds in 4 per cent. railway stock at 132, he increases his income by £12 a year? (No charge for brokerage.) (C.W.)

53. By a reduction of the interest on Exchequer Bills from $2\frac{1}{2}d.$ to $1\frac{1}{2}d.$ per cent. per day, a person loses at the rate of £152 7*s.* 9*d.* per annum. What amount of Exchequer Bills does he hold? (C.S.)

54. A man owns £840 stock in 3 per cent. stock, and £675 in $2\frac{1}{2}$ per cent. stock. He is obliged to realize £1002 sterling when the former stands at 105 and the latter at 80. How much of each stock must he sell so as to have the smallest possible loss of income? (C.M.)

55. A trustee invests £1465 12*s.* 6*d.* in the purchase of $2\frac{3}{4}$ per cent. Consols at $112\frac{1}{2}$ (the cost of brokerage and other expenses of the purchase being paid separately). How much stock does he buy, and what will be the quarterly income after deducting income-tax at 8*d.* in the £? (C.H.L.)

56. Investing a certain sum in a railway paying £6 per £100 share, at a time when each share cost £125, I find my income £10 13*s.* 6*d.* better than when I invested the same amount in 3 per cent. Consols at 95. Determine the amount. (C.P.)

With Brokerage.

57. A broker is employed to invest a sum of money in the funds, and receives as brokerage £45, being $\frac{1}{8}$ per cent. on the purchase-money, and $\frac{1}{10}$ per cent. on the nominal value of the stock. What was the amount of stock purchased, and at what price per cent. was it standing? (C.W.)

58. A sells some stock, and B buys it, each paying the broker $\frac{1}{8}$ per cent. If the broker makes £8 8*s.* 3*d.* by the transactions, what is the amount of stock dealt with? (C.S.)

59. A bankrupt's assets consist of (1) some stock saleable at $92\frac{1}{2}$ (brokerage $\frac{1}{8}$ per cent.); (2) a bill for £2700 due 6 mths. hence, which can be discounted at the rate of $2\frac{1}{2}$ per cent. per annum; (3) £65 4*s.* 2*d.* in cash. If his debts are £80000, and he can pay 13*s.* 6*d.* in the £, determine the amount of stock he holds. (O.J.)

60. A person sold a certain quantity of 3 per cent. stock at 95. He invested half the proceeds in 5 per cent. railway bonds at 106, and the other half on a mortgage at $4\frac{1}{2}$ per cent. The brokerage for transfer of the stock amounted to £2 9*s.* $8\frac{1}{2}$*d.*, and the cost of drafting the mortgage deed £15. After these charges had been paid out of the first year's income, there remained 15*s.* 6*d.* more than the previous income from the 3 per cents. How much of this stock was sold? (C.S.)

To find the Price of Stock.

61. A railway stock pays a dividend of $6\frac{3}{4}$ per cent. What price should a person pay for £100 of the stock that he may have 4 per cent. interest on his money? (L.M.)

62. A man buys £4575 three per cent. stock at 75, and pays for it with borrowed money, for which he gives 8 per cent. per annum. At the end of a year he sells out, and after repaying his loan has £30 10*s.* At what price was the stock sold? (C.H.L.)

63. A man sells £6000 $2\frac{1}{2}$ per cent. Consols, and invests the proceeds in railway stock at 165. The railway pays a dividend of $5\frac{1}{2}$ per cent., and the man finds that he has added £50 to his income by the transaction. At what price did he sell? (C.M.)

64. A stock yields 5 per cent. to an investor; a fall of 5 in its price causes it to yield $5\frac{1}{3}$ per cent. What was its price? (S.A.)

65. A person wishes to sell out £10000 three per cent. stock. On December 1 the stock is at 85, but by waiting till December 17 he is able to clear £1100 more than if he had sold out on the former day. Required the price of the stock on December 17. (O.R.)

66. A person buys railway shares at £18 when £15 is paid up, and sells when £24 is paid up, gaining $11\frac{1}{9}$ per cent. on his invested money. Find the price at which he sold out. (I.I.)

67. A man invests 1200 guins. in 3 per cent. stock at $94\frac{1}{2}$. After receiving the first half-year's dividend he sold out. If the whole increase of his capital, including the dividend, amounted to £65, what was the price of stock when he sold out? (K.S.)

68. A man sells £1000 new 3 per cents. at 95, and buys shares and gains £10 per annum. If the dividend on the shares is at the rate of 8 per cent. per annum, what is the price of each £100 share? (I.I.)

69. A person buys 6 per cent. bonds, the interest on which is payable yearly, and which are to be paid off at par three years after the time of purchase. If money be worth 5 per cent., what price should he give for the bonds? (C.PR.)

70. A man possesses £10000 of $3\frac{1}{2}$ per cent. stock, and £7500 of $4\frac{1}{2}$ per cent. stock. He sells out when the stocks are at 126 and 168 respectively, and invests the proceeds in $2\frac{3}{4}$ per cent. Consols. What price must he pay in order that his total income may be the same as before? (L.M.)

With Brokerage.

71. I pay income-tax at the rate of 6*d.* in the £ on the income arising from 4 per cent. stock. When the income-tax is raised to 8*d.* I sell out at $146\frac{3}{8}$, and buy $3\frac{1}{2}$ per cent. stock. At what price must I buy this stock if my net income remains the same as before? (Brokerage $\frac{1}{8}$ per cent.) (O.J.)

72. A man bought $2\frac{3}{4}$ per cent. Consols at $97\frac{3}{8}$, and sold them 2 months afterwards. If his profit (including a quarterly dividend) amounted to 5 per cent. per annum on the sum invested, at what price did he sell the stock? (Brokerage $\frac{1}{8}$ per cent.) (C.P.)

73. If the 3 per cent. Consols are at $81\frac{7}{8}$, what must be the price of the 5 per cents. that there may be no loss of income in selling out the former and re-investing in the latter, allowing for the usual brokerage of $\frac{1}{8}$ per cent. of stock on each transaction? (C.S.)

74. By selling out of the 3 per cents. when they are at $91\frac{1}{8}$, and then buying $3\frac{1}{4}$ per cent. stock, I find that I obtain £1 0*s.* $1\frac{3}{7}$*d.* of income for every £1 which I previously received. Find the price of the $3\frac{1}{4}$ per cent. stock, $\frac{1}{8}$ per cent. brokerage being charged in each transaction. (C.S.)

To find Rate per Cent.

75. How much stock at $97\frac{1}{2}$ must I sell out in order to realize £1300? And if, by investing this £1300 in $4\frac{1}{4}$ per cent. stock at 102, I increase my income by £8 6*s.* 8*d.* per annum, what rate of interest was I receiving on the former stock? (R.M.A.)

76. One-half of a certain sum of money is invested so as to return 5 per cent., one-third so as to return 6 per cent., and the remainder so as to return 8 per cent. interest. What is the average rate of interest on the whole? (K.S.)

77. A man invests the following sums of money, viz. £2500 in 3 per cent. stock at $95\frac{1}{2}$, £8000 in 4 per cent. stock at par, and £6000 in 5 per cent. stock at $92\frac{1}{2}$. What rate of interest does he receive on his whole investment? (C.S.L.)

78. A man buys a certain stock at $112\frac{1}{2}$. For the first half-year the dividend was at the rate of 4 per cent. per annum per £100 stock, and for

the next half-year at the rate of 3 per cent. per annum. What is the total interest per cent. per annum on his outlay? (Work out to the nearest penny.) (L.M.)

79. If I invest £4925 in the 3 per cents. at $98\frac{1}{2}$, and at the end of 6 mths., having received a half-year's dividend, sell out at $87\frac{3}{4}$, and invest the proceeds of the sale for another 6 mths. at 5 per cent., what income shall I have for the year, and what rate per cent. will it be on my original investment? (C.S.)

80. A man invested a sum of money in the 3 per cent. Consols at 88, and at the end of $4\frac{1}{2}$ mths., after receiving one half-year's dividend, sold out at $87\frac{5}{8}$. At what rate per cent. per annum did he receive interest on his capital? (C.S.L.)

81. A man has stock in the 3 per cents. which brings him £240 a year. He sells out $\frac{1}{4}$ of the stock at $87\frac{1}{2}$, and invests the proceeds in railway stock at $174\frac{1}{2}$. What dividend per cent. per annum ought the railway stock to pay, so that he may increase his income £40 per annum by the transaction? (C.S.L.)

82. A invests £4625 in railway stock at 125, B invests £4920 in another at 120. A receives an annual dividend at the rate of $4\frac{1}{4}$ per cent., and receives £3 10*s.* more than B. At what rate was the annual dividend paid to B calculated? (C.W.)

83. A person has a sum of money invested in Consols (3 per cent.). He sells out at $87\frac{1}{2}$, and invests the proceeds in railway shares, when £100 sells for $£174\frac{1}{2}$. He thus increases his income from £120 to £200. What dividend per cent. does the railway pay? (R.M.A.)

84. A man, having £4400 to invest, buys £1000 stock in the 3 per cents. at 81, £1500 stock in the $3\frac{1}{2}$ per cents. at 90, and invests the remainder of his capital in 4 per cents. at 112. What interest does he get per cent. on his whole investment? (C.S.)

With Brokerage.

85. If I buy railway stock, paying a dividend of $6\frac{1}{2}$ per cent., at $129\frac{7}{8}$, what rate per cent. do I get for my money, the usual brokerage being charged? (P.T.)

86. £500 $4\frac{1}{2}$ per cent. preference stock is bought at $111\frac{1}{2}$, the stamp and fee cost £2 10*s.*, and commission is charged at the rate of $\frac{1}{2}$ per cent. (per £100 stock). What is the actual interest produced by the money invested? (C.M.)

87. An investment of £7440 is made in the 3 per cent. Consols at $92\frac{7}{8}$ 2 mths. before the half-yearly dividend is paid. Immediately after the receipt of this dividend the stock is sold out at 92. If a brokerage of $\frac{1}{8}$ per cent. is paid both on the purchase and sale, find the rate of interest per cent. per annum obtained by the investor on this transaction. (C.S.L.)

88. A sum of money is divided into two parts in the ratio of 37 to 35; the first part is invested in $2\frac{3}{4}$ per cent. stock at $101\frac{5}{8}$, the second in a 4 per cent. stock at $109\frac{1}{8}$. Brokerage being $\frac{1}{8}$ per cent., find the average rate of interest obtained. (S.A.)

To find Comparative Value.

89. Which is the better investment, the $3\frac{1}{2}$ per cents. at 91, or the 4 per cents. at 112; and what is the difference per cent. between them? (C.S.)

90. Which of the following stocks is most profitable for investment: the 3 per cents. at $91\frac{1}{2}$, the $3\frac{1}{2}$ per cents. at $108\frac{1}{2}$, the 4 per cents. at 118? (K.S.)

91. Compare the relative advantages to the purchaser of the following investments: A gas company guarantees 5 per cent. on shares of £100 each; a railway company offers $4\frac{5}{6}$ per cent. on shares of £7 10*s.* each. The price of the former is £124 10*s.*, and of the latter £8 10*s.* (P.T.)

92. Which of the following stocks is most profitable for investment: the 3 per cents. at $91\frac{1}{2}$, the $3\frac{1}{2}$ per cents. at $108\frac{1}{2}$, the 4 per cents. at 118? Find the yearly income produced by investing £5217 16*s.* 3*d.* in the most advantageous of the three. (K.S.)

93. Compare the incomes derivable from the two investments that follow: A invests £1491 15*s.* in $3\frac{1}{4}$ per cents. at a discount of $12\frac{1}{4}$ per cent.; B invests the same sum in $4\frac{3}{4}$ per cent. railway stock at a premium of $14\frac{3}{4}$ per cent. (C.W.)

94. What proportion does the income obtained by investing in the 3 per cents. at $82\frac{1}{2}$ bear to that obtained by investing in the $3\frac{1}{2}$ per cents. at $93\frac{1}{2}$? and what will be the difference of income if the sum invested be £30000? (P.T.)

95. A person is considering two investments—a stock paying $3\frac{1}{2}$ per cent. at 88, and another paying 4 per cent. at $96\frac{2}{3}$. By how much per cent. is one better than the other? (P.T.)

96. If the 3 per cents. are at $87\frac{1}{8}$, and railway debentures paying $4\frac{1}{2}$ per cent. are at a premium of 25 per cent., by which investment can the larger income be secured? And by how much will it exceed the smaller if the sum invested be £2613 15*s.*? (C.W.)

97. A person sells out of the 3 per cents. and realizes £18000. Would it be more to his advantage to re-invest this money in the $3\frac{1}{2}$ per cents. at $101\frac{7}{8}$, or in a mining company paying a dividend of $1\frac{3}{4}$ per cent. per annum, of which the £10 shares were quoted at $5\frac{3}{8}$? No brokerage is to be charged. N.B.—Show all your calculations. (C.S.)

98. When Spanish 4 per cents. are at $63\frac{1}{16}$, Portuguese 3 per cents. at $52\frac{1}{4}$, and Italian 5 per cents. at $99\frac{1}{2}$, which offers the best investment, supposing all to be equally secure? If I invest £1000 in each of the three stocks, what annual dividend shall I receive? (C.W.)

Miscellaneous Problems.

99. A man invests £10000 in the $3\frac{1}{2}$ per cents. at 105, receiving the interest yearly, and giving his son whatever interest there is over and above £300. The son invests this gift *each year*, immediately, at $2\frac{1}{2}$ per cent. How much has the son acquired on receiving the third payment from his father? (N.B.—The son has invested *twice*.) (K.S.)

100. I invest two sums of money, one at 3 per cent. and the other at 5 per cent. compound interest. At the end of three years they together amount to £7715, and the sum at 3 per cent. is double that at 5 per cent. What, to the nearest shilling, were the amounts invested? (C.S.)

101. An investor orders some stock quoted at 95. Before the order can be executed the stock rises to 96. Find the ratio of the rate of interest which the investor actually receives to that which he hoped to obtain. (You may assume a rate of interest on the stock, but it is unnecessary to do this.) (K.S.)

102. Divide £5300 into two parts, such that if one part is invested in the $3\frac{1}{2}$ per cents. at 98, and the other in the 4 per cents. at par, the resulting incomes are equal. (Neglect brokerage.) (C.M.)

103. A man invests £9600 in the 4 per cents. at 80, and at the end of each year invests the dividend which becomes due in the same stock. Find his dividend at the end of the third year, the funds remaining all the time at 80. (P.T.)

104. One company pays $6\frac{1}{2}$ per cent. on shares of £100 each; another pays at the rate of $3\frac{1}{2}$ per cent. on shares of £20 each. If the price of the former be £137 10*s.*, and of the latter £18 5*s.*, compare the rates of interest which the shares return to purchasers. (C.M.)

105. A person who has £1515 stock in a railway company guaranteed to pay 5 per cent. per annum, receives for his half-year's dividend the guaranteed interest and a further bonus of 5*s.* 3*d.* per cent. for the half-year. Find the amount of his dividend when an income-tax of 5*d.* in the £ has been deducted from it. (C.W.)

106. A person sells out £3162 10*s.* in 3 per cent. stock at $97\frac{5}{8}$, and buys in bank stock at $287\frac{3}{8}$, paying his broker 2*s.* 6*d.* on each £100 of stock sold or bought. He receives on his investment an April dividend of 4 per cent., and another in October of 5 per cent. How far is his income affected by the exchange? (C.W.)

107. A speculator invested in bank stock at 285, and sold out at $289\frac{1}{4}$; he re-invested in French Rentes at $81\frac{3}{4}$, and sold out at $82\frac{1}{2}$; then in Sheffield railway stock at 55, and sold at $54\frac{1}{2}$; then in Spanish 2 per cents. at $45\frac{1}{2}$, and sold at $42\frac{3}{4}$; then he bought Consols at $100\frac{1}{8}$, and sold at 105. Did he gain or lose on the transactions? (C.M.)

108. A person buys five shares in a concern at 40; next year the price is 45, and every succeeding year there is a fall of 4. Each year he sells out one share, and finds at the end he had neither gained nor lost. What interest did the concern pay? (C.M.)

109. A man invests £1000 in 3 per cent. stock at the price of 105. At the end of the first year he invests the year's dividend in the same stock at 107; at the end of the second year he invests the whole dividend for that year in the same stock at 108. What is his dividend for the third year to the nearest penny? (C.H.L.)

110. An investor placed a sum of money in a stock A, and a sum £1000 greater in a stock B, getting £30 a year more from the second than from the first. Had he invested the whole in A, he would have obtained £5 a year more than if he had invested it in B. Find what income would arise from an investment of £2000, half in A and half in B. (R.M.A.)

111. A man invests £16213, partly in shares at 105 bearing a dividend (free from income-tax) of 8 per cent., and partly in a mortgage at 5 per

cent. interest, on which he pays income-tax at 8*d.* in the £. His net income from each investment is the same. What is his whole income? (C.H.L.)

112. A man invests one-third of his capital in 2½ per cent. stock at 96, one quarter in 4 per cent. stock at 110, and the remainder in 5 per cent. stock at 120. Find, correct to the nearest farthing, the average rate per cent. returned on the capital. (L.M.)

113. If the 3 per cents. are at 95, and Government offer to receive a loan of £5000000, the lender to receive five millions in the 3 per cents., together with a certain sum in the 3¼ per cent. stock, what sum in the 3¼ per cents. ought the lender to accept? (C.S.)

114. How much must I invest in the 3 per cent. stock at 90, in order that when I sell £1000 stock at 93½, and the remainder at 84¼, I may gain £6 5*s.* And if I invest the proceeds in the 4 per cents. at par, find the difference in my income. (C.S.)

115. The half-yearly dividends on £18600 3 per cent. stock are invested as they become due in the same fund in augmentation of the stock at 96, 94½, and 93 respectively. By how much will the dividend for the fourth half-year exceed that for the first? (C.S.)

116. I invest in 2¾ per cent. stock at 110. After 2 yrs. the stock is converted into 2½ per cent. stock; after 2 yrs. more I sell out at par. Show that if brokerage of ⅛ per cent. be paid on investing and selling out, my capital, with accumulated simple interest, has increased only by about 4*s.* 6½*d.* per £100. (C.S.)

117. A certain income can be obtained by an investment in the 3¼ per cents. at 93⅜. The same income can also be obtained by investing £1595 less in the 4¼ per cents. at 105⅝, no brokerage being charged on either transaction. What is the income? (C.S.L.)

118. A person, holding £4205 2½ per cent. stock, calculates that by selling at 84 and investing the proceeds in 5 per cent. debentures at the current price, he will improve his income by £63 1*s.* 6*d.* What is the price of the debentures? He has omitted to observe that he will have to pay ⅛ per cent. (calculated on the par value) as brokerage on each transaction. Having this to pay, what actual improvement in income will he obtain? (C.S.)

119. An investment in 3 per cent. Consols, standing at 102, is partially converted into 2½ per cent. Consols at 94. The part which is converted is half again as large as the part which remains in the 3 per cents., and the loss in net income is £155, after the deduction in each case of an income-tax at 6*d.* in the £. Write down an arithmetical expression for the amount of the original investment, but do not evaluate it. (C.S.)

120. Given that 2¾ per cent. stock is at 94¾, and 2½ per cent. stock at 92½, determine (1) what annual interest will be derived from £1423 2*s.* 6*d.* invested in 2¾ per cent. stock, and (2) what fall in price of the latter stock (the price of the former stock remaining unaltered) will make the rates of interest yielded to the investor the same in both cases. (Brokerage ⅛ per cent.) (C.P.)

121. At the beginning of a year a man invests £9000 in 3 per cent.

stock at 117, and £7200 in $2\frac{3}{4}$ per cent. stock at 108. At the beginning of the next year, when both stocks are at 111, he sells them. Assuming him to have received one year's interest on each year's stock, and to have paid a brokerage of 5*s.* per £100 of stock on the latter transaction, but not on the former, find how much he has gained on the whole.

(O. and C.H.C.)

122. A friend lent me £425, raising the money by selling out $2\frac{3}{4}$ per cent. Consols at 91. I repaid the money at the end of 6 mths., but meanwhile Consols had risen to $96\frac{1}{8}$. How much should I pay to my friend to cover the loss of two quarterly dividends on the Consols sold, and to enable him to buy back the stock he previously held? (O. and C.S.)

123. A man invests £23610 in the purchase of 5 per cent. stock at $98\frac{3}{8}$. After a time this stock is converted to $3\frac{1}{2}$ per cent., whereupon a bonus of £9 on every £100 stock is returned to him, and this bonus he invests in 3 per cent. stock at 54. What is his loss of income?

(O. and C.H.C.)

124. A man has £12000 to invest. He invests $\frac{1}{3}$ of this sum in the 3 per cent. Consols at 96, $\frac{7}{24}$ in railway debenture 4 per cent. stock at 105, and the remainder in French 3 per cent. Rentes at $67\frac{1}{2}$. Find his whole income, and the average rate of interest he makes on his investments. Find also the increase in his capital, when each stock has risen 1 per cent. (C.H.L.)

125. A gentleman, deriving all his income from a railway investment paying 4 per cent. on his capital, spends $\frac{2}{3}$ of his gross income on personal expenses, and pays an income-tax of 6*d.* in the £. Next year the railway dividends are $4\frac{1}{4}$ per cent., his personal expenses are 10 per cent. more, and the income-tax is reduced to 4*d.* in the £, and he finds that he is £5 richer. What was his original income? (C.S.L.)

126. By buying 3 per cent. Consols at a certain price, I find I obtain $3\frac{3}{7}$ per cent. for my money, and derive a net income therefrom, after paying an income-tax of 6*d.* in the £, of £421 4*s.* Find the amount of stock and the price at which I bought it. (R.M.A.)

127. A capitalist has 15 preference shares of £10 each, £800 railway debenture stock, and ordinary railway stock, which he purchased for £1237 10*s.* when it was $82\frac{1}{2}$. What capital sum does he realize by selling all out, the market prices being—preference shares, 12; debenture stock at 105, and ordinary at $84\frac{1}{4}$? (I.I.)

128. The accrued interest on November 30, due on £35000 6 per cent. mortgage debentures, the interest on which is paid half-yearly, is £166 10*s.* On what day was the previous interest paid? (S.A.)

129. The owner of some $3\frac{1}{4}$ per cent. stock spends in a certain year 75 per cent. of his net income after an income-tax of 6*d.* in the £ has been deducted from the interest due to him. The next year the income-tax is reduced to 3*d.*, but, the rate of interest being also reduced to 3 per cent., he spends only $68\frac{3}{4}$ per cent. of his net income. He then finds his surplus to be £8 11*s.* more than in the previous year. How much stock did he possess? (C.S.)

130. A invests $\frac{1}{4}$ of his money in the $2\frac{1}{2}$ per cents. at 90; the other $\frac{3}{4}$ he lends to B at 4 per cent. simple interest. B pays the interest duly for

4 yrs.; at the end of another 4 yrs. he becomes bankrupt, and pays a dividend of 13*s.* 4*d.* in the £ to A on the principal and interest owing. At the same time A sells out his stock at 80, and then finds that the whole sum received during the 8 yrs. as principal and interest is £10 less than that with which he began. What was his original amount? (C.S.)

131. A person sold a certain quantity of 3 per cent. stock at 95. He invested half the proceeds in 5 per cent. railway bonds at 106, and the other half on a mortgage at $4\frac{1}{2}$ per cent. The brokerage for transfer of the stock amounted to £2 9*s.* $8\frac{1}{4}$*d.*, and the cost of drafting and executing the mortgage deed was £15. After these charges had been paid out of the first year's income, there remained 15*s.* 6*d.* more than the previous income from the 3 per cent. stock. How much of this stock was sold? (C.S.)

132. A person possessed of £2500 invested £348 13*s.* 9*d.* in 3 per cent. stock at $99\frac{1}{2}$, £1651 17*s.* 6*d.* in foreign 6 per cent. securities at 110, and the remainder on loan at 5 per cent. At the end of a year he sold the stock at 97, the securities at $102\frac{1}{4}$, and recovered only 75 per cent. of the loan. He paid $\frac{1}{8}$ per cent. on both purchase and sale of the stock and securities, and received the year's dividends on them as well as the interest on the loan. How much was he possessed of at the end of the year? (C.S.)

133. A person had £6900 stock paying 3 per cent. in a company, and was offered the choice of being paid off at par, or of receiving £110 of a new $2\frac{1}{2}$ per cent. stock for every £100. He chose the former alternative, and invested his money in the 3 per cent. Consols at 92. Find the amount of his stock in Consols, and the excess of his income above what it would have been if he had agreed to the proposed conversion. (C.S.)

134. A man has stock in the $3\frac{1}{2}$ per cents. which brings him in £43 15*s.* a year. He sells out $\frac{1}{3}$ of the whole at $112\frac{7}{8}$, and invests the proceeds in railway debentures at $128\frac{3}{4}$. How much per cent. must the dividend of the debentures be in order that his annual income may be diminished by 10*s.* $11\frac{1}{2}$*d.*, no brokerage being charged for the sale of the stock, but $\frac{1}{4}$ per cent. being charged for the purchase of the debentures? (C.S.)

135. A man invests £2500 in a business paying 25 per cent. profit; at the end of each year he invests his profits in railway shares at par paying 6 per cent. The business fails after the end of the sixth year, and he loses the whole of his original capital. By how much greater is his present capital than it would have been if he had invested the whole in railway shares at first, calculating simple interest? (K.S.)

136. I wish to transfer £50000 in 4 per cent. preference stock at 105 to 5 per cent. debentures at 131. It is calculated that the result of each sale of £5000 will be to lower the price of the stock which I am selling by $\frac{1}{16}$ per cent. (*i.e.* the first £5000 will be sold at 105, the next at $104\frac{15}{16}$, etc.), and that each purchase of £5000 will raise the price of that which I am buying by $\frac{1}{8}$ per cent. Find my change of income to the nearest penny, assuming that the charges for brokerage amount altogether to £62. (C.S.)

137. A man invests a sum of money in 3 per cent. stock at $87\frac{1}{4}$, and at the end of 3 mths. sells out at 90, having in the interval received a

half-year's dividend. He then spends 20 guins. in taking a holiday, and finds that the sum remaining for re-investment (including the dividend) is to the sum originally invested in the ratio of 25 to 24. If the brokerage was 2*s.* 6*d.* per £100 on both buying and selling the stock, and if the income-tax deducted from the dividend was at the rate of 5*d.* in the £, how much was originally invested? (C.S.)

138. A man buys £5000 railway stock on May 1 at $155\frac{3}{4}$, and, after receiving a half-year's dividend at the rate of $5\frac{3}{4}$ per cent. per annum, sells out in August at $155\frac{1}{8}$. What profit has he made? If he at once invests the proceeds of the sale, together with the dividend received, in $2\frac{1}{4}$ per cent. Consols at par, and sells out at $99\frac{5}{8}$ on November 1, after having received a quarter's dividend in October, calculate the total profit on the whole transaction, and the rate per cent. per annum returned on the original outlay. (All prices given are nett.) (C.H.L.)

139. A man invests £3600 in 3 per cent. stock at 90. He sells out at 80, and lends $\frac{2}{3}$ of his money at 4 per cent. and $\frac{1}{3}$ at 5 per cent. How long must the loan last, so that when he re-invests his money in 3 per cents. at 90, his gain on interest (simple) may exactly equal his loss upon principal? (S.K.)

140. A man invested £2000 in a 4 per cent. guaranteed stock when the price of £100 of stock was $£97\frac{5}{8}$ in money. He sells out when the price of £100 of stock has risen to £154, and invests the proceeds in a $2\frac{1}{2}$ per cent. stock, the price of £100 of which is £98. Neglecting the commissions, find whether his income is increased or diminished, and to the nearest penny by how much. (S.K.)

CHAPTER XXXIV.

EQUATION OF PAYMENTS.

Equated Time is that time when several debts, due at different times, may be equitably discharged by one payment.

As a result some debts would be paid before they were due, and some after. To ensure accuracy, interest ought to be added to the latter, and true discount subtracted from the former. But in practice, banker's discount is allowed, and not true discount, and this influences the rule for these problems.

The **Term of Credit** is the period of time before a bill falls due.

The **Average Term of Credit** is the equated time when banker's discount is paid.

The true average term of credit is the equated time when *true* discount is calculated, and is only theoretically interesting. Practically, since banker's discount is always allowed, the average term of credit is the term required, and not the *true* average.

To find Equated Time or the Average Term of Credit.

Example.—*If* £100 *be due in* 4 *mths.*, £200 *in* 15 *mths.*, £300 *in* 12 *mths., and* £400 *in* 9 *mths., what is the equated time for paying the whole?*

The S.I. on £100 for 4 mths. = S.I. on £400 for 1 mth.
and S.I. ,, £200 ,, 15 ,, = S.I. ,, £3000 ,,
,, S.I. ,, £300 ,, 12 ,, = S.I. ,, £3600 ,,
,, S.I. ,, £400 ,, 9 ,, = S.I. ,, £3600 ,,

If the sum of £1000 has to be paid in one amount, the problem is to find how much time should be allowed so that the S.I. on £1000 for that time may be equal to the S.I. on £(400 + 3000 + 3600 + 3600), or £10600 for 1 mth. Hence the following proportion :—

£1000 : £10600 :: 1 mth. : time required

$$\text{Time required} = \frac{10600 \times 1}{1000} = 10\tfrac{3}{5} \text{ mths.} \quad \textit{Ans.}$$

Reason for the Process.—This rests on the following assumption :—

The sum of the interests of the several debts for their respective times } = { the interest of the sum of the debts for the equated time.

Rule.—

1. Multiply each debt by its term of credit.

All the debts must be in the same unit, and all the terms of credit in the same unit.

2. Add these products.
3. Divide the sum of these products by the sum of the debts.
4. The quotient is the equated time.

Note.—1. This is the method used in business.
2. It is only approximately correct.
3. To be equitable the various times of payment must not be widely apart.
4. The method is in favour of the *payer*, and *not* of the *payee*, for—

Banker's discount (instead of true discount) is deducted from the payments before they fall due, and banker's discount is greater than true discount.

To find True Equated Time or the True Average Term of Credit.

Example.—*Find the true equated time of* £410 *due* 6 *mths. hence, and* £810 *due* 3 *mths. hence at* 4 %.

Take the year as a unit.

Then 6 mths. at 5 % = 12 mths. at $2\frac{1}{2}$ %
and 3 mths. at 5 % = 12 mths. at $1\frac{1}{4}$ %

Now, £$102\frac{1}{2}$: £410 :: £100 : P.V. of £410

$$\text{P.V.} = £\frac{410 \times 100}{102\frac{1}{2}} = £400$$

and £$101\frac{1}{4}$: £810 :: £100 : P.V. of £810

$$\text{P.V.} = £\frac{810 \times 100 \times 4}{405} = £800$$

Sum of face values = £410 + £810 = £1220
Sum of present values = £400 + £800 = £1200
T.D. on £1220 = £1220 − £1200 = £20

Required to find in what time £1200 will amount to £1220 at 4 % per annum.

$$\text{T} = \frac{100 \times \text{I}}{\text{P} \times \text{R}} = \frac{100 \times 20}{1200 \times 4} = \frac{5}{12}$$

Ans. $\frac{5}{12}$ yr.

Rule.—(1) Find the P.V. of each debt, and add these values.
(2) Find the sum of the face values of the debts.
(3) Then find the time necessary for the sum of the present values to amount to the sum of the debts.

THEORY.

1. Define equated time, term of credit, average term of credit, and true average term of credit.

2. State and explain the rule for equation of payments. (R.M.A.)

3. Give the rule for finding the true average term of credit, and illustrate your answer by an example.

4. Show that the rule for the equation of payments is in favour of the payee.

EXERCISE LI.

Equation of Payments.

1. A owes B £2000 due in 9 mths. He pays £900 in 4 mths., £400 in 6 mths., and £100 in 7 mths. When is the remainder due?

2. Give the equated time for paying the following bills: £500 at 3 mths., £250 at 4 mths., £150 at 8 mths., £800 at 16 mths.

3. A has to pay £100 per annum half-yearly for two years. Find the equated time. (R.M.A.)

4. Find the equated time of payment of £750, of which half is due in 4 mths., $\frac{3}{5}$ in 5 mths., and the rest in 6 mths.? (C.S.)

5. If £40 be due at the end of 6 mths., £60 at the end of 12 mths., and £80 at the end of 15 mths., what is the equated time for paying the whole? (L.M.)

6. Three debts of £200, £300, £400 are due respectively in 3, 4, 5 mths. If the interest is 5 per cent. per annum, what amount will pay them off at the end of 4 mths.? (C.S.)

7. £135 is due 2 yrs. hence, and £145 due 4 yrs. hence. What single sum to be paid 3 yrs. hence may they be commuted for at 4 per cent. simple interest? (C.S.)

8. Find the sum which, paid 2 yrs. hence, will be an equivalent for £250 due in 6 mths., £40 due now, and £800 due in 18 mths., the rate of interest being 5 per cent. in each case.

9. A man owes £10000, which is due in 9 mths. If he pays $\frac{1}{4}$ in 4 mths., $\frac{1}{10}$ in 5 mths., and $\frac{2}{3}$ of the remainder in 6 mths., when should he make the final payment?

10. A man owes four debts, the equated time of payment of which is 6 mths. hence. The first debt of £500 is due in 3 mths., the second debt of £450 in 4 mths., the third of £300 in 5 mths. When was the fourth debt of £1000 due?

11. Five debts are in the proportion of 2 : 4 : 6 : 8 : 10, and they are respectively due in 3, 5, 7, 9, and 11 mths. What is the equated time of payment?

12. Find the true average term of credit of £610 due in 4 mths., £620 due in 8 mths., and of £1225 due in 5 mths., the rate of interest being 5 per cent.

13. Find the true average term of credit of £505 in 3 mths., £612 in 6 mths., and £1030 due in 9 mths. at 4 per cent.

14. Find the true equated time of £318 due 18 mths. hence, and £824 due 9 mths. hence at 4 per cent.

CHAPTER XXXV.

AVERAGES.

A "**mean**" or "**average**" implies, in its most widely accepted sense, a number that is intermediate to other numbers; and this number is of such magnitude that if it were substituted for each of the given numbers, the sum total would be the same in each case.

Example.—*Find the average of* 1, 2, 3, 4, 5, 6, 7, 0, 8.

$$\text{Sum} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 0 + 8 = 36$$

Number of terms = 9

$\therefore$ the average $= 36 \div 9 = 4$ *Ans.*

Proof.—

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 0 + 8 = 36$$

and $4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$

Rule.—Add the given terms, and divide their sum by the number of terms.

The above is a case of simple averages, but more involved cases may present themselves. Consider the following :—

Example.—*A man keeps four grades of bicycles, their prices being respectively* £22, £18, £15, *and* £10. *Supposing his stock to consist of* 70, 200, 350, *and* 500 *machines respectively, find the average price of each machine.*

Total price $= £(22\times70) + £(18\times200) + £(15\times350) + £(10\times500)$

$£1540 + £3600 + £5250 + £5000 = £15390$

and the average price $= £15390 \div (70 + 200 + 350 + 500)$

$= £15390 \div 1120$

$= £13\frac{83}{112}$ *Ans.*

THEORY.

1. Define average. (C.M., K.S.)
2. What is meant by "the average of 7 numbers"? (P.T.)
3. Give the rule for ascertaining "average." (C.W.)

EXERCISE LII. 52

To find Average.

1. Find the average of £27, £34, £6, £49, £53, £19, £23, and £5.

2. A cricketer made the following scores in successive innings: 96, 57, 45, 18, 0, 9, 57, and 6. What was his average?

3. Find the average temperature for a week, when the daily readings of the thermometer were 59°, 62°, 65°, 57°, 58°, 61°, 65°.

4. At a certain examination there were four candidates at the age of 19, three at 20, two at 21, and three at 23. Find the average age of the candidates. (P.T.)

5. The average daily number of persons passing a certain point on Sunday, Monday, Friday, and Saturday is 1765. The average daily number passing on Tuesday, Wednesday, and Thursday is 1541. What is the daily average for the whole week? (C.S.)

6. Three fields are bought for £240, £270, and £470 respectively; they contain 4 ac. 3 rds., 5 ac. 2 rds., 9 ac. 3 rds. respectively. Find the average cost per acre, and the highest-priced field. (K.S.)

7. My travelling expenses for a week are as follows: Monday, 17 mi., 1*s.* 6*d.*; Tuesday, 15 mi., 1*s.* 10*d.*; Wednesday, 18 mi., 1*s.* 7*d.*; Thursday, 10 mi., 1*s.* 3*d.*; Friday, 10 mi., 6*s.*; Saturday, 12 mi., 4*s.* 4*d.* Find the average cost to the nearest farthing per mile for my travelling. (P.T.)

8. Find the average per year of the following incomes (a year = 12 mths.; a month = 4 wks.; a week = 7 days): (1) $8\frac{1}{2}d.$ per day; (2) half a guinea a week; (3) £5 a month; (4) 50 guins. a year. (P.T.)

9. At a sale of cattle this year, 354 head of cattle were sold for £8732. What was the average price paid per head? At a similar sale last year the average price per head was £3 16*s.* 10*d.* greater than that obtained this year, and 279 head of cattle were sold. For what sum were the cattle sold last year? (C.P.)

10. Taking the prices per quarter of wheat for the five years commencing 1837 to be £2 15*s.* 10*d.*, £3 4*s.* 6*d.*, £3 10*s.* 8*d.*, £3 6*s.* 4*d.*, and £3 4*s.* 4*d.*; also for the five years ending 1889 to be £1 12*s.* 10*d.*, £1 11*s.*, £1 12*s.* 6*d.*, £1 11*s.* 10*d.*, and £1 9*s.* 9*d*, determine how much less the average price per quarter for the latter 5 yrs. is than half the average price per quarter for the former 5 yrs. (C.P.)

To find Number or Amount.

11. The population of five parishes being 1236, 452, 364, 516, 3430 respectively, find what the population of a sixth parish must be, that the average population of the six may be 1256·5. (P.T.)

12. A sum of £132 18*s.* 4*d.* was distributed among a certain number of persons. One half of the number received £1 3*s.* 9*d.* each, and the other half £1 4*s.* 7*d.* each. Find the number of persons. (K.S.S.)

13. An army of 12,000 consists of black men and white men. The average height of a black man is 5 ft. 11 in., that of a white man 5 ft. 10 in., that of the army 5 ft. $10\frac{3}{4}$ in. Find the number of white men in the army. Reason clearly. (P.T.)

14. The weights of four rowers in a boat are respectively 10 st. 6 lbs., 10 st. 9 lbs., 11 st. 1 lb., and 11 st. 3 lbs., and the average weight of the whole crew, including the coxswain, is 10 st. Find the weight of the coxswain. (K.S.)

15. A cricketer's average of runs after a certain number of innings, calculated to two places of decimals, was 33·38 . . . per innings. If he afterwards plays two more innings and makes in them 21 and 58 runs respectively, and so raises his average to 33·91 . . . , how many innings has he now played altogether? (C.H.L.)

16. How many pages of a magazine will a manuscript occupy which has 123 pages, each of 28 lines, with seven words per line on an average, while a page of the magazine has 46 lines, with ten words per line on an average? (S.L.C.)

17. Six men A, B, C, D, E, F agree with a seventh man G to provide a sum of money among them. A, B, C, D, E, F are to subscribe £10 each, and G is to pay £2 more than the average of the seven. What is the whole sum to be provided? (K.S.)

18. One-sixth of an estate is let at £2 12*s.* 6*d.* per acre, five-twelfths at £1 11*s.* 6*d.*, and one-fourth at £1 1*s.*; the remainder being let for £31 10*s.*, the average rate of rent for the whole becomes 28*s.* per acre. Find the rate per acre at which the remainder is let, and the number of acres in the estate. (C.S.)

To find Average Percentage.

19. If a man receives on the fourth of his capital 3 per cent., on two-thirds 5 per cent., and on the remainder 11 per cent., what percentage does he receive on the whole? (K.S.)

20. One-half of a certain sum of money is invested so as to return 5 per cent., one-third so as to return 6 per cent., and the remainder so as to return 8 per cent. interest. What is the average rate of interest on the whole? (K.S.)

21. A man lends £1000 in four sums. If he gets 3 per cent. for £200, 3½ per cent. for £400, and 5 per cent. for £250, what per cent. must he get for the remainder, if his average interest is £3 17*s.* per cent.? (P.T.)

22. A carpenter supplies a customer with ten deal boards, whose average length is 12 ft. 6 in. Two of these boards, 11 ft. 6 in. and 12 ft. 3 in., are objected to, and their places supplied by two others, measuring 12 ft. 9 in. and 13 ft. 6 in. respectively. What is the average length of the boards now supplied, and the increase per cent. in the total length? (P.T.)

23. The inhabitants of four towns, numbering 6000, 4250, 3500, and 1250 five years ago, have since increased 5, 10, 15, and 20 per cent. respectively. Find (*a*) the present average population of the four towns, and (*b*) the average increase per cent. (K.S.S.)

24. Out of 175 samples of milk analyzed, 160 are found genuine and 15 adulterated with water; viz. six watered 10 per cent. (*i.e.* 10 per cent. water and 90 per cent. milk), one 20 per cent., three 15 per cent., and the rest 5 per cent. Find the average watering per cent. of the samples analyzed, including the genuine samples. (C.P.)

Problems on Schools.

25. The average age of a school of 600 scholars is 10·75 yrs.; by the enrolment of 40 new scholars the average age is reduced to 10·4375 yrs. Find the average age of the new scholars. (K.S.S.)

26. In a school the numbers for the week were: Monday morning, 67; Tuesday morning, 60; Wednesday morning, 65; Thursday morning, 68; Friday morning, 62; Monday afternoon, ·5 more than the average of Monday and Tuesday mornings; Tuesday afternoon, 59; Wednesday afternoon, ·5 less than the average of Tuesday; Thursday, the average of Monday morning and Tuesday afternoon; Friday afternoon, 60. Find the average attendance for the week. (C.S.)

27. A school opens at 9 a.m. No child is punctual; one comes in at 1 min. past 9, two more at 2 min. past 9, three more at 3 min. past 9, and so on, the number coming in being always equal to the number of minutes past 9. The last six children come in at 6 min. past 9. The school is dismissed at 12. Find the average time spent by a child in school, and the average fine paid by a child at the rate of one penny for every minute he is late. (P.T.)

28. In a certain school, 80 per cent. of the scholars examined pass in arithmetic, $\frac{1}{10}$ more in writing than in arithmetic, and $\frac{1}{11}$ more in reading than in writing. If the passes in reading number 120, how many scholars were examined? (C.W.)

29. It is found in an elementary school that $\frac{1}{3}$ of the children are absent on an average once in 20 times, $\frac{1}{3}$ once in 10 times, and the remaining 40 children once in 4 times. If the number of children on the school registers is the same throughout the year, find the average attendance for the year. (K.S.)

30. A school with 78 boys and 72 girls on the books meet 432 times in a year. If each boy loses one meeting in 9, and each girl one in 8, find (1) the average attendance of each sex; (2) the total number of attendances made by the whole school. (C.W.)

31. In a certain town 9 schoolmasters receive annual salaries amounting to £1109 2*s.*; 17 mistresses and 19 pupil teachers receive salaries amounting to £1602 0*s.* 8*d.*, the average salary of a pupil teacher being one-fifth of that of a mistress. Compare the average salary of a schoolmaster and of a pupil teacher. (K.S.)

32. The average attendance in a school that has met 440 times in the year is 120. At six meetings, when the weather was very stormy, the actual attendance was 25, 35, 40, 20, 15, 45. If these meetings be discarded, and the average of the 434 remaining meetings of the school be calculated, find correctly, to one place of decimals, the increase in the average attendance. (P.T.)

33. A school is open 457 times in the course of the year, and the total number of attendances made by the scholars in the year is 41,587; but by error the attendances are added up as 41,563, and the number of times the school was open as 467; the consequent loss in grant was £1 15*s.* What would be the correct grant, paid upon the unit of average attendance? (C.M.)

34. In a school which has on its books 250 children uniformly, 18 per cent. are absent during the former half of the year, 12½ per cent. during the latter half; in the former there are three weeks of holidays, in the latter five. Reckoning the full weekly attendances of a child as

10, find the average attendance for the year, and the average number of attendances made by each child, reckoning 52 weeks to the year. (K.S.)

Miscellaneous.

35. The mean temperature from the 9th to the 16th of January, both days inclusive, was 34·6°, and from the 10th to the 17th it was 39·8°; the mean temperature on the 9th was 30·5°. What was it on the 17th? (C.S.)

36. The average weight of the 8 oarsmen in a boat is increased by $2\frac{1}{2}$ lbs. when one of the crew, who weighs 11 st. 12 lbs., is replaced by a new man. What is the weight of the new man? (K.S.)

37. Show that rabbits, pheasants, fowls, and hares can be bought (some of each kind) at 1*s.* 2*d.*, 2*s.* 9*d.*, 3*s.* 9*d.*, and 4*s.* each respectively, and the average price be 2*s.* per head. (C.P.)

38. The average age of a committee of seven trustees is the same as it was five years ago, a younger man having been substituted for one of them. How much younger was he than the trustee whose place he took? (K.S.)

39. A bicyclist ran $4\frac{1}{4}$ mi. in 17 min., the distance made in the last minute was $\frac{3}{5}$ of that made in the first minute; and the distance in each successive minute was less than that made in the preceding minute by the same quantity. Find the average and the decrease per minute. (C.M.)

40. One-fifth of an estate is let at £1 12*s.* 6*d.* per acre, $\frac{2}{13}$ at £1 10*s.* per acre, $\frac{5}{12}$ at £1 17*s.* 6*d.* per acre, and the remainder for £106 10*s.* The average of the whole is £2 an acre. Find the rent at which the remainder is let per acre, and the number of acres in the estate. (S.A.)

41. For a certain year in Bombay the rainfall was 22·7 in. in June, 23·09 in. in July, 25·8 in. in August, and 12·03 in September. What was the total rainfall for those months? The similar rainfall for the three succeeding years was respectively 84·06, 80·2, and 81·3. What was the average rainfall of those four years for those months? (P.T.)

42. The rainfall at a certain place for the first 2 mths. of 1895 was 2·05, and for the same period of 1896 it was 1·18 in.; the difference from the mean average for the same period of all previous years on record was a deficiency of 1·8 and 2·52 in. respectively. Find for how many years previous to 1895 the rainfall had been recorded. (C.P.)

43. The annual average depth of rainfall for the three years 1879, 1880, 1881 at a certain place was 24·98 in.; the succeeding three years it was 29·62. The year 1883 was the rainiest, when there fell 4·8 in. more than in 1882, 6·36 in. more than in 1884, and 7·47 more than in 1880. The year 1881 was short of the preceding year by only ·17 in. Find the depth of rain that fell in each of the six years. (B.M.A.)

44. The rainfall of London in the months of April, May, and June, 1876 was 1·90, 0·94, and 1·27 in. respectively, being in April 0·77 in. above, and in May and June 1·46 and 1·78 in. respectively below, the average of those months for the previous six years. Find the average for these three months, both for 1876 and for the whole seven years, to two places of decimals. (K.S.)

45. If the cost of a passenger to a railway company is 7·56*d.* independent of distance in addition to ·58*d.* per mile, while the average total cost is ·76*d.* per mile, what is the average length of journey? (S.A.)

CHAPTER XXXVI.

ALLIGATION OR MIXTURES.

Alligation (*alligo* = I bind or link) is the rule that determines, by a process of linking, the quantities and values of the various ingredients in a mixture or compound.

The rule embraces, not only problems dealing with the quantities of ingredients and their prices, but mixtures of all kinds, including alloys. It derives its name from the process of linking ingredients in the solution of its problems, which may be roughly grouped into two main classes—

1. To find the *mean* or *average* of a mixture of several ingredients at different prices.
2. To find the *proportion* in which ingredients must be mixed at specified prices to produce a given mixture at a given price.

I. To find the Mean or Average Price of the Mixture.

Example.—*A coal merchant mixes* 100 *tons of coal at* 25*s. a ton,* 200 *tons at* 16*s., and* 150 *tons at* 18*s. Find the selling price per ton of the mixture.*

Cost of 100 tons at 25*s.* a ton = 2500*s.*
„ 200 „ 16*s.* „ = 3200*s.*
„ 150 „ 18*s.* „ = 2700*s.*
Then 450 tons of the mixture cost 8400*s.*
∴ 1 ton sells for 8400*s.* ÷ 450 = 18*s.* 8*d.*

Ans. 18*s.* 8*d.*

Example.—*A grocer has equal quantities of two kinds of tea, worth* 5*s. and* 4*s. a lb. respectively. He takes* $\frac{1}{3}$ *of each kind and mixes it with the other. At what price per lb. ought the two mixtures to be respectively sold?* (L.M.)

First mixture—
$\frac{2}{3}$ of each lb. will be at 5*s.*, and $\frac{1}{3}$ at 4*s.* per lb.
∴ (5*s.* × $\frac{2}{3}$) + (4*s.* × $\frac{1}{3}$) = 3*s.* 4*d.* + 1*s.* 4*d.* = 4*s.* 8*d.*

Second mixture—
$\frac{1}{3}$ of each lb. will be at 5*s.*, and $\frac{2}{3}$ at 4*s.* per lb.
∴ (5*s.* × $\frac{1}{3}$) + (4*s.* × $\frac{2}{3}$) = 1*s.* 8*d.* + 2*s.* 8*d.* = 4*s.* 4*d.*

Ans. (*a*) 4*s.* 8*d.*; (*b*) 4*s.* 4*d.*

II. To find the Proportion of the Ingredients.

1. Two Ingredients being given.

Example.—*How must a grocer mix teas at* 1*s.* 2*d. a lb. and at* 2*s.* 10*d. a lb. to make the mixture worth* 2*s.* 6*d. a lb.?* (C.P.)

Gain per lb. by use of inferior tea = cost of mixture per lb. − cost of inferior tea per lb.
= 2*s.* 6*d.* − 1*s.* 2*d.*
= 1*s.* 4*d.* = 16*d.*

Loss per lb. by use of better tea = cost of better tea per lb. − cost of mixture per lb.
= 2*s.* 10*d.* − 2*s.* 6*d.*
= 4*d.*

Now, the loss on the better tea must balance the gain on the inferior tea. If he takes 4 lbs. of the inferior tea and 16 lbs. of the better—

The *gain* would equal 16*d.* × 4 = 64*d.* and the *loss* ,, 4*d.* × 16 = 64*d.* } *i.e.* the loss or gain on the whole would be *nil.*

∴ he could take 4 lbs. at 1*s.* 2*d.* and 16 lbs. at 2*s.* 10*d.*; or
1 lb. ,, 1*s.* 2*d.* ,, 4 lbs. ,, 2*s.* 10*d.* *Ans.*

Note that—

Gain on inferior tea : loss on better tea :: 16*d.* : 4*d.*
or ,, ,, : ,, ,, :: 4*d.* : 1*d.*
and that { No. of lbs. of inferior tea } : { No. of lbs. of better tea } :: 1*d.* : 4*d.*

i.e. the quantities taken to form the mixture are in *inverse ratio* to the prices of the quantities above or below the average price of the mixture.

Example.—*A person bought* 40 *lbs. of sugar of two different sorts for* £1 5*s.* 4*d. The better sort cost* 10*d. per lb. and the worse* 7*d. per lb. Find how many pounds there were of each sort.* (L.M.)

$$\text{Average price per lb.} = \frac{£1\ 5s.\ 4d.}{40} = \frac{324}{40}d. = 7\tfrac{3}{5}d.$$

Gain on better sort = $10d. - 7\frac{3}{5}d. = 2\frac{2}{5}d.$
Loss on worse sort = $7\frac{3}{5}d. - 7d. = \frac{3}{5}d.$

Then, mixing in inverse ratio, we have—

$\frac{3}{5}$ of better : $2\frac{2}{5}$ of worse
i.e. 3 ,, : 12 ,,
or 1 ,, : 4 ,,

Number of parts = 1 + 4 = 5
hence amount of better sort = $\frac{1}{5}$ of 40 lbs. = 8 lbs.
and ,, worse ,, = $\frac{4}{5}$ of 40 lbs. = 32 lbs.
Ans. 8 lbs. of better sort and 32 lbs. of worse sort.

2. Three Ingredients being given.

Example.—*What is the least quantity of coffee at* 1*s.* 4*d.*, 2*s.*, *and* 2*s.* 6*d. per lb. respectively which must be mixed to sell at* 2*s.* 2*d. per lb.?*

Selling or mean price is 26*d*.
Gain on 1 lb. at 1*s*. 4*d*. = 10*d*.
,, 1 lb. at 2*s*. = 2*d*.
∴ total gain = 12*d*.
Loss on 1 lb. at 2*s*. 6*d*. = 4*d*.
and ,, 3 lbs. at 2*s*. 6*d*. = 12*d*. = total gain given above.
∴ mix 1 lb. at 1*s*. 4*d*.
1 lb. at 2*s*.
3 lbs. at 2*s*. 6*d*. } *Ans*.

3. Four or more Ingredients.

Example.—*How may a coal merchant mix coals at* 14*s*., 16*s*., 18*s*., 24*s*., 25*s*., *and* 30*s*. *a ton so as to have a mixture worth* 19*s*. *a ton*?

Mean price in shillings.	Prices in shillings.		
19	14	19*s*. − 14*s*. = 5*s*.	gain
	16	19*s*. − 16*s*. = 3*s*.	,,
	18	19*s*. − 18*s*. = 1*s*.	,,
	24	24*s*. − 19*s*. = 5*s*.	loss
	25	25*s*. − 19*s*. = 6*s*.	,,
	30	30*s*. − 19*s*. = 11*s*.	,,

1st and 6th are mixed	1st gains 5*s*. per ton 6th loses 11*s*. ,,	}	Hence 11 tons of 1st go with 5 of 6th, for 11 × 5*s*. = 5 × 11*s*.
2nd and 5th ,,	2nd gains 3*s*. ,, 5th loses 6*s*. ,,	}	Hence 6 tons of 2nd go with 3 of 5th, for 6 × 3*s*. = 3 × 6*s*.
3rd and 4th ,,	3rd gains 1*s*. ,, 4th loses 5*s*. ,,	}	Hence 5 tons of 3rd go with 1 of 4th, for 5 × 1*s*. = 1 × 5*s*.

Ans. 11 tons of the 1st; 6 of the 2nd; 5 of the 3rd; 1 of the 4th; 3 of the 5th; 5 of the 6th.

Linking and Balancing.

Any number of linkings can be made, so long as the balancing is preserved. Each different linking produces a different combination, but all the results will be correct; *e.g.* take the above example and make other "linkings" and "balancings," *one* of which is shown below.

Mean price in shillings.	Prices in shillings.		
19	14	19*s*. − 14*s*. = 5*s*.	gain
	16	19*s*. − 16*s*. = 3*s*.	,,
	18	19*s*. − 18*s*. = 1*s*.	,,
	24	24*s*. − 19*s*. = 5*s*.	loss
	25	25*s*. − 19*s*. = 6*s*.	,,
	30	30*s*. − 19*s*. = 11*s*.	,,

On mixing 1st and 5th, there is a gain of 5*s*. and a loss of 6*s*.

The mixture will thus be in inverse ratio of 5 : 6, viz. 6 tons at 14*s*. and 5 tons at 25*s*.

Similarly, there will be 5 tons at 16*s*. and 3 tons at 24*s*.; 11 tons at 18*s*. and 1 ton at 30*s*.

Ans. 6 tons at 14*s*.; 5 at 25*s*.; 5 at 16*s*.; 3 at 24*s*.; 11 at 18*s*.; 1 at 30*s*.

From this it will be seen that questions containing 3 or more ingredients are indeterminate, *i.e.* they admit of a variety of solutions.

The Method of Linkages.—The above method is known as the method of linkages, and may be described as follows:—

1. Bring all rates or prices to the same denomination.
2. Put the given prices under one another in order of magnitude, and write the mean price to the left of these given prices.
3. Compare the prices with the mean price, and put against each price the difference between the price with which it is linked and the mean price.
4. Link the prices in pairs, so that one under and one over the mean price shall always go together. If there is an odd price, some one price must be linked with two other prices.
5. The amount of each ingredient in the pair will be in inverse ratio to the money which each ingredient of the pair is above or below the mean price.

4. When the Quantity of one Ingredient is fixed.

Example.—*How should coffee at* 21*d.*, 24*d.*, 29*d.*, *and* 32*d. per lb. be mixed to make a mixture worth* 26*d. a lb.? There are* 30 *lbs. at* 24*d. a lb.*

$$26 \left\{ \begin{array}{ll} 21- & 32 - 26 = 6 \\ 24- & 29 - 26 = 3 \\ 29- & 26 - 24 = 2 \\ 32- & 26 - 21 = 5 \end{array} \right\} \text{Then 3 lbs. : 30 lbs. :: 1 lb. : } x, \text{ where } x = 10, \text{ } i.e. \text{ multiply each quantity by 10.}$$

Ans. $6 \times 10 = 60$ lbs. at 21*d.*
$3 \times 10 = 30$ lbs. ,, 24*d.*
$2 \times 10 = 20$ lbs. ,, 29*d.*
$5 \times 10 = 50$ lbs. ,, 32*d.*

In such cases it will be noticed that linkages are used, and then proportion.

5. When the whole Quantity is fixed.

In the above example, let the whole quantity be 160 lbs. Linking the quantities and solving in the usual way, we get 6, 3, 2, and 5 lbs. respectively. This gives a total of 16 lbs. Then the 160 lbs. must be composed of $(160 \div 16)$ groups, or 10 groups of 16 lbs. each; hence the total at each price will be 10×6, 10×3, 10×2, 10×5 respectively.

Or, as before, we could proceed by proportion—

16 lbs. : 160 lbs. :: 6 lbs. : x, where $x = 60$ lbs.
16 lbs. : 160 lbs. :: 3 lbs. : x, where $x = 30$ lbs., and so on.

Mixing Spirits.

Proof Spirit.—Common spirits (brandy, whiskey, etc.) are about half alcohol and half water; but *proof spirit* contains 57·27 % of alcohol by volume, and 49·50 % by weight.

A spirit is said to be *above proof* when it contains a greater percentage, and *below proof* when it contains a smaller percentage of alcohol than that contained in proof spirit.

In working problems, "proof" is taken as unity, and the "above-proof" strengths are added to, and the "under-proof" strengths taken from unity, and the results are expressed as decimal quantities; *e.g.*—

49 above proof becomes $1 + {\cdot}49 = 1{\cdot}49$
35 under ,, $1 - {\cdot}35 = {\cdot}65$

Example.—*How much spirit* 10 % *above proof must be added to a blend of* 40 *gals.* 20 % *above proof and* 50 *gals.* 5 % *above proof to produce a mixture* 11 % *above proof?*

The blend gives (40 + 50) gals., or 90 gals.
The strength of the blend is—

$$\frac{(40 \times 120) + (50 \times 105)}{90} = \frac{4800 + 5250}{90} = \frac{10050}{90} = 111\tfrac{2}{3}$$

The strength of the mixture is to be 100 + 11 = 111

$$111 \begin{cases} 110 & 111 - 110 = 1 \text{ below mean strength} \\ 111\frac{2}{3} & 111\frac{2}{3} - 111 = \frac{2}{3} \text{ above} \quad ,, \end{cases}$$

The spirit to be mixed is 1 below the *mean* strength, and the "blend" is $\frac{2}{3}$ above the mean strength, therefore the quantities will be in *inverse* ratio, viz. $\frac{2}{3}$ of a gal. of the spirit must be mixed with 1 gal. of the blend.

Ans. $\frac{2}{3}$ gal. 10 % above proof with 1 gal. of the blend.

Pouring from One Vessel into Another.

Example.—*I have one bottle full of brandy, and a second of the same size half full of water. I fill up the second bottle from the first, shake it, and then pour back into the first bottle till it is full. Again I shake the first bottle, and pour from it into the second till it is full. What is the proportion of brandy in the second bottle?*

Let B = bottle full of brandy; and let W = bottle full of water.

	Brandy Bottle.	*Water Bottle.*
1st operation,	1B	$\frac{1}{2}$W
2nd ,,	$\frac{1}{2}$B	$\frac{1}{2}$W + $\frac{1}{2}$B
3rd ,,	$\frac{1}{2}$B + $\frac{1}{2}$($\frac{1}{2}$W + $\frac{1}{2}$B)	$\frac{1}{2}$($\frac{1}{2}$W + $\frac{1}{2}$B)
4th ,,	$\frac{1}{2}$\{$\frac{1}{2}$B + $\frac{1}{2}$($\frac{1}{2}$W + $\frac{1}{2}$B)\}	[$\frac{1}{2}$($\frac{1}{2}$W + $\frac{1}{2}$B) + $\frac{1}{2}$\{$\frac{1}{2}$B + $\frac{1}{2}$($\frac{1}{2}$W + $\frac{1}{2}$B)\}] which gives the proportions of liquids in the second bottle.

Simplifying the quantity on the right-hand side, we get—

$$[\tfrac{1}{4}W + \tfrac{1}{4}B + \tfrac{1}{2}\{\tfrac{1}{2}B + \tfrac{1}{4}W + \tfrac{1}{4}B\}] = \tfrac{1}{4}W + \tfrac{1}{4}B + \tfrac{1}{4}B + \tfrac{1}{8}W + \tfrac{1}{8}B$$

in the second bottle

$$\therefore \text{ proportion of brandy} = \tfrac{1}{4}B + \tfrac{1}{4}B + \tfrac{1}{8}B = \tfrac{5}{8}B$$

Ans. $\frac{5}{8}$ brandy in the second bottle.

Example.—*An alloy of silver is mixed with an alloy of gold in the proportion of* 11·4 *to* 2·6. *The percentage of dross in the silver alloy is* 1·5, *and that in the gold* 17·35. *Find the percentage of dross.*

$$\text{Sum of proportional parts} = 11{\cdot}4 + 2{\cdot}6 = 14$$

$$\text{The proportional part of dross in silver} = \frac{11{\cdot}4 \times 1{\cdot}5}{100} = 1{\cdot}539$$

$$\text{and in gold} = \frac{2{\cdot}6 \times 17{\cdot}35}{100} = {\cdot}4511$$

$$\text{total proportional parts of dross} = 1{\cdot}539 + {\cdot}4511 = 1{\cdot}9901$$

If there are 1·9901 parts dross in 14 parts alloy, how many parts dross are there in 100 parts of alloy?

$$14 : 100 :: 1{\cdot}9901 : \text{percentage of dross} = \frac{199{\cdot}01}{14} = 14{\cdot}215$$

Ans. 14·215 %.

THEORY.

1. Define alligation. What does the rule embrace, and into how many classes may its problems be roughly divided?

2. Name and define the various kinds of alligation.

3. Explain fully what is meant by the method of linkages.

4. When are problems in alligation said to be indeterminate? Show by an example what is meant by the statement.

5. What is the meaning of "proof spirit"? Define, with examples, "above proof" and "under proof."

EXERCISE LIII.

ALLIGATION.

To find Price of Part or Whole of a Mixture.

1. A grocer buys 30 lbs. of tea at 2*s.* a pound, and 50 lbs. of tea at 2*s.* 8*d.* a pound, and, after mixing them, sells 40 lbs. of the mixture at 2*s.* 4*d.* a pound. At what price per pound must he sell the remainder that he may neither gain nor lose? (C.W.S.)

2. If I purchase 2276 lbs. of coffee at 1*s.* 9*d.* per pound, and mix it with chicory at 4½*d.* per pound in the proportion of 3 parts by weight of the former to 2 parts of the latter, at what price per pound must I sell the mixture to gain 25 per cent., and what will be the gross amount of my profit on the transaction? (I.I.)

3. A grocer has three kinds of tea. He sells some of one kind at 1*s.* 3*d.* per pound, and half that quantity of another kind at 1*s.* 6*d.* per pound. His sales of the third kind equal in weight those of the other two kinds together. The average price of all the tea sold being 1*s.* 2*d.* per pound, find the price of the third kind. (C.S.)

4. Strong spirit is mixed with inferior spirit, valued at 5*s.* per gallon, in the proportion of 6 to 1. The mixture is then worth 9*s.* the gallon. Find the value of a gallon of the strong spirit. (R.M.A.)

5. Two mixtures of milk and water, containing 34 per cent. and 46 per cent. of milk respectively, are mixed together in the proportion of 2 gals. of the latter to 1 gal. of the former, and the resulting mixture is sold at 4*d.* a quart. If 9 gals. of milk cost 10*s.*, what percentage is gained on the original outlay, after deducting from the profits the cost of delivering the mixture to customers, which amounts to 2*d.* a gallon? (C.J.)

6. A merchant mixes two qualities of tea in the proportion of 2 parts of the cheaper to 1 part of the dearer, and, by selling the mixture at 2*s.* 11*d.* per pound, he gains a profit of 25 per cent.; on mixing them in the proportion of 3 parts of the cheaper to 2 parts of the dearer, and selling the mixture at 3*s.* per pound, he gains the same rate of profit. Find the prices at which he bought. (L.M.)

7. A butt of sherry is made up of two wines, A and B, mixed in the ratio of 1 : 3, and the price is £70. Another butt is made up of the same wines, mixed in the ratio 3 : 1, and the price is £50. What are the prices of the two wines A and B? (L.M.)

8. Three kinds of tea are mixed in the proportion of 1 : 2 : 3. The

value of the mixture is 2*s.* 9*d.* per pound. If only half of the second kind had been used the mixture would have been worth 2*s.* 6*d.* per pound. But if all the second kind had been used and only half of the third kind, the mixture would have been worth 3*s.* per pound. Find the worth per pound of each kind of tea. (C.H.L.)

9. A grocer directs his boy to mix 20 lbs. of tea, bought at 1*s.* 6*d.* a pound, with 30 lbs. bought at 2*s.* a pound, and proceeds to sell the mixture made, at a price per pound intended to produce 66⅔ per cent. profit. After selling 20 lbs., the grocer finds that the boy has used tea bought at 2*s.* 6*d.* a pound instead of tea bought at 2*s.* At what price per pound must the rest of the mixture be sold, if a net profit of 66⅔ per cent. is to be made on the whole actual transaction? (C.W.)

10. A mixture of 16 parts of gin costing 10*s.* 1½*d.* a gallon with 5 parts of water (costing nothing) is sold at 10*s.* 8*d.* a gallon, while one of 4 parts of rum costing 15*s.* a gallon with 1 part of water is sold at 16*s.* a gallon. What proportion of water must be mixed with brandy costing 30*s.* a gallon so that, if the mixture be sold at 33*s.* 6*d.* a gallon, the whole profit gained may be 40 per cent., supposing the quantity of diluted gin sold to be thrice, and that of the diluted rum twice, that of the diluted brandy? (C.S.)

To find Proportion of Ingredients in a Mixture.

11. If coffee and chicory cost £8 10*s.* and £2 10*s.* per cwt. respectively, what is the proportion of coffee and chicory in a mixture of which 7 lbs. are worth 7*s.* 6*d.*? (R.M.A.)

12. A grocer buys two kinds of tea costing respectively 1*s.* 1½*d.* and 1*s.* 7*d.* per pound. In what proportion must he mix them so as to make a profit of 15 per cent. by selling the mixture at the rate of 4 lbs. for 5*s.* 9*d.*? (C.H.L.)

13. A tradesman professes to retail his goods at a profit of 10 per cent. He mixes with them ¼ of their weight of an inferior article which cost him only ⅓ of the price of the better. How much per cent. profit does he make? Also, in what proportion must he mix them in order to gain 20 per cent.? (C.S.)

14. What weight of tea worth 3*s.* 6*d.* per pound should be mixed with 60 lbs. of tea worth 2*s.* 3*d.* per pound, so that, when the mixture is sold at 2*s.* 9*d.* per pound, there may be (1) neither gain nor loss, (2) a gain of 10 per cent.? (C.P.)

15. How many pounds of tobacco at 5*s.* 3*d.* per pound must a tobacconist mix with 4 lbs. at 6*s.* 6*d.* that he may sell the mixture at 7*s.* 10*d.* per pound and gain 33⅓ per cent. upon his outlay? (C.PR.)

16. How many pounds of water must be added to 100 lbs. of moist tobacco, which already contains 14 per cent. of water, in order that it may contain 30 per cent. of water? (S.A.)

17. Green tea is mixed with 11 cwt. 14 lbs. of black in the proportion of ⅛ to the pound of the mixture. Determine how much green is used. If the black tea be worth 2*s.* 6*d.* a pound, and the green 4*s.* a pound, and the mixture be sold at 3*s.* a pound, what extra profit will be gained?

(C.P.)

18. A wine merchant has bought a number of bottles of claret at 35*s.* per dozen, and another number at 21*s.* per dozen, which he mixes and sells at 30*s.* per dozen, without gaining or losing by the transaction. In what proportion were the two wines mixed? (C.M.S.)

19. To three hogsheads of natural sherry, containing 14·2 per cent. of alcohol, a wine merchant adds 100 pints of brandy containing 43·5 per cent. of alcohol. What is the approximate percentage of alcohol in the fortified wine? (1 hogshead of wine = 63 gals.) (C.S.)

20. One gallon of spirit, which contains 11 per cent. of water, is added to 3 gals. containing 7 per cent. of water, and to this mixture $\frac{1}{2}$ gal. of water is added. Find the percentage of water in the mixture. (R.M.A.)

21. One liquid contains $22\frac{1}{2}$ per cent. of water, another 27 per cent. A glass is filled with 5 parts of the one liquid and 7 parts of the other. What percentage of water is in the glass? (C.S.)

22. Two gallons of a mixture of spirit and water contain 12 per cent. of water, and are added to 3 gals. of another mixture containing 7 per cent. of water; to the whole is then added $\frac{1}{2}$ gal. of water. Find the percentage of water in the resulting mixture. (K.S.)

Problems on pouring Liquids from One Vessel into Another.

23. Two casks, A and B, contain 8 and 3 gals. of wine, worth 18*s.* and 15*s.* per gallon respectively. One gallon is taken from A and put into B, and then 1 gal. is taken from B and put into A. Find the worth per gallon of the wine in each cask. (C.M.)

24. A wine-glass is filled with a mixture of spirit and water in the ratio of 1 of spirit to 4 of water; another glass of the same size is filled in the ratio of 2 of spirit to 5 of water. Find the ratio of spirit to water when the contents of the two glasses are mixed in a tumbler. (K.S.)

25. A person fills a glass with medicine and drinks $\frac{1}{4}$ of it. He then fills up the glass with water and drinks $\frac{1}{3}$ of it. Again he fills up with water and drinks $\frac{1}{2}$. How much of the medicine does he drink altogether, and how much each time? (C.S.)

26. A vessel is full of a mixture of spirit and water, in which there is found to be 17 per cent. by measure of spirit. Ten gallons are drawn off, and the vessel is filled up with water. The proportion of spirit is now found to be $15\frac{1}{3}$ per cent. How much does the vessel hold? (C.S.)

27. Two pint measures are filled with milk and water; in one the ratio of milk to water is 6 to 1; in the other 7 : 1. If the contents of each be emptied into a quart measure, find the ratio of milk to water in it. (K.S.)

28. Two casks of 48 and 42 gals. are filled with mixtures of wine and water, the proportions being 13 : 7 and 18 : 17 in the two casks respectively. If the contents of the two casks be mixed, and 20 gals. of water added to the whole, what will be the proportion of wine to water in the result? (C.S.)

29. A wine merchant mixes in a vat 27 gals. of wine at 15*s.* a gallon, 33 at 20*s.*, and 25 at 22*s.* He then draws off 15 gals. of the mixture, and adds to the remainder 10 more gallons at 15*s.* a gallon. What is the value per gallon of the mixture now in the vat? (C.S.)

30. One vessel contains 12 gals. of wine and another 6 gals. of water. One gallon is taken from each and transferred to the other. Find the proportion of water to wine in the first vessel, after the operation of transferring liquid from one to the other has been performed three times in all. (C.S.L.)

31. From a cask of wine, containing 64 gals., 8 gals. are drawn out, and the cask is filled up with water; if the same process is repeated a second, then a third time, what will be the number of gallons of wine left in the cask? (O.S.L.)

32. There are two vessels, one containing 12 gals. of water, the other 6 gals. of wine. If a gallon be taken out of each and poured into the other, and if this be repeated four times, find how much wine and water will be in each. (C.S.)

33. A person has two casks, with wine in each; in order to have an equal quantity of wine in each, he pours out of the first, which contains the larger quantity, into the second as much as the second already contains; he next pours from the second into the first as much as the first then contains; and again pours from the first into the second as much as the second contains; each now has 16 gals. How much had each at first? (C.S.)

34. A wine merchant purchases an 18-gal. cask of spirit at 12*s.* per gallon, from which he draws off 2 gals. and fills it up with water. He then draws off enough to fill a $4\frac{1}{2}$-gal. cask and fills it up again with water. From this a 9-gal. cask is next filled, after which, having returned the 2 gals. first taken, he fills it up with water for the last time. What profit does he make on each cask by selling the smallest at 16*s.* per gallon, the largest at 9*s.*, and the other at 14*s.*? What is the gain per cent. on the whole transaction?

35. A cask is filled with spirits; $\frac{1}{2}$ of the contents is drawn off into another equal cask, which is then filled up with water. The first cask is then filled up again out of the second one, and then the second one is filled up again out of the first. Find what proportion of spirits the second cask finally contains. (L.M.)

36. From a vessel filled with alcohol, $\frac{1}{3}$ of its contents is removed, and the vessel is then filled up with water. If this be done five times in succession, what proportion of the alcohol originally contained in the vessel will have been removed from it? (L.M.)

37. There are four vessels of equal capacity; the first is filled with spirit to the extent of $\frac{1}{5}$, the second to $\frac{1}{6}$, the third to $\frac{1}{4}$, and the last to $\frac{1}{3}$. The first is then filled up with water, and from this mixture the second is filled up, again from this second mixture the third is filled up, and in like manner the fourth from the third. What proportion of spirit to water is there in the fourth vessel? (R.M.A.)

38. There are four vessels of equal capacity: $\frac{1}{6}$ part of the first, $\frac{1}{5}$ of the second, $\frac{1}{4}$ of the third, and $\frac{1}{3}$ of the fourth are filled with spirit; the first is then filled up with water, then the second is filled from the first, next the third from the second, and lastly the fourth from the third. If the mixture in the fourth vessel contains now 2 gals. of water, what is the capacity of each vessel? (O.S.L.)

Problems on Alloys.

39. If 9 oz. of gold, 10 carats fine, and 5 oz., 11 carats fine, be mixed with 6 oz. of unknown fineness, and the fineness of the resulting mixture be 12 carats, what was the unknown fineness? (C.S.L.)

40. In a sovereign 1 part in 12 of the metal used consists of an alloy worth $\frac{1}{2}d.$ an ounce, and the rest is pure gold worth £3 17*s.* $10\frac{1}{2}d.$ an ounce. If the metal in a sovereign is just worth £1, what is the weight of pure gold in 257 sovereigns? (O.S.L.)

41. If the silver coin of Britain is made of 37 parts of pure silver and 3 parts of copper, and a pound Troy of this metal is coined into 66*s.*, find the weight of pure silver in a shilling. (C.S.)

42. An alloy containing 7 times as much gold as silver is worth 3 times as much as one that contains 3 times as much silver as gold. In what ratio must these alloys be mixed to form one worth £2 10*s.* an ounce, gold being worth £3 17*s.* 11*d.* an ounce? (O.S.L.)

43. If 1 lb. of metal, containing copper and zinc in the ratio of ·84 to ·16, be mixed with 2 lbs. containing the same metals in the ratio of ·75 to ·25, find how much copper and zinc there is in the mixture. (O.S.L.)

CHAPTER XXXVII.

FOREIGN MONEYS.

The Latin Union.—France, Italy, Belgium, Switzerland, and Greece form what is known as the Latin Union.

In these countries the gold and silver coins (with the exception of the Italian 2 lire, 1 lire, and ½ lire) are alike in weight, size, and fineness, and (with the exception of the Italian pieces mentioned) the coins of one country are legal tender in the others.

Spain, Servia, Roumania, and Bulgaria have virtually the same coinage as the Latin Union, but have never formally joined it.

The coinage of the Latin Union is based upon the French coinage.

French Coinage.—The unit is the *franc* (fr.), which is equal to 100 centimes (c.). The coins are: *Gold*, 20 fr., 10 fr.; *silver*, 5 fr., 2 fr., 1 fr., 50 c.; *bronze*, 10 c., 5 c.

The *bronze* coins are not accepted in the Latin Union; only the gold and silver ones.

Roughly speaking, £1 = 25 francs, and 1 franc = $9\frac{1}{2}d.$

The units and sub-units have different names in different countries.

France, Belgium, Switzerland	Francs	Centimes
Italy	Lires	Centesimi
Greece	Drachmai	Lepta
Spain	Pesetas	Centimos
Servia	Diners	Paras
Roumania	Leïs	Banis
Bulgaria	Leva	Stotinkis

Practically all South Europe, except Portugal and Turkey, have a coinage based on the French coinage.

American Coinage.—Just as the monetary system of the greater part of South Europe is based on the French system, so the money of the New World may be said to be based upon

the coinage of the United States, with this important proviso, however, that the coins of one country are *not* legal tender in the others.

United States.—Unit: the dollar (worth nearly 4*s.* 2*d.*), which is divided into 100 cents.

Chief Coins.—*Gold*, 20 dollars (double eagle), 10 dollars (eagle), 5 dollars ($\frac{1}{2}$ eagle). *Silver*, dollar, $\frac{1}{2}$ dollar, quarter (25 cents), dime (10 cents). *Nickel*, 5 cents. *Copper*, 1 cent.

Note.—The use of paper money, from the one-dollar bill upwards, is universal in the States. The *cent* of the United States is very different from the *centime* of the Latin Union.

Canada has the same system as the United States.

Central and South America (except Brazil and Mexico) have a similar system. The unit is the *peso* (worth about a dollar), and this is divided into 100 *centavos*.

Mexico.—The Mexican dollar has practically the same amount of silver in it as the United States dollar, but is not equal to it in value, because Mexico is a "silver standard" country.

Germany.—The unit is the *mark* (worth just a little less than a shilling), which is divided into 100 *pfennigs*.

Chief Coins.—*Gold*, 20, 10, and 5 marks. *Silver*, 5, 2, and 1 marks, 50 and 20 pfennigs. *Nickel*, 10 and 5 pfennigs.

The Chief Coins of other Countries are shown in the table below:—

Country.	Unit.	Chief subdivisions.	Value of unit in English money.
			s. *d.*
Spain	peseta	= 100 centimos	0 $9\frac{1}{2}$
Denmark	kröna	= 100 öre	1 1
Sweden			
Norway			
Holland	florin or gulden	= 100 cents	1 8
Austria	krone	= 100 heller	0 $9\frac{1}{2}$
Russia	rouble	= 100 kopeks	3 0
Turkey	pound	= 100 piastres	18 0
Japan	yen	= 100 sen	4 0
Portugal	milreis	= 1000 reis	4 5
China	tael	= 1000 cash	6 6

British Empire.—English money is current in Australia and New Zealand, the West Indies, and South Africa. Canada has a special currency (p. 350). In the Straits Settlements and Hong Kong a British dollar (equal to the Mexican dollar and the Japanese yen) circulates. Besides the English, the Indian is the only important non-decimal coinage.

Indian Coinage.—Accounts are kept in *rupees*, *annas*, and *pies*.

Money Table.

12 pies	= 1 anna
16 annas	= 1 rupee
100000 rupees	= 1 lac
100 lacs	= 1 crore

The nominal value of the rupee is 2*s*. Its *actual* value is much less, about 1*s*. 4*d*.

Ceylon Coinage.—In Ceylon the rupee is divided on the decimal system into 100 cents.

Exchange.

If Mr. Black (of London) owes M. Noir (of Paris) the sum of £1000, he could discharge the debt—

(1) By sending *bullion* (*i.e.* bars of gold) to the value of £1000.
(2) By sending *goods* to M. Noir to that amount (*i.e.* by *barter*).
(3) By sending *specie* (*i.e.* gold coins).
(4) By means of a *bill of exchange*.

Methods (1) and (2) are rarely employed, so we will proceed to examine methods (3) and (4).

The *standard coin* of England is the *sovereign*; of France, the *napoleon* (or 20-franc gold piece). Mr. Black owes a debt to M. Noir, and, assuming that this debt is to be discharged in specie, it becomes necessary to make a comparison between the standard coin of England and that of France.

This has been done by carefully estimating the quantity of pure gold in the standard coin of each country. Applying this test, it is found that one English sovereign contains as much pure gold as is contained in 1·261075 French napoleons, *i.e.* £1 = 25·2215 francs, and this is termed the par of exchange.

Par of Exchange.—Since the value of a digit in the third decimal place is only so many thousandths of a coin worth about 9½*d*., in business, the fact that £1 = 25·22 francs is said to be the par of exchange, and the other decimals are disregarded. The par of exchange shows the *intrinsic* value of one coin as compared with another, and depends entirely on the weight of pure gold in the standard coins of the two countries concerned.

If exchange is at par, Mr. Black can settle his debt of £1000 with M. Noir by presentation in Paris of 25220 francs.

Rate of Exchange.—But owing to various causes, it rarely happens that exchange is at par. M. Noir might be willing to accept 25000 francs in payment of the debt, or he might require 25300 francs. In the former case the rate of exchange would be said to be *low* in Paris, and in the latter case *high*.

Bills of Exchange.—Specie is rarely transmitted from one merchant to another. A bill of exchange is the commonest medium for the settlement of accounts.

Thus, Herr Schwartz (of Hamburg) buys goods from Mr. Black (of London) to the value of £100. Mr. Black draws a bill on Herr Schwartz for £100 in the form shown below. He sends Schwartz the bill, asking him to accept it, which Schwartz does by writing across the bill, "Accepted, payable at (name of Schwartz's bankers), J. Schwartz."

Form of a Foreign Bill.

£100. London, 8th January, 1907.

Three months after date pay to my Order the sum of One Hundred pounds, value received.

WILLIAM BLACK.

To Herr J. Schwartz,
16, Steinbuhlerstrasse,
Hamburg.

Now, we will further suppose that Mr. Black buys goods to the value of £100 from Herr Weiss (of Hamburg). Black hands Weiss the bill as payment, and, when it becomes due, Weiss presents it to Schwartz for payment.

Thus Schwartz has settled his debt with Black, and Black has settled his debt with Weiss, and no money has passed from England to Germany, or from Germany to England.

But it is only *in theory* that things work out so easily. Black might find it difficult to discover a man in Germany from whom he could buy exactly £100 worth of goods. In *practice*, Black would take his bill against Schwartz to a banker (or to a bill broker), who would exchange it for money, and to discharge his debt to Weiss he would probably buy a bill from the same broker.

The amount Black would get for his £100 bill, and the amount he would have to pay for a bill, would be regulated by the rate of exchange less a deduction for the banker's or bill broker's commission.

Long and Short Rates of Exchange.—A bill which is payable on presentation (*i.e.* without delay) is said to be payable "at sight." Quotations for bills payable at sight are quoted at "short rates."

Quotations for bills payable at some distant date are given at "long rates."

The difference between "long rate" and "short rate" is the *discount* (at the current rate per cent. for discount) for the time named in the "long rate."

The rates of exchange are published in the press, and are given for London and important foreign trade centres. Below are given some rates of exchange from a daily paper.

	Previous quotation.	Latest quotation.
Paris (cheque)	25.11½	25.11½
Berlin (sight)	20.45–44½	20.40½
Hamburg (sight)	20.43½	20.44
Amsterdam (sight)	12.12¾	12.12¾
Brussels (cheque)	25.17½	25.18
Vienna (sight)	23 92½	23.92
Barcelona (sight)	33.70	33.90
Lisbon (sight)	42 1/16	42 1/16
Rome (sight)	25.11	25.11
St. Petersburg (three months)	93.80	—
Constantinople (three months)	110.50	Holiday.
Calcutta (T.T.)	1/4	1/4
Bombay (T.T.)	1/3 31–32	1/3 31–32
Hong Kong (T.T.)	1/7½	1/7 5/8
Shanghai (T.T.)	2/2⅞	2/2¾
Singapore (four months)	1/7⅝	1/7 7/16
Yokohama (four months)	2/1	2/1
Rio de Janeiro (90 days' sight)	12 1-32*d.*	12 1-32*d.*
Valparaiso (90 days' commission)	16¼*d.*	16 9-32*d.*
Buenos Ayres Gold Premium	127.30	127.40
New York Cable Transfers	4.87⅞	4.87⅞

A few notes on the above will be of interest to the student.

In some cases the value of £1 *in the foreign coinage is given: vide* Paris, Berlin, Hamburg, Amsterdam, Brussels, Vienna, Barcelona, Rome, New York. Thus, Paris 25.11½ means 25 francs 11½ centimes for £1; Hamburg 20.45 means 20 marks 45 pfennig for £1; New York 4.87⅞ means 4 dollars 87⅞ cents. for £1.

In other cases the value of the principal foreign coin is given in pence, or shillings and pence. Thus, Lisbon (sight) 42 1/16, means 42 1/16*d.* for 1 milreis; Calcutta 1/4, means 1*s.* 4*d.* for 1 rupee. T.T. means telegraphic transfers.

Hong Kong 1/7½, means price per dollar; similarly Singapore, but with discount for four months. Yokohama 2/1, means 2*s.* 1*d.* for 1 yen; Shanghai,

price per tael (Mexican dollar); Rio, price per milreis; Valparaiso, price per silver dollar.

In the case of St. Petersburg, the quotation shows the number of roubles and kopecks per £10.

Buenos Ayres Gold Premium 127.30 means that 100 in gold = 227.30 face value in paper.

Example.—*Convert* 550 *Italian lire into Dutch florins, a lire being equivalent to* $9\frac{1}{2}d.$, *and a florin to* $20d.$

$$550 \text{ lire} = 9\tfrac{1}{2}d. \times 550$$
$$20d. = 1 \text{ fl.}$$
$$1d. = \tfrac{1}{20} \text{ fl.}$$
$$9\tfrac{1}{2}d. \times 550 = \frac{1 \times 9\frac{1}{2} \times 550}{20} \text{ fl.} = 261\tfrac{1}{4} \text{ fl.} \quad Ans.$$

Example.—*A sight draft on New York for* \$7200 *sold in London for* £1477 10*s.* *What was the course of exchange?*

$$\$7200 = £1477\tfrac{1}{2}$$
$$\$1 = £\frac{1477\frac{1}{2}}{7200} = 4s.\ 1\tfrac{1}{4}d.$$
$$\$1 = 4s.\ 1\tfrac{1}{4}d. \quad Ans.$$

Example.—*Find the face value of a bill due at sight on Paris which sold in London for* £2000, *the exchange on Paris being* 25·17 *fr.*

If £1 = 25·17 fr.
then £2000 = 2000 × 25·17 fr. = 50340 fr.

Ans. 50340 fr.

Example.—*Sight bills on London are quoted at* 25·30 *fr. in Paris, and bills at* 2 *months at* 25·10 *fr. Find the rate of discount.*

If the market discount for $\frac{1}{6}$ yr. on 25·30 fr. be ·20 fr.
then „ „ 1 „ „ is 1·2 fr.

Hence the following proportion:—

25·30 : 100 :: 1·2 : rate of discount
= 4·74 . . . %. *Ans.*

Example.—A *person in London owes another in Petersburg* 460 *roub., which must be remitted through Paris. He pays the requisite sum to his broker when the exchange between London and Paris is* 23 *fr. for* £1, *and between Paris and Petersburg* 2 *fr. for* 1 *roub. The remittance is delayed until the rates of exchange are* 24 *fr. for* £1 *and* 3 *fr. for* 2 *roub. What does the broker gain or lose by the delay?*

At first rate of exchange—

London.		Paris.		Petersburg.
£2	=	46 fr.	=	23 roub.
£40	=	920 fr.	=	460 roub.

At second rate of exchange—

London. Paris. Petersburg.
£2 = 48 fr. = 32 roub.

If 32 roub. = £2

then 460 roub. = $£\frac{2 \times 460}{32}$ = £28 15*s.*

Loss = £40 − £28 15*s.* = £11 5*s.* *Ans.*

THEORY.

1. What is a bill of exchange? (C.S.)

2. What is meant by the "course of exchange" between two countries? (R.M.A.)

3. What is meant by the "par of exchange" between two countries? When is the exchange said to be against a country? (C.S.)

4. What is the meaning of a foreign bill of exchange? Explain the variation of conditions which makes the exchange on one day £1 = 25·23 francs, and on another day £1 = 25·25 francs. Give an example of the state of the exchanges in three cities which would make it advantageous to settle claims in the first city upon the third through the agency of the second, and work out any case you choose. (C.S.)

5. Give the relative values of a shilling (English), a franc (French), and a mark (German); and the relative magnitudes of a yard (English) and a metre (French). (C.S.)

6. Give as nearly as possible the present English value of the following foreign moneys and measures, and say in what countries they are used respectively: (*a*) florin, centime, pfennig, peseta, anna; (*b*) metre, hectare, league, chetwert. (O. and C.S.C.)

7. Give as nearly as possible the present English value of the following foreign moneys and measures, and say in what country or countries they are used: (*a*) franc, rupee, centesimo, Mexican dollar; (*b*) pood, kilometre, maund, quintal. (O. and C.S.C.)

8. Explain the following exchange quotations: "Both the Chinese exchanges came $\frac{1}{8}d.$ higher, Hongkong being called 3*s.* $2\frac{1}{2}d.$, and Shanghai 4*s.* $4\frac{1}{4}d.$ Paris: Sight exchange on London, 25·35 at $25·37\frac{1}{2}$; bank notes, 25·30. San Francisco: We quote best commercial paper $49\frac{3}{4}d.$"

(O. and C.S.C.)

9. Explain the following quotations of exchange: "India demand rates were cabled 1*s.* $6\frac{1}{2}$ for Bombay and 1*s.* $6\frac{9}{16}$ for Calcutta. Berlin: foreign bills weaker, both long and short on London being half a pfennig lower. Hamburg: 3 mos. 20·54 to 20·58. St. Petersburg: 3 mos. $27\frac{3}{4}$ to $27\frac{7}{8}$. New York: 60 days. $48\frac{5}{8}$ to $48\frac{3}{4}$." (O. and C.S.C.)

EXERCISE LIV.

Foreign Money and Exchange.

1. Assuming that £1 is equal to 25 fr., how much English money will be required to discharge a debt of 24875 fr.? (O.R.)

2. When the pound sterling was worth 24 fr. 75 c., how many shillings should have been given in exchange for 20 fr.? (P.T.)

3. What, in English money and decimal fractions, is the value of the franc at the exchange of 25·57 fr. per £ sterling? (C.J.)

4. In Paris, how many francs should be given in exchange for £3 7*s.* 5½*d.*, four francs being worth 3*s.* 3*d.*? (C.P.)

5. Find the value, in British money, of goods sold for 7889 fr. 90 c.; exchange, 24 fr. 41½ c. per £. (C.J.)

6. Find the value in English money of 1572·4185 fr. when the exchange is at 26·675 fr. per £. (O. and C.S.C.)

7. To 80 men are given 693 fr. to be equally divided. How much does each man receive when the money is changed into British currency at 25⅛ fr. to the sovereign? (O. and C.H.C.)

8. A debt of £65 is paid in francs valued at 9¾*d.* each, at a time when 25 fr. are worth £1. What does the creditor gain or lose? (C.M.)

9. If a sovereign be worth 25 fr. 15 c., what per cent. does a man lose in getting only 25 fr. for a sovereign? (P.T.)

10. If 25 fr. are worth a sovereign, what will a person gain or lose who receives payment of a debt of £16 5*s.* in francs valued at 9¾ each? (P.T.)

11. In the early part of the year the Government offered to take French bronze pennies from the public at the rate of 13 for a shilling. The Government would be able to exchange them again at the rate of 252 for a sovereign. What would the Exchequer gain or lose on every million pennies so received? (C.S.L.)

12. On a certain continental tour the sum spent amounted to £8 6*s.* 11*d.* in English money, and 312·34 m. in German money; while on another tour it amounted to £11 7*s.* 2*d.* and 733·32 m. If the second tour cost altogether twice as much as the first, find the number of marks which is equal in value to £100. (C.H.L.)

13. If £1 be equal to 11·9 Austrian florins or to 20·3 German marks, and if I spend 6 days in Austria at 10 florins a day, and 7 days in Germany at 18 m. a day, what value ought I to have left in English money out of a £20 note? (C.P.)

14. A man exchanged £32 at Dresden for marks, at 20¼ m. for £1. He stayed there 28 days, spending on the average 12½ m. a day. He then travelled to London, the journey costing 120 m., and on his arrival he changed what he had left at the rate of a mark for 10½*d.* How much English money will he have out of the original £32? (C.W.)

15. A sovereign is worth 4 dol. 7 c. in American, and 25 fr. 2 c. in French money. How many cents will a man lose who changes £15 at the rate of 25 fr. for £1? (C.PR.)

16. A person enters France with 33 sovs., 1 half-sov., and 7 fl. He spends 577 fr. 50 c. How many florins will he receive in exchange for the remainder, if £1 be worth 26 fr. 25 c.? (C.S.)

17. If a dollar worth 4*s.* 3½*d.* be equivalent to 5·439 fr., how many francs are in £1? (P.E.A.)

18. If exchange be at the rate of 25·50 fr. for £1, and of 57·75 fl. for 119 fr., find the value of £4760 in florins. (K.S.)

19. What would £1 be worth at Paris, if a bill at Amsterdam of 12 fl. 15 c. sells for £1, and 9 fl. 50 c. are worth 20 fr. at Paris? (C.S.)

20. What is the value in napoleons of 16 guins., if 25 fr. are given for a sovereign, and 20 fr. are equal to 1 napoleon? (O.S.L.)

21. If 10 pias. are equivalent to £1 13*s.* 9*d.*, and 25·3 fr. to £1, how many francs are equivalent to 7·5 pias.? (P.T.)

22. If £3 are equal to 20 thal.; 25 thal. equal to 93 fr.; 27 fr. to 5 scu.; 62 scu. equal to 135 gul.; how many gulden in £2? (C.S.)

23. If an English sovereign is equal to 25·25 fr., or to $6\frac{2}{3}$ thal., find how many thalers there are in 16000 fr., and reduce the remainder into English money. (C.S.)

24. English money bearing a premium of 5 per cent. in the United States, how much sterling should be given for 750 dol., each worth 4*s.* 6*d.* at par? (P.T.)

25. A merchant in New York wishes to remit to London 5110 dol., a dollar being equal to 4*s.* 6*d.* English. For what sum in English money must he draw his bill when bills on London are at a premium of $9\frac{1}{2}$ per cent.? (R.M.A.)

26. An Indian officer, whose annual pay was estimated in rupees, lost £41 12*s.* 6*d.* in one year by a fall in the value of a rupee from 1*s.* $11\frac{1}{2}d.$ to 1*s.* $10\frac{1}{4}d.$ What was his salary, estimated in rupees? (C.J.)

27. A person sends from India to England 390 rup. per calendar month. If a rupee be worth 1*s.* $9\frac{1}{2}d.$, what will be the value in £ *s.* *d.* of the sum sent in a year? If a rupee be worth only 1*s.* $7\frac{1}{2}d.$, how many rupees per month must be sent that the value in £ *s.* *d.* of the sum sent in a year may remain the same as before? (O.R.)

28. If 11·65 Dutch florins are given for 24·89 fr., 383 Dutch florins for 437 m., and $68\frac{1}{4}$ m. for 32 silver roub., how many francs should be given for 932 silver roub.? (O.S.)

29. If 250 rup. have the same value as 84 dol., and 36 dol. have the same value as 7 guins., what is the value of a rupee in £ *s.* *d.*?
(O. and C.H.C.)

30. If 12 pies = 1 anna, 16 annas = 1 rupee; and 16 chataks = 1 seer, 5 seers = 1 pusseree, 8 pusserees = 1 maund, find the cost of 5 maunds 6 pusserees 3 seers 8 chataks at 23 rupees 5 annas 4 pies per maund. (S.A.)

31. If 1*s.* is worth 3·66 dg. of pure gold, 1 rup. is worth 106·92 dg. of pure silver, and 1 oz. of gold is worth 33·31 oz. of silver, what is the value of a rupee in English money? (S.A.)

32. Which will be more profitable for me, to sell 10050 kg. of asphalte in Germany at 3 m. 20 pf. per 100 kg., the exchange being 20 m. 25 pf. to the £; or to sell it in England at 40*s.* per ton, 1015 kg. being equivalent to 1 ton? Show that the difference thus calculated will be not quite £1.
(C.P.)

33. How many English sovereigns must I have in order that, if I take them to France and change them into francs, etc., at an exchange of 25 fr. 45 c. for each sovereign, I can spend as much of them as is equal to £16 while there, and by changing the rest into sovereigns on leaving, at an exchange of 25 fr. 25 c., have as many sovereigns as I began with?
(C.S.)

34. In Berlin 500 g. of fine gold is worth 1392 m., and in London an ounce of standard gold is worth £3 17*s.* 9*d.* What should be the bank par of exchange between Berlin and London? (An ounce is 31·1035 g.)
(S.A.)

35. What is the least price in £ per ton that is an exact number of cents per kilogram; 1 dol. being 100 c., 4·85 dol. being £1, and 2 lbs. 3 oz. being 1 kg.? (S.A.)

CHAPTER XXXVIII.

MOTION: RACES.

The Rate, Velocity, or Speed of a body is measured by the distance over which the body moves uniformly in a unit of time.

Units of Time.—The usual ones are—

1. Miles per hour. 2. Feet per second.

Relative Rate or Velocity.—The relative rate of two bodies moving in the same straight line is the speed with which they approach or recede from each other. If the bodies move—

(1) *In the same direction*, the relative velocity is the *difference* of their rates.

(2) *In opposite directions*, it is the *sum* of their rates.

Formulæ.—For moving bodies we have the following relationships :—

1. Single body moving uniformly; time $= \frac{\text{distance}}{\text{rate}}$.

2. Two bodies moving in *opposite* directions; time $= \frac{\text{distance}}{\text{sum of rates}}$.

3. Two bodies moving in *same* direction; time $= \frac{\text{distance}}{\text{difference of rates}}$.

I. Rectilinear Motion.

(*a*) Opposite and Distant.

Example.—*A and B, who are* 45 *mi. apart, approach each other, A at the rate of* 5 *mi. an hour, and B at* 4 *mi. per hour. When and where will they meet?*

$$\text{Time} = \frac{\text{distance}}{\text{sum of rates}} = \frac{45}{5+4} = \tfrac{45}{9} = 5 \text{ hrs.}$$

Time of meeting = 5 hrs.
Then A's distance is 5 mi. × 5, or 25 mi.
and B's ,, 4 mi. × 5, or 20 ,,

The meeting-place, then, is 25 mi. from A's starting-point, or 20 mi. from B's.

Ans. They meet in 5 hrs.; 25 mi. from A's starting-point.

(*b*) Following and Distant.

Example.—*Two cyclists, A and B, are riding to the same place. A rides at the rate of* 10 *mi. an hour, while B averages* $12\frac{1}{2}$ *mi. an*

hour; A started $2\frac{1}{2}$ hrs. before B, and B just overtakes A as they reach their goal. How long was B on the road, and how far did he ride?

A has a start of $(2\frac{1}{2} \times 10)$ mi., or 25 mi.

$$\text{Time} = \frac{\text{distance}}{\text{difference of rates}} = \frac{25}{12\frac{1}{2} - 10} = \frac{25}{2\frac{1}{2}} = 10 \text{ hrs.}$$

Distance $= 12\frac{1}{2}$ mi. $\times 10 = 125$ mi.

Ans. B was 10 hrs. on the road, and rode 125 mi.

II. Circular Motion.

(*a*) Common Starting-point—Opposite Directions.

Example.—*Suppose it to be 40 mi. round the circumference of a circle; A and B start from S and go in opposite directions, A at the rate of 5 mi. an hour, and B at the rate of 3 mi. an hour. When and where do they meet?*

1. *To find when they meet.*—If the circumference of the circle be stretched out into a straight line, the problem becomes the same as that for a straight line, opposite and distant (p. 358). Hence—

$$\text{Time in hours} = \frac{\text{distance}}{\text{sum of rates}} = \frac{40}{5+3} = \frac{40}{8} = 5$$

2. *To find where they meet.*

A goes 5 mi. $\times 5 = 25$ mi.
B ,, 5 ,, $\times 3 = 15$,,
i.e. 25 mi. round the circle once from A's starting-point
or 15 ,, ,, ,, B's ,,

Ans. They meet 5 hrs. after starting; 25 mi. from A's starting-point.

(*b*) Different Starting-points—Same Direction.

Example.—*Suppose the rates and distance to be as in the previous example, but that A gives B 10 mi. start. When and where does A catch B?*

Straighten out the circumference of the circle into a straight line; the problem then becomes the same as that for a straight line, "distant and following" (p. 358).

$$\text{Time} = \frac{\text{distance}}{\text{difference of rates}} = \frac{10}{5-3} = \frac{10}{2} = 5 \text{ hrs.}$$

How far does A go before he catches B?

A's distance $= 5$ mi. $\times 5 = 25$ mi.
for if B travels for 5 hrs. at 3 mi. per hour, B's distance $= 3$ mi. $\times 5 = 15$ mi.
and the difference between the two distances $(25 - 15) = 10$ mi., which was B's start.

Ans. A catches B in 5 hrs., after riding 25 mi.

Handicap Problems.

Example.—*A gives B 150 yds. start in a race of 2 mi., and he loses by 10 yds. How many yards ought A to have given B?*

Since A is beaten by 10 yds.; he has run is $(2 \times 1760) - 10$ } the distance in yds. } $= 3520 - 10 = 3510$ yds.

Distance in yds. run by B $= 3520 - 150 = 3370$ yds.

Then A runs 3510 yds. while B runs 3370 yds.

and A ,, 1 yd. ,, B ,, $\frac{3370}{3510}$ yd.

and A ,, 3520 yds. ,, B ,, $\frac{3520 \times 3370}{3510}$ or 3379·6 yds.

Start to be given = 3520 yds. − 3379·6 yds. = 140·4 yds. *Ans.*

Example.—*A hare starts to run when a dog is* 44 *yds. off, at the rate of* 12 *mi. an hour. After* $\frac{1}{2}$ *min. the dog sees her, and pursues at the rate of* 16 *mi. an hour. How soon will he overtake her, and how far will the dog have run?* (R.M.A.)

Distance hare travels in $\frac{1}{2}$ min. $= \frac{12}{120}$ mi. $= \frac{1}{10}$ mi. $= 176$ yds.

$\therefore$ hare's total start $= 44$ yds. $+ 176$ yds. $= 220$ yds. $= \frac{1}{8}$ mi.

The dog's rate is 4 mi. per hour faster than the hare's.

If dog gains 4 mi. in 1 hr., how long would he take to gain $\frac{1}{8}$ mi.?

Time in hours $= \frac{\frac{1}{8}}{4} = \frac{1}{32}$, and $\frac{1}{32}$ hr. $= 1\frac{7}{8}$ min.

Distance run by the dog in $\frac{1}{32}$ hr. $= \frac{16}{32}$ mi. $= \frac{1}{2}$ mi.

Ans. Dog overtakes hare in $1\frac{7}{8}$ min., after running $\frac{1}{2}$ mi.

Problems connected with Streams.

Example.—*A person can row* 9 *mi. an hour on still water, and he finds it takes him just* 3 *times as long to row up as to row down the river. Find the rate of the stream.* (C.S.)

His rate *down* the stream in miles per hour } $=$ { his rate in miles per hour on still water + rate of stream in miles per hour

His rate *up* the stream in miles per hour } $=$ { his rate in miles per hour on still water − rate of stream in miles per hour

By question, the "up" rate is $\frac{1}{3}$ of the "down" rate.

i.e. (9 + rate of stream) miles per hour $= 3(9 -$ rate of stream) miles per hour

$= (27 - 3 \times$ rate of stream) miles per hour

i.e. $4 \times$ rate of stream $= 18$ mi. per hour

and rate of stream $= 18$ mi. $\div 4 = 4\frac{1}{2}$ mi.

Ans. $4\frac{1}{2}$ mi. per hour.

Example.—*Two watermen, rowing with a tide flowing at the rate of* $2\frac{1}{4}$ *mi. an hour, accomplish* 20 *mi. in* $3\frac{1}{5}$ *hrs. In what time could they row this distance against the ebb of the same tide?* (P.T.)

Number of miles due to tide in $3\frac{1}{5}$ hrs. $= 3\frac{1}{5} \times 2\frac{1}{4} = \frac{16}{5} \times \frac{9}{4} = \frac{36}{5}$

Then number of miles due to rowing $= 20 - \frac{36}{5} = \frac{64}{5}$

$\therefore$ rate of rowing in miles per hour $= \frac{64}{5} \div 3\frac{1}{5} = \frac{64}{5} \times \frac{5}{16} = 4$

Then net rate against tide in miles per hour $= 4 - 2\frac{1}{4} = 1\frac{3}{4}$

$\therefore$ time in hours for the return journey $= 20 \div 1\frac{3}{4} = 20 \times \frac{4}{7} = 11\frac{3}{7}$

Ans. $11\frac{3}{7}$ hrs.

THEORY.

1. Define rate, unit of time, relative rate, with examples.

2. Give formulæ for finding time: (*a*) single body moving uniformly; (*b*) two bodies moving in opposite directions; (*c*) two bodies moving in the same direction.

3. Show how problems on circular motion can be adapted for solution to similar problems on rectilinear motion. Give examples.

EXERCISE LV.

MOTION: RACES.

Motion in Straight Lines.

1. Two boats, A and B, row on the same course, A having the front place, the distance between A's stern and B's bow being 240 ft. at starting. If A row 34 strokes to the minute, clearing 29 ft. each stroke, and B row 33 strokes to the minute, clearing 32 ft. each stroke, find the shortest time which must elapse before B's bow touches A's stern. (C.P.)

2. A bicyclist, travelling with uniform velocity along a straight level road on a foggy day when objects were just visible a furlong off, saw a milestone in front of him for half a minute. Afterwards he overtook a carriage, the back of which he saw for a minute and a half: with what velocity (supposed uniform) was the carriage travelling? (C.P.)

3. On August 1, at noon, a steamer, A, was $12\frac{1}{2}$ mi. ahead of another, B; A got over 220 mi. in a day, while B travelled at the rate of $9\frac{1}{2}$ mi. an hour. On what date and at what hour was the distance between them again $12\frac{1}{2}$ mi., the course being the same and the speeds uniform? (C.S.)

4. A car and a cyclist start together on a journey of $5\frac{1}{2}$ mi., the car travelling uniformly. The cyclist while on the footpath travels at the rate of 38 yds. to 33 of the car, but on the road only at the rate of 10 yds. to 11 of the car. If they finish together, what distance of the journey did the cyclist ride on the road? (I.I.)

5. A and B run a mile race; at first A runs 11 yds. to B's 10, but after A has run $\frac{1}{4}$ mi. he gets tired, and runs 9 yds. in the time in which he at first ran 11, B running at his original rate. Which will win, and by how much? (P.T.)

Motion in Circles.

6. Three riders pass a given point on a circular track at the same instant, with velocities 10, 12, and 15 miles per hour respectively. How long, from that instant, must they continue to move round the track with those velocities before they next pass the same point abreast (the distance round the track for each rider being 704 yds.)? (C.P.)

7. The rates of two bicyclists are as 11 to 8. They start together from the winning-post and race round and round a circuit of 480 yds. It is noticed that the best man passes the other every 4 min., and that at the moment when the race ends they are passing the winning-post

together for the first time. How long did the race last, and how many minutes' start can the best man afford to give in a 9-mile race without losing? (C.H.L.)

8. Three bicyclists, A, B, and C, ride round a circular track half a mile in length, B receiving 240 yds. start, and C 720 yds. If A travels at the rate of 18 mi. an hour, and B at the rate of 16 mi. an hour, find at what rate C must travel in order that B may overtake him at the instant when A overtakes B. Find also what time will elapse before they are again all three abreast. (C.S.L.)

9. A and B run a race twice round a circular course of 450 yds. circumference. They start together, and run opposite ways. The faster runner A passes B for the second time at 9 yds. from the starting-post. B then quickens his pace by $\frac{1}{40}$ of his previous pace; while A slackens down to $\frac{39}{31}$ of his previous pace. Which will win the race, and by how many yards? (C.H.L.)

10. Two men, A and B, begin at the same time to run opposite ways once round a circular course, starting from the same point. From the moment at which they first meet A takes 63 sec. to finish the round, B 84 sec. But whereas A ran at a uniform rate the whole way, B mended his pace in the ratio of 4 : 3 from the moment at which they first met. Find the time each takes to run the whole course. (C.S.)

11. A and B start from the same point to run in opposite directions round a circular racecourse, 4324 ft. circumference, A not starting till B has run 716 ft. They pass each other when A has run 1927 ft. Which will come first to the starting-point, and what distance will they then be apart? (C.M.)

Streams.

12. A boat moves down stream at the rate of a mile in 6 min., and up stream at the rate of 6 mi. an hour. What is the velocity of the current? (C.S.L.)

13. A and B row on a river, starting simultaneously from the same point, A down stream, B up stream. In $3\frac{3}{4}$ min. they are 5 fur. apart; A then turns to follow B, and at the end of $12\frac{1}{2}$ min. from that time the boats have together rowed 1 mi. 7 fur. in all. If the speeds of A, B, and the stream are constant, how many miles an hour does the river flow? (C.S.)

14. I row against a stream flowing $1\frac{1}{2}$ mi. an hour to a certain point, and then turn back, stopping 2 mi. short of the place whence I originally started. If the whole time occupied in rowing be 2 hrs. 10 min., and my uniform speed in still water be $4\frac{1}{2}$ mi. an hour, find how far upstream I went. (R.M.A.)

15. A man starts to row on the still water of a canal, and completes a mile in $\frac{1}{4}$ of an hour; he then reaches a river, and, entering it, rows with the current $1\frac{1}{2}$ mi. in the next $\frac{1}{4}$ of an hour. How long will he take to return to the starting-point, if he rows back with the same strength as on his outward course? (C.S.)

16. A prisoner of war, in attempting to escape to a friendly ship moored within sight of land, swims at the rate of 3 mi. per hour, and is assisted by the tide, which ebbs at the rate of 200 yds. per hour. After

swimming for $\frac{3}{4}$ hr. the tide turns, and impedes him as much as it had previously assisted him. He also slackens his speed from 3 to $2\frac{1}{2}$ mi. per hour, and succeeds in reaching the vessel in another hour. How far was the vessel from the shore? (C.M.)

Handicaps.

17. At a game of skill A can give B 20 in 100, B can give C 15 in 100. How many can A give C? (C.M.)

18. A can walk 3 mi. while B walks 5 mi., and C can walk 6 mi. while A walks $3\frac{1}{2}$ mi. What start can C give B in a 3 mi. walk? (C.S.)

19. In a race B starts a certain distance ahead of A, and C an equal distance ahead of B. If A passes B in 12 min., and 9 min. afterwards passes C, in what time from the start will B pass C? (C.H.L.)

20. Three men run a mile race; the first beats the third by $76\frac{12}{23}$ yds.; the second beats the last by 11 sec.; and the pace of the first is to that of the second as 45 : 44. In what time does each run the mile? (C.S.)

21. In a half-mile race, if A gives B 10 yds. start, A wins by $10\frac{1}{2}$ sec., but if A gives B 15 sec. start, B wins by 22 yds. Find the time that each takes to run the $\frac{1}{2}$ mi. (O.S.L.)

22. A hare 50 leaps in advance of a greyhound, and taking 3 leaps for every 2 of the latter, is overtaken in 300 leaps of the greyhound. What is the proportion between the length of the leaps? (L.M.)

23. A greyhound pursues a hare which is 95 springs ahead; the greyhound makes 6 springs to the hare's 8, and 4 springs of the greyhound are equivalent to 7 of the hare. How many springs must the greyhound make to catch the hare? (I.I.)

Miscellaneous.

24. If two boats, A and B, row in a race at their usual speed, A will win by 80 yds.; but, the day proving unfavourable, A can only row at $\frac{8}{9}$ of its usual speed, while B rows at $\frac{9}{10}$ of its usual speed. A wins by 26 yds. Find the length of the course. (C.S.L.)

25. Two boats start for a race at 3 o'clock. The race is over at $6\frac{2}{3}$ min. after 3, the losing boat being 40 yds. behind at the finish: at 4 min. past 3 this boat was 700 yds. from the winning-post. Find the length of the course, and the speed (supposed uniform) of each boat in miles per hour. (R.M.A.)

26. A walks to a place at the rate of $4\frac{1}{2}$ mi. an hour; at 8 mi. from his destination he meets B, and turns back with him (walking at B's rate) for a mile; if A is half an hour late at his destination, what is B's rate, and at what rate should A have walked after parting with B so as to arrive at the proper time? (C.S.L.)

27. A man, with his dog by his side, is walking at $4\frac{1}{2}$ mi. an hour. The dog suddenly sees another man on the road before them, walking in the same direction at 4 mi. an hour, and 240 yds. off. The dog runs to this man, and (without stopping) back to his own master, at 10 mi. an hour. How long a time (in minutes) does this incident occupy? (C.W.)

28. A, B, and C run a race, the course consisting of three laps of a $\frac{1}{4}$ mi. each. A runs the first lap in 68 sec., and when he completes it he is $27\frac{1}{2}$ yds. behind C; B, when he begins the second lap, increases his pace by $\frac{1}{159}$ of his former pace, and when he completes it he is 4 yds. in front of A. Show that A and B start at the same pace, and find by how many seconds C will beat B. (C.P.)

29. A starts in pursuit of B, who has had $1\frac{1}{2}$ min. start. He overtakes and passes B, who at once stops. After proceeding further for 363 yds., A returns in pursuit of B, who at once starts off back again. If B be overtaken at the starting-place, find where he is first overtaken, his speed being $7\frac{1}{2}$ mi. an hour. (C.P.R.)

30. A and B walk at different uniform rates, such that, if A's rate were doubled and B's rate halved, then A could walk half as far again as B in the same time. If they start together, and walk along a straight road in the same direction, how far must B walk so as to be 1 mi. ahead of A? (C.P.)

31. Assuming that sound travels at the rate of 1100 ft. per second, and that a man walking towards a gun fired once a minute hears the fourteenth report $12\frac{27}{29}$ min. after hearing the first, find the rate in miles per hour at which the man is walking. (C.S.L.)

32. A, B, and C run a mile race; B has 85 yds. start, and C has 150 yds.: $3\frac{1}{4}$ min. after the start B passes C, and $\frac{1}{2}$ min. later A passes C; 200 yds. beyond the spot where A passed C he passes B; when A is 100 yds. from the winning-post he increases his pace by 6 ft. a second. Which will win, and what will be the position of the other two when the winner passes the winning-post? (O.S.L.)

33. A person started at half-past two and walked to a village, arriving there when the church clock indicated a quarter-past three. After staying 25 min., he drove back by a road one-fourth as long again, at a rate twice as fast as he had walked, and reached home by four o'clock. How far wrong was the church clock? (C.M.)

34. In a bicycle race, the rates of speed of A, B, and C are as $7 : 6\frac{3}{8} : 5\frac{3}{4}$, for the first half of the course. A slackens on completing this, and C increases his speed on doing so, B travelling the same rate throughout. The rates of speed consequently for the second half of the course are as $5\frac{1}{2} : 6\frac{3}{8} : 6\frac{3}{4}$. In what order do they reach the goal? (C.P.)

35. During the first part of a boat race $1\frac{1}{2}$ mi. long, the two crews are each rowing 34 strokes a minute, and when the leading crew has just completed half the distance the other crew is 40 ft. behind. The former now quickens its stroke to 35 a minute, and the latter takes a slower stroke of 33 a minute; but at each stroke the former boat travels a shorter distance than before by 4 per cent., and the latter boat a longer distance by 4 per cent. Which boat wins, and by how much (to the nearest inch)? (C.S.)

CHAPTER XXXIX.

MOTION: CLOCKS AND WATCHES.

Most problems on clocks and watches may be regarded as special forms of "race" questions between the minute-hand (M.H.) and the hour-hand (H.H.).

On the face of a clock or watch there are 12 hour-spaces (h.s.) and 60 minute-spaces (m.s.), and the M.H. moves 12 times as fast as the H.H., since it passes through 60 m.s. whilst the H.H. is passing through 5 m.s.

Hence gain of M.H. per hour $= 60 - 5 = 55$ m.s.

$\therefore$ M.H. gains 1 m.s. in $\frac{1}{55}$ hr., or in $1\frac{2}{11}$ min.

and it gains 60 m.s. in $\frac{60}{55}$ hr., or in $1\frac{1}{11}$ hr.

Hands coincident.

Example.—*At what time between* 3 *and* 4 *o'clock are the H.H. and M.H. of a watch together?* (O.S.L.)

At 3 the M.H. points to XII. } They are therefore
and the H.H. ,, III. } 15 m.s. apart.
Then M.H. has to gain 15 m.s.
But M.H. gains 55 m.s. in 1 hr.

$\therefore$ M.H. ,, 15 m.s. in $\frac{1 \times 15}{55}$ hrs. $= 16\frac{4}{11}$ min.

Ans. $16\frac{4}{11}$ past 3.

Hands opposite.—Remember that the hands can be opposite, *i.e.* pointing in the same straight line, *once* only within the space of any hour.

Example.—*At what time will the M.H. and H.H. of a clock be in one line opposite to each other between* 9 *and* 10 *o'clock?* (C.W.)

The hands are opposite when M.H. is 30 m.s. behind the H.H.
But at 9 o'clock the M.H. is 45 m.s. behind,
$\therefore$ M.H. must gain $(45 - 30)$ or 15 m.s.
Now, M.H. gains 55 m.s. in 1 hr.

$\therefore$ M.H. gains 15 m.s. in $\frac{1 \times 15}{55}$ hrs. $= 16\frac{4}{11}$ min.

Ans. $16\frac{4}{11}$ min. past 9.

Hands at Right Angles.—The hands may be at right angles *twice* within an hour, once when the M.H. is behind, and once when it is in front of the H.H.

Example.—*At what times between 2 and 3 o'clock are the H.H. and M.H. of a clock at right angles?* (L.M.)

The hands are at right angles when they are 15 m.s. apart.

(1) The case in which the M.H. is *behind* the H.H.

At 2 o'clock the M.H. is 10 m.s. behind H.H. Hence it will be at right angles when it has gained $(10 + 15)$ m.s.

$$\text{Time} = \frac{25 \times 60}{55} \text{ min.} = \tfrac{300}{11} \text{ min.} = 27\tfrac{3}{11} \text{ min.}$$

Ans. $27\frac{3}{11}$ min. past 2.

(2) The case in which the M.H. is *in front of* the H.H.

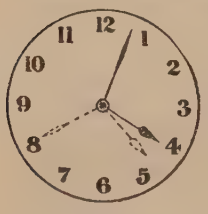
At $27\frac{3}{11}$ min. past 2 the M.H. is 15 m.s. in front of the H.H.

i.e. it has 30 m.s. more to gain, or, reckoning from 2 o'clock, the total to be gained is $(10 + 15 + 30)$ min. $= 55$ min.

$$\text{Time} = \frac{55 \times 60}{55} \text{ min.} = 60 \text{ min.}$$

Ans. 3 o'clock.

Hands a Given Number of Spaces apart.—The hands may be twice apart within the hour for any number of spaces less than 30.



Example.—*At what times between 4 and 5 o'clock will the hands of a watch be 17 m.s. apart?*

(1) At 4 o'clock M.H. is 20 m.s. *behind* the H.H., and it has to gain $(20 - 17)$, or 3 m.s.

$$\therefore \text{ time} = \frac{3 \times 60}{55} \text{ min.} = \tfrac{180}{11} \text{ min.} = 3\tfrac{3}{11} \text{ min.}$$

(2) When the M.H. is *in front of* the H.H. it has to gain $(20 + 17)$, or 37 m.s.

$$\therefore \text{ time} = \frac{37 \times 60}{55} \text{ min.} = \tfrac{444}{11} \text{ min.} = 40\tfrac{4}{11} \text{ min.}$$

Ans. $3\frac{3}{11}$ min. and $40\frac{4}{11}$ min. past 4.

Hands making a Given Angle, *e.g.* 60°.—Reduce the angle to m.s.

The easiest method of conversion is to remember that a circle contains 360°, and that therefore each m.s. $= 360° \div 60 = 6°$.

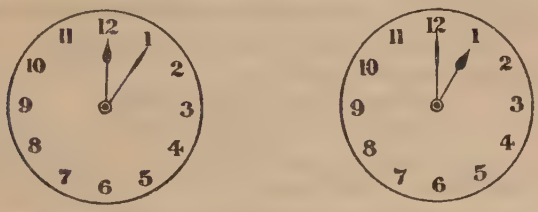
Then $60° = 10$ m.s.

The method of working is then similar to the last.

Position of Hands interchanged.

Suppose the H.H. to be behind the M.H. It moves on and occupies the position of the M.H. Meantime the M.H. has moved on round the clock, and

has taken up the position of the H.H. The hands have changed places, and *together have done the round of the clock.* But since the M.H. moves 12 times as fast as the H.H., the two hands must have travelled in the ratio of 12 : 1;



i.e. the M.H. must have journeyed $\frac{12}{13}$ of 60 m.s., and the H.H. $\frac{1}{13}$ of 60 m.s. But $\frac{1 \times 60}{13} = 4\frac{8}{13}$ m.s., and this must have been the original distance separating the hands.

Now, we know that the hands are together at 12, at 1 hr. $5\frac{5}{11}$ min., etc. Take any one of these coincident positions (say 12); then the M.H. has to gain $4\frac{8}{13}$ m.s.

M.H. gains 55 m.s. per hour
M.H. ,, 1 m.s. in $\frac{60}{55}$ min.
M.H. ,, $4\frac{8}{13}$ m.s. in $\frac{60}{55} \times 4\frac{8}{13}$ min. $= 5\frac{5}{143}$ min.

Then 12 hrs. $5\frac{5}{143}$ min. is the first position in which the hands can be interchanged.

Example.—*The hands of a clock show the time to be* 12 *hrs.* $5\frac{5}{143}$ *min. If the positions of the hands were interchanged, what would be the time then?*

Since 12 o'clock the M.H. has moved $5\frac{5}{143}$ m.s. How many m.s. has the H.H. moved?

Now, the M.H. moves 60 m.s. while H.H. moves 5
and M.H. ,, 1 m.s. ,, H.H. ,, $\frac{5}{60}$
∴ M.H. ,, $5\frac{5}{143}$ m.s. while H.H. moves $\frac{5}{60} \times 5\frac{5}{143}$ m.s.
$= \frac{60}{143}$ m.s.

The position of the hands, therefore, is M.H. $5\frac{5}{143}$, H.H. $\frac{60}{143}$ min. And interchanging we have—

M.H. $\frac{60}{143}$ m.s., H.H. $5\frac{5}{143}$ m.s. after 12 o'clock
∴ the time $= \frac{60}{143}$ min. past 1 o'clock. *Ans.*

To find Correct Time.—These problems, although complicated at times, are often mere applications of the Rule of Three.

Example.—*A watch, which is set correctly at noon, indicates* 10 *min. past* 9 *the same evening when the true time is* 10. *What is the true time when the watch indicates* 11 *on the same evening?* (C.M.)

When watch indicates 9 hrs. 10 min., the true time is 10 hrs.

,, ,, 1 hr. ,, ,, $\frac{10}{9\frac{1}{6}}$ hrs.

,, ,, 11 hrs. ,, ,, $\frac{10 \times 11}{9\frac{1}{6}} = 12$ hrs.

Ans. 12 p.m.

Example.—*A watch is* 70 *sec. slow at* 5 *p.m. on Sunday. On the next Sunday at noon it was* 100 *sec. fast. When was it correct?* (C.S.)

Sunday 5 p.m. to Sunday next at 12 noon $\Big\} = 1$ week $- 5$ hrs. $= (7 \times 24)$ hrs. $- 5$ hrs. $= 163$ hrs.

Hence the gain is $(70 + 100)$ sec., or 170 sec. in 163 hrs.

It was correct when it had gained 70 sec.

$$170 \text{ sec. : } 70 \text{ sec. :: } 163 \text{ hrs. : time}$$

$$\text{Time in hours} = \frac{70 \times 163}{170} = 67\tfrac{8}{17}$$

The watch was correct $67\frac{8}{17}$ hrs. after 5 p.m. on the first Sunday, or $72\frac{8}{17}$ hrs. after noon on the first Sunday.

$$\text{Now, } 72\tfrac{8}{17} \text{ hrs.} = 3 \text{ days } 0 \text{ hrs. } 7 \text{ min. } 3\tfrac{9}{17} \text{ sec.}$$

$\therefore$ watch was correct at 7 min. $3\frac{9}{17}$ sec. past noon on Wednesday. *Ans.*

EXERCISE LVI.

Motion: Clocks and Watches.

1. At what exact times will the hands of a clock be *at right angles?*—(*a*) Between 5 and 6 o'clock (C.S., C.P.); (*b*) between 6 and 7 o'clock (P.T.); (*c*) between 10 and 12 o'clock (C.S.).

2. At what time will the hands of a clock be *opposite* to each other?—(*a*) Between 9 and 10 o'clock (C.W.); (*b*) between 10 and 12 o'clock (C.S.); (*c*) between 3 and 4 o'clock (K.S.S.).

3. At what moment between 9 and 10 o'clock is the H.H. of a clock $15\frac{2}{3}$ minute divisions in advance of the M.H.? (C.S.)

4. Find the times between 3 and 4 o'clock when the angle between the hands of a watch is $\frac{1}{3}$ of a right angle. (C.M.)

5. What time between 4 and 5 o'clock will the hands of a watch be 17 minute divisions apart? (P.P.)

6. Find the exact time between 5 and 6 o'clock when the M.H. is $\frac{1}{6}$ of the circumference of the dial in advance of the H.H. (C.S.)

7. Find the exact time when the hands of a watch will coincide—(*a*) Between 7 and 8 o'clock (C.W.); (*b*) between 6 and 7 o'clock (K.S.S.); (*c*) between 3 and 4 o'clock (K.S.S.); (*d*) next after 12 o'clock (L.M.).

8. Two clocks are together at 12 o'clock; one loses 7 sec. and the other gains 8 sec. in 12 hrs. When will one be half an hour before the other, and what o'clock will it then show? (R.M.A.)

9. A watch at 10 o'clock on Monday morning is 10 min. 54 sec. slow, but it gains ·2625 of a minute in 3·5 hrs. What time will it show at 10 o'clock on the following Saturday night? (C.W.)

10. If a watch, which is 10 min. too fast at noon on Monday, gains 3 min. 12 sec. every 24 hrs., what will be the time by it at a quarter-past ten on the following Saturday morning? (C.W.)

11. Two clocks, one of which gains as much as the other loses—viz. 1 min. 15 sec. in 24 hrs.—are set right at noon on April 1. When is there a difference of 48 min. between them? (K.S.S.)

12. Three watches are set together. The first gains 5 min. a week, the second gains 8 min. a week, whilst the third loses 4 min. a week. When will they again indicate the same time? (L.M.)

13. The hands of a clock are put back $\frac{1}{4}$ hr. between 7 and 8 o'clock, thereby making the hands exactly coincide. What will be the true time when they are next together? (C.S.)

14. One clock gains 4 min. in 24 hrs., and another loses 2 min. in 36 hrs. Both are set right at noon. When one is 15 min. before the other, what will be the correct time? (C.S.)

15. A watch set accurately at noon indicates 10 min. to 7 at 7 o'clock p.m. What is the true time when the watch indicates 7 p.m.; and when will the watch first indicate the true time again? (C.S.)

16. A watch shows the true time at 6 o'clock on Sunday morning, and at noon on Tuesday has gained 24 min. Find what the true time will be when the watch shows 1 o'clock on Friday afternoon. (C.W.S.)

17. A clock which was 1·4 min. fast at a quarter to 11 p.m. on December 2 was 8 min. slow at 9 a.m. on December 7. When was it exactly right? (C.PR.)

18. A timepiece started at 12 o'clock loses at the rate of 4 per cent. What is the real time when the hands are next again together at 12? (C.W.)

19. A clock which goes steadily is noticed on Sunday at 10 a.m. to be 140 sec. slow. On the succeeding Sunday afternoon, at 24 min. past 3, it is found to be 109 sec. fast. When was it exactly right? (C.P.)

20. A watch which gains 5 sec. in every 3 min. was set right at 6 a.m. What was the true time in the afternoon of the same day when the watch indicated a quarter-past 3 o'clock? (O. and C.S.C.)

21. One clock gains 7 sec. a day, a second loses 8 sec. a day. If they are set right at noon, what is the true time when one of them is 5 min. before the other? (C.W.)

22. If a clock gains 2 min. 24 sec. in 24 hrs. of true time, and you wish to have it right at 6 p.m. on Thursday, how would you arrange the hands at 11 a.m. on the previous Tuesday? (I.I.)

23. At noon on Monday, one clock is 1 min. slow, and another 2 min. fast; at 8 on Wednesday morning the first is right, and the second is 1 min. slow. When was the second right, and when were they together? (S.L.C.)

24. If the hands of a clock coincide every 66 min., how much does the clock lose or gain in a day? (C.PR.)

25. The minute-hand of a clock overtakes the hour-hand at intervals of 64 min. of true time. How much a day does the clock gain or lose? (O. and C.H.C.)

26. The hands of a clock are at right angles to one another every 33 min. Does the clock gain or lose, and in what correct time will it gain or lose half an hour? (C.W.)

27. Of two clocks set right at 12 o'clock, one gains 50 sec. and the other loses 40 sec. per hour. What will be the time by the second clock when that shown by the first is 26 min. past 2? (C.M.)

28. A clock's rate is ·9 sec. per hour faster when a fire is not alight than when the fire is alight. If it neither gains nor loses on the whole when the fire is alight for 17 hrs. out of 24, find how much it will gain in a week if the fire is not lit at all. (C.H.L.)

29. Two clocks strike 9 together on Monday morning. On Tuesday morning one wants 8 min. to 11 when the other strikes 11. How much must the slower be put on that they may strike 9 together in the evening? (C.S.)

30. A clock loses at the rate of 8·5 sec. an hour when the fire is alight, and gains at the rate of 5·1 sec. an hour when the fire is not burning; but on the whole it neither loses nor gains. How long in the 24 hrs. is the fire burning? (C.S.L.)

31. A clock is so made that its hands go round in directions which are opposite to each other. At noon these hands agree in position with those of an ordinary clock. At what time between 3 and 4 o'clock are the hands together? (K.S.)

32. What is the exact time between 6 and 7 o'clock at which the *hour* and *minute* hands of a clock occupy the same positions as were occupied by the *minute* and *hour* hands respectively at a certain instant between 2 and 3 o'clock? (C.P.)

33. Two clocks point to 2 o'clock at the same instant on the afternoon of Christmas Day; one loses 8 sec., and the other gains 9 sec. in 24 hrs. When will one be half an hour before the other, and what time will each clock then show? (C.S.)

34. I have two watches, one of which gains 1 min. in one revolution of its H.H., and the other loses 1 min. in one revolution of its H.H. At noon to-day, the first denotes 10 min. to 12, and the second 10 min. past 12. Find at what true time they will first indicate the same time. (C.P.)

35. Two clocks, A and B, have seconds pendulums. In 11 min. B makes three ticks more than A (the 11 min. being on the slower clock). Find B's daily rate of gain or loss if A lose 10 min. a day. (C.S.)

CHAPTER XL.

MOTION: PROBLEMS ON TRAINS.

Many problems on Trains are applications of those on Motion. *The peculiarity of the problems is the introduction of bodies with length as the moving units.* The difficulty lies in the many relative positions two trains may assume to each other, and in the failure of some students to distinguish between absolute and relative velocity; *e.g.* if a train passes a stationary man, the train's rate is actual or absolute; but if the man be walking, or if he be in a moving train, then the rate is relative.

Perhaps the chief difficulty in these problems is to recognize the distance moved by two trains when clearing each other; hence the student is advised to carefully study the diagrams given, and to note the following points:—

1. A long and a short train are chosen for greater ease of distinction, but, of course, the two trains could be of the same length.
2. The effective rate of two trains—
   (*a*) Is the *sum* of their respective rates when travelling in *opposite* directions.
   (*b*) Is the *difference* of their respective rates when travelling in the *same* direction.
3. Make one of the trains a fixed "point," and move the other to its new position. Then, to estimate the distance moved, compare the head (or end) of the moving train in its first position with its head (or end) in its new position.
4. As a train is a long body, it is necessary to add the lengths of the two trains to the distance to be covered to get the *total* distance.
5. The ordinary formula for motion is—

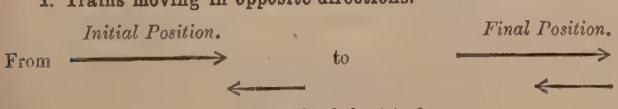
$$\text{Time} = \frac{\text{distance}}{\text{effective rate}}$$

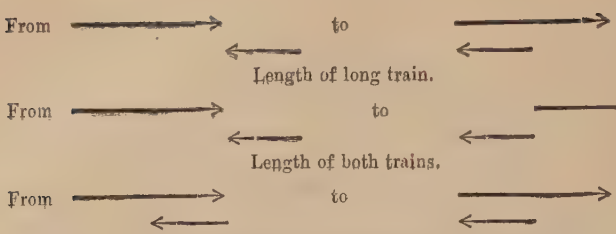
and this formula is useful in the solution of many of these problems.

Diagrams illustrating the Relative Positions of Two Trains, and the Distance moved in each case.

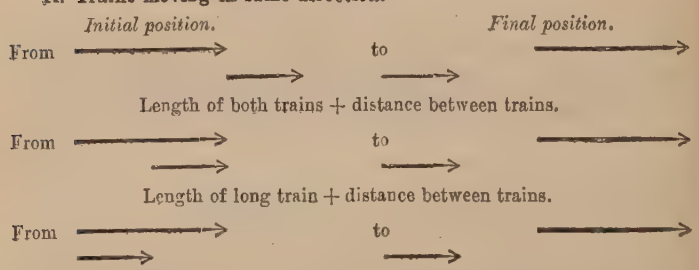
The wording under each position shows the distance moved in each case.

I. Trains moving in opposite directions.





II. Trains moving in same direction.



The diagrams given are typical. Diagrams might also be drawn, and the distance moved in each case determined in each of the following cases:—

Trains moving in **opposite** *directions.*

1. *Initial position*—some distance apart.
   Final ,, —just met, head to head.
2. *Initial* ,, —the end of the short train just level with the head of the long train.
   Final ,, —the ends together.
3. *Initial* ,, —head of short train level with end of long train.
   Final ,, —ends together.

Trains moving in the **same** *direction.*

1. *Initial position*—end of short train level with head of long train.
   Final ,, —heads of the two trains level.
2. *Initial* ,, as in 1.
   Final ,, —ends of the two trains level.
3. *Initial* ,, as in 1.
   Final ,, —head of short train level with end of long train.
4. *Initial* ,, —heads level.
   Final ,, —ends level.
5. *Initial* ,, —heads level.
   Final ,, —head of short train level with end of long train.
6. *Initial* ,, —ends level.
   Final ,, —head of short train level with end of long train.
7. *Initial* ,, —head of short train level with end of long train.
   Final ,, —head of short train some distance behind the end of long train.

Motion in Opposite Directions.

Example.—*Two trains, one going* 40 *and the other* 30 *miles an hour, meet on parallel lines, each with an engine and tender* 40 *feet long; the first has* 12, *the second* 17 *carriages, each carriage being* 32 *feet, and the coupling spaces* 5 *feet. Find the time from the moment the engines meet until the last carriages have passed each other.* (C.S.)

The relative positions are as follows:—

From ⟶ to ⟵ ⟵ ⟶

Hence the distance moved is the length of both trains.

Length of short train $= (40 + 12 \times 32 + 11 \times 5)$ ft.

,, long ,, $= (40 + 17 \times 32 + 16 \times 5)$ ft.

Then time in hours $= \frac{\text{distance}}{\text{rate}} = \frac{80 + 29 \times 32 + 27 \times 5}{(40 + 30) \times 5280}$

$$= \frac{(80 + 928 + 135)60 \times 60}{70 \times 5280} \text{ sec.}$$

$$= \frac{1143 \times 60 \times 60}{70 \times 5280} \text{ sec.} = \tfrac{82420}{308} \text{ sec.}$$

$$= 11\tfrac{41}{308} \text{ sec.} \quad \textit{Ans.}$$

Example.—*Two trains start at the same time from London and York, and proceed towards each other at the rates of* 30 *and* 50 *miles an hour respectively. When they meet, the quicker train has run* 50 *miles farther than the other. What is the distance between London and York?* (C.P.)

Difference in rates $= (50 - 30)$ mi. $= 20$ mi. per hour

Time of meeting in hours $= 50 \div 20 = 2\frac{1}{2}$

Distance by fast train $= 50$ mi. $\times 2\frac{1}{2} = 125$ mi.

,, slow ,, $= 30$ mi. $\times 2\frac{1}{2} = 75$ mi.

whole distance $= (125 + 75)$ mi. $= 200$ mi. *Ans.*

Motion in Same Direction.

Example.—*If two trains on the separate parallel lines of a railway be respectively* 150 *ft. and* 200 *ft. long, and travel uniformly, the former at* 25 *and the latter at* 20 *miles an hour, in what time will they clear each other exactly when travelling in the same direction?* (C.P.)

The relative positions of the trains are—

From ⟶ to ⟶ ⟶

Hence the faster and smaller train has to gain the length of the two trains.

The effective rate $= (25 - 20)$ mi. $= 5$ mi. per hour $= \frac{22}{3}$ ft. per second

then time in seconds $= \frac{\text{distance}}{\text{rate}} = \frac{150 + 200}{\frac{22}{3}} = \frac{350 \times 3}{22} = \frac{525}{11} = 47\frac{8}{11}$

Ans. $47\frac{8}{11}$ sec.

Example.—*If a train* 90 *yds. long overtakes a person walking at the rate of* $3\frac{3}{4}$ *mi. an hour along the line, and passes him completely in* 10 *sec., what is the rate of the train per hour?*

In 10 sec. the man walks $\frac{10}{60 \times 60} \times \frac{15}{4}$ mi.

$\therefore$ the whole distance the train travels in 10 sec. $= \left(\frac{90}{1760} + \frac{10 \times 15}{60 \times 60 \times 4}\right)$ mi.

$= (\frac{9}{176} + \frac{1}{960})$ mi. $= \frac{54 + 11}{1056} = \frac{65}{1056}$ mi.

and time $= 10$ sec. $= \frac{1}{360}$ hr.

then rate in miles per hour $= \frac{\text{distance}}{\text{time}} = \frac{65}{1056} \div \frac{1}{360}$

$= \frac{65 \times 360}{1056} = \frac{975}{44} = 22\frac{7}{44}$

Ans. $22\frac{7}{44}$ mi. per hour.

THEORY.

1. What is the peculiarity of problems on trains, and wherein does their chief difficulty lie?

2. Distinguish between absolute and relative velocity in connection with problems on trains.

3. How do long bodies, like trains, affect the solution of "train" problems generally both for distance and rate?

EXERCISE LVII.

Motion: Trains.

1. Two trains start at the same time from Sheffield and Leicester, and proceed towards each other at the rates of 30 and 50 mi. per hour respectively. When they meet it is found that one train has run $14\frac{1}{2}$ mi. farther than the other. Find the distance between Sheffield and Leicester. (P.T.)

2. Two trains start at the same time from London and Edinburgh, and proceed towards each other at the rates of 30 and 50 mi. an hour respectively. When they meet the quicker train has run 100 mi. farther than the other. What is the distance between London and Edinburgh?

3. A passenger train moving at the rate of 45 mi. per hour overtakes a mineral train $1\frac{1}{2}$ times as long moving at the rate of 27 mi. per hour, and passes it completely in 25 sec. How long would the passenger train take to pass completely through a station 165 yds. in length? (C.S.L.)

4. A train 66 yds. long, moving at a uniform rate, overtakes a man who is walking along the line at the rate of 4 mi. an hour, and passes him in 6 sec. Twenty minutes after it has passed him it meets a man

walking in the opposite direction, and passes him in $4\frac{1}{2}$ sec. How long will it be after this before the two men meet? (O.S.L.)

5. A train going at the rate of 60 mi. an hour overtakes another train, 160 yds. long, going 45 mi. an hour in the same direction on a parallel line, and passes it in half a minute. In what time would the first train pass another, of its own length, going in the opposite direction at the rate of 30 mi. an hour? (C.S.)

6. Grantham is 105 mi. from London, and two trains start simultaneously at noon, one from London for Grantham, and the other from Grantham for London. After 48 min. they are still 23 mi. apart. Find when they pass one another. If the passing-place is $2\frac{2}{11}$ mi. nearer to Grantham than to London, find the time at which each train arrives at its destination. (C.H.L.)

7. Two men walk with uniform speed along a railway in the same direction. A train, travelling uniformly, overtakes one of them, who walks at 4 mi. per hour, and the whole train, which is 132 yds. long, passes him in $7\frac{1}{2}$ sec. $6\frac{3}{4}$ min. after it began to pass the above-named man, the train begins to pass the other, and occupies $7\frac{1}{3}$ sec. in passing him. When will one man overtake the other? (C.S.)

8. A train leaves London for Brighton at 9 a.m., travelling at a uniform rate of 15 mi. an hour. An express train leaves Brighton for London at 10 a.m., and travels at a rate of 40 mi. an hour. At what time will they pass each other, and at what distance from London, the distance from London to Brighton being 50 mi.? (R.M.A.)

9. Three trains, 200, 528, and 130 yds. long, are travelling on parallel lines at 15, 24, and 30 mi. per hour respectively, the first two in the same direction, the third meeting them; the second reaches the rear of the first train at the same moment as the third reaches its front. Find the interval in time between the moments when each of the two trains clears the third. (C.M.)

10. A train passed a man standing at the entrance of a tunnel in $11\frac{1}{4}$ sec., and passed completely through the tunnel, which was 682 yds. long, in 50 sec. What was the train's length and speed per hour? (C.M.)

11. A passenger and a goods train start from the ends of a line 120 mi. long, and meet in the middle of it; the goods train started an hour earlier than the other, and goes at $\frac{2}{3}$ its speed. Find the rate of each. (C.S.)

12. A man sitting in a train which is travelling at the rate of 50 mi. an hour observes that it takes 6 sec. for a luggage train travelling in the opposite direction to pass him. If the luggage train is 220 yds. long, find the rate at which it is travelling. (K.S.)

13. A train 88 yds. long overtook a person walking along the line at the rate of 4 mi. an hour, and passed him completely in 10 sec.; it afterwards overtook another person, and passed him in 9 sec. At what rate per hour was this second person walking? (O.R.)

14. A train 88 yds. long, and travelling at a uniform speed, passes a telegraph post by the line in 6 sec. It then overtakes a train 110 yds.

long, travelling in the same direction on a straight parallel track, and passes it in 81 sec. What was the speed in miles per hour of the second train? (C.M.)

15. A person standing on a railway platform 264 yds. long noticed that a train passed the platform in 20 sec., and himself in 8 sec. Find the length of the train and its rate per hour. (K.S.S.)

16. A train travelling at the rate of 40 mi. an hour whilst inside a tunnel meets another train of half its length travelling at 60 mi. an hour, and passes it completely in $4\frac{1}{2}$ sec. Find the length of the tunnel, if the first train passes completely through it in 4 min. $37\frac{1}{2}$ sec. (C.J.)

17. An observer on the platform of a railway station noticed that a train passed him in 7 sec., and that it passed completely through the station, which was 396 yds. long, in 25 sec. How long was the train, and how fast was it moving? (P.T.)

18. If a train 90 yds. long, going 40 mi. an hour, meets another train going 35 mi. an hour, and they pass one another in 4·8 sec., what is the length of the other train? And how long would the faster train have taken to pass the other if they had been going in the same direction?

19. If two trains on the separate parallel lines of a railway be respectively 150 ft. and 200 ft. long, and travel uniformly, the former at 25, and the latter at 20 mi. an hour, in what time will they clear each other exactly when travelling (i.) in the same, and (ii.) in contrary directions? Explain your process. (C.P.)

20. A fast train leaves a place, A, for another place, B, at the same moment that a slow train leaves B for A. The fast train takes 2 hrs. on the journey, and if the slow train kept its time they would meet at a distance from B equal to $\frac{3}{8}$ the whole distance from A to B. Instead of this, the place where they meet is at a distance from B equal to $\frac{1}{4}$ the distance from A to B. How much behind time will the slow train be when it arrives at A? (C.S.)

CHAPTER XLI.

THE CALENDAR.

THE **Calendar** is the name given to the orderly arrangement of the natural divisions of time with respect to one another for the purposes of civil life.

Leap Year.—The earth makes a complete revolution round the sun in 365 days 5 hrs. 48 min. 46 sec. Hence the ordinary or calendar year of 365 days is about $\frac{1}{4}$ day too short. Julius Cæsar, assuming $365\frac{1}{4}$ days to be the true length, added a day in every fourth year, which became known as a Leap Year; but accuracy was not obtained even then, for—

The length of the mean solar year = 365·242218 days
and $365\frac{1}{4}$ days = 365·25 ,,
∴ the excess = ·007782 ,,

Now, in four centuries this excess amounts to—

400 × ·007782 or 3·1128 days

The Gregorian Calendar.

In 1582 Pope Gregory XIII. introduced a new calendar—now known as the New Style—and corrected this error by calling three leap years in every four centuries common or ordinary years. By this arrangement the error is reduced to 3·1124 days − 3 days, or ·1124 of a day in 400 yrs. The leap years thus missed are the centuries whose first two digits are not divisible by 4, such as 1700, 1800, 1900, etc.; for 17, 18, etc., are not divisible by 4. Allowance must be made for this in the working of some problems.

Rules for the Determination of Leap Year.

1. The last two digits of the given date should be exactly divisible by 4; *e.g.* 1896 is a leap year, since 96 is divisible by 4.

2. If the date is a multiple of 100, then the digits denoting the number of the century must be divisible by 4.

1600 is a leap year, since 16 is divisible by 4; 1900 is not a leap year, although divisible by 4, since 19 is not divisible by 4.

Example.—*If the length of the year be* 365·242264 *days, and if there be no leap year in a century year, unless the number of the century be a multiple of four, find the error that will have accumulated under the Gregorian system between the beginning of* 1852 *and the end of the nineteenth century.* (C.P.)

Number of years from January 1, 1852, to December 31, 1900 } = 49

Number of days in 49 yrs. = 365·242264 × 49 = 17896·870936

Number of leap years in 49 yrs. = 49 ÷ 4 = 12

∴ number of days in 49 calendar years = (365 × 49) + 12 = 17885 + 12 = 17897

Difference = 17897 days − 17896·870936 days
= ·129064 day

and ·129064 day = 3 hrs. 5 min. 51·1296 sec. *Ans.*

Example.—*Find on which day in the week March* 19, 1853, *fell, given that November* 10, 1902, *was on a Monday.*

From 19 iii. 53 to 10 xi. 53 = (12+30+31+30+31+31+30+31+10 days)
= 236 days

From 1853 to 1902 = 49 yrs.
= 38 ordinary + 11 leap years

Then number of days from March to November = 236
,, ,, in 38 yrs. (365 days) = 13870
,, ,, in 11 ,, (366 days) = 4026

7)18132 days

2590 wks. + 2 days

Ans. Saturday (2 days before Monday).

Example.—*The periodical "Nature" is published every Thursday. On June* 1, 1893, *No.* 1231 *appeared. On what day of the month in what year did No.* 1 *appear?* (C.P.)

1231 wks. × 7 = 8617 days

There is a period of 7 days between each issue *after* the first, ∴ 7 days must be subtracted from 8617, which leaves 8610,

and 8610 ÷ 365 = 23 yrs. + 215 days

1893 − 23 = 1870

But number of leap years from 1870 to 1893 = 6

∴ 6 days must be subtracted from 215, which leaves 209

And 209 days back from June 1, 1870, gives—

1 + 31 + 30 + 31 + 28 + 31 + 31 + 26

This takes the date back 26 days in November, 1869; hence the required date is November 4, 1869. *Ans.*

THEORY.

1. Define the term "calendar."

2. What do you understand by the term "leap year"? Give its origin and comment on the degree of accuracy attaching to it.

3. What is the "New Style," and what influence has it upon the accurate measurement of time?

4. How would you find on which day in the week a certain date fell in the last century?

5. What was the "Old Style," and what amount of error did it involve in four centuries?

6. Give the two rules for the determination of leap year. (S.L.C.)

EXERCISE LVIII.

1. On what day of the week did January 1 fall in 1890, in 1897, and in 1900 respectively? (C.S.)

2. On what day will a child born June 1, 1897, have lived exactly 4000 days? (S.L.C.)

3. Saturday, December 31, 1870, was the last of 40000 days. Find the year, month, day of the month, and day of the week on which the first of the said 40000 days fell. (C.S.)

4. Show that the same days of the month will fall on the same days of the week in March and November in any one year.

5. Find the number of days between June 1, 1794, and October 21, 1804, and explain how you get your result. (S.K.)

6. In what years in the nineteenth century did February 29 fall on a Sunday? (L.M.)

7. The first day of June, 1901, was on a Saturday. Find the day of the week upon which the first day of June fell in 1850, and explain how you get your result. (S.K.)

8. On May 15, 1872, a person had lived 30000 days. Find the year and day of the month of his birth. (C.W.)

9. In 1892 January 1 was a Friday. When did this occur again in that century, and when in the next, 1900 not being a leap year? Give all the steps of your reasoning. (C.S.)

10. A lamp burnt for 36720 min., and then went out at 3 a.m. on Monday, January 1, 1900. On what day of the week, and at what hour, was it lighted, and was it lighted a.m. or p.m.? (P.T.)

11. No. 1585 of a weekly publication came out on March 13, 1879. At what date did No. 1 appear? (C.W.)

12. On what day of the week did September 21, 1860, fall?

13. The *Times* issued on the morning of January 1, 1880, was numbered 29766. Find the number of the same journal that was published on the second Saturday in December, 1877. (C.S.)

14. If a boy receive 1*d.* on Saturday, January 1, 2*d.* on the following Saturday, 3*d.* the third Saturday, and so on to the last Saturday in the following December, how much will he have received within the time?

15. December 12, 1876, fell on a Tuesday, and the copy of the *Times* issued on December 8, 1877, was No. 29120. What day of the week was December 8, 1877, and what number was the copy of the *Times* published December 12, 1876? (C.S.)

16. The solar year contains 365·242218 days. Find to the nearest second the error produced in 2000 years by our calendar, which, by means of leap year, intercalates 97 days in 400 years. (P.E.A.)

17. The sidereal day, which is subdivided in the same way as an ordinary day, is equivalent to 23 hrs. 56 min. 4·09 sec. of ordinary time. Prove that the difference between 100 sidereal seconds and 100 ordinary seconds is a little more than a quarter of an ordinary second. (P.E.A.)

18. The solar year is determined to contain 365·242218 mean solar days, while the ordinary year contains 365 days. Show that it would be a closer correction to add 8 days in 33 yrs. than, as at present, to add 1 day in 4 yrs. Show, further, that with this correction the accumulated errors in 4000 yrs. would not amount to 1 day. (C.S.)

19. The length of the solar year being 365 days 5 hrs. 48 min. 50 sec., and the Chinese civil year consisting of 12 mths. of 29 and 30 days alternately, how many months of 30 days must be added in the course of 60 yrs. that the Chinese civil year and the solar year may agree most nearly? (K.S.)

20. In Old Style the mean solar year was $365\frac{1}{4}$ days, but in New Style, introduced in A.D. 1752, three leap years in four centuries were omitted. Find the average length of a year in New Style. If the year be taken actually to be 365·242218 days, show that the accumulated error in new style will not amount to 1 day in the year A.D. 5252. (C.W.)

CHAPTER XLII.

EQUATIONS.

AN **Equation** is a mathematical statement that two expressions are equal to each other.

Certain problems in arithmetic assume the form of equations, and as such they may be easily solved. Such problems also admit of solutions which are more or less pure arithmetic, and it is suggested that each problem should be worked by both methods.

An equation consists of two sets of terms—

(*a*) "*Known terms*," *i.e.* terms expressed in terms of some known unit.

(*b*) "*Unknown terms*," *i.e.* terms not so expressed.

The values of the unknown terms, when found, are said to "satisfy" the equation, and the process of finding the value of these unknown terms is called "solving the equation."

Kinds of Equations.

1. **Identical Equations**, or "Identities," as they are usually called. In these cases the statements are *universally true*, whatever values may be given to the unknown quantities; *e.g.* the statement that $(a + b)^2 = a^2 + 2ab + b^2$ is always true, whatever may be the values given to a and b.

2. **Conditional Equations**, or "Equations" simply, as they are generally known. The statements in this type of equation are true for certain values only of the unknown quantity; *e.g.* $a + 12 = 20$ is true only when $a = 8$. This kind only will concern us in this chapter.

A few Axioms connected with Equations.

1. If equals be added to equals the wholes are equal.
2. If equals be taken from equals the remainders are equal.
3. If equals be taken from unequals the remainders are unequal.
4. If equals are multiplied by equals the products are equal.
5. If equals are divided by equals the quotients are equal.

Processes based on the above Axioms.

1. *A term can be removed from one side of an equation to the other provided the sign preceding that term is changed.*

Let $a - b = \mathrm{N}$.

Then, adding b to each side—

$$a - b + b = \mathrm{N} + b \text{ (Axiom 1)}$$
$$\therefore a = \mathrm{N} + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

i.e. b has been moved from one side of the equation to the other. Note that it appears as $-b$ on the left-hand side, and reappears as $+b$ on the right-hand side.

Again, let $a + b = \mathrm{N}$.

Subtracting b from each side—

$$a + b - b = \mathrm{N} - b \text{ (Axiom 2)}$$
$$\therefore a = \mathrm{N} - b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

It will be noted that $+b$ is removed from the left-hand side of the equation, and reappears as $-b$ on the right-hand side.

2. *All the terms containing the unknown quantities should be placed on one side of the equation (usually the left), and all the terms containing known quantities should be placed on the other side (usually the right).*

This follows as a corollary of Process 1.

In the equations just given, if a is an unknown quantity, and N and b are known quantities, the unknown quantity can be arranged on one side, and the known quantities on the other. See (i.) and (ii.) above.

3. *It is advisable to clear off* (i.e. *to get rid of*) *fractions before endeavouring to solve an equation.*

This is done by multiplying each side of the equation by the L.C.M. of the denominators of the fractions.

If $\frac{2}{5}a + \frac{3}{4}b = \mathrm{N}$, where a is an unknown, and b and N are known quantities, then, multiplying by 12 (the L.C.M. of 3 and 4), axiom 4, we get—

$$8a + 9b = 12\mathrm{N}$$
$$\text{Then } 8a = 12\mathrm{N} - 9b$$
$$\text{and } a = \frac{12\mathrm{N} - 9b}{8a}$$

Example.—*An equal number of sovereigns, half-crowns, and pence amounts to* £19 3*s.* 11*d.* *What is the number?* (K.S.)

Let N = the number.

$$\text{Then } (240\mathrm{N} + 30\mathrm{N} + \mathrm{N})d = £19\ 3s.\ 11d. = 4607d.$$
$$\text{and } 271\mathrm{N} = 4607$$
$$\therefore \mathrm{N} = \frac{4607}{271} = 17$$

Ans. 17 of each.

Or, reducing all terms to the lowest denomination—

1 sovereign + 1 half-crown + 1*d.* = 240*d.* + 30*d.* + 1*d.* = 271*d.*
and £19 3*s.* 11*d.* = 4607*d.*
Then number of coins of each kind = 4607 ÷ 271 = 17

Ans. 17 of each kind.

Example.—*The sum of the ages of A and B is now* 60 *yrs., and their ages* 10 *yrs. ago were as* 5 : 3. *Find their present ages.* (C.PR.)

If N years be A's age, then (60 − N) yrs. is B's age. Ten years ago A's age was (N − 10) yrs., and B's age was (50 − N) yrs.

By question, N − 10 : 50 − N :: 5 : 3

$\therefore 3(N - 10) = (50 - N)5$

and $3N - 30 = 250 - 5N$

and $8N = 280$

$\therefore N = 280 \div 8 = 35$

$\therefore$ A's age = 35, and B's age = 25

Ans. A, 35 yrs.; B, 25 yrs.

THEORY.

1. What is an equation? Distinguish between identical and conditional equations.

2. On what axioms or first principles do equations depend for their solution?

3. Any term can be removed from one side of an equation to the other. Prove this.

4. What is meant by (*a*) satisfying an equation, (*b*) solving an equation?

5. Supposing the terms of an equation to contain fractions, what is the method of work?

EXERCISE LIX.

Equations.

1. Each member of a society subscribes as many pence as there are members in the society; the total subscription is £29 8s. How many members are there? (L.M.)

2. The sum of £23 is made up of half-sovereigns, half-crowns, and sixpences. There are three times as many half-crowns as half-sovereigns, and the number of half-sovereigns is to the number of sixpences as 3 : 10. How many are there of each coin? (K.S.S.)

3. If the ratio of threepenny to fourpenny pieces in a given sum, which consists entirely of those coins, were altered from 3 : 7 to 7 : 3 the sum would be diminished by £20. Find the sum. (K.S.)

4. A man finds that he has in his purse half-crowns, shillings, and sixpences. He calculates that the number of the half-crowns and the number of sixpences added together will exceed the number of the shillings by two; but that the value of the half-crowns and sixpences together is a shilling less than twice the value of the shillings; whilst the number of shillings in the value of the sixpences and shillings together is one less than double the number of half-crowns. How many coins of each kind has he? (L.M.)

5. If a person spends £50 more than $\frac{19}{20}$ of a fixed income in a certain year, £35 less than $\frac{14}{15}$ of it in the next year, and $1\frac{1}{2}$ of it in the year after that, and his savings amount to £705, what is his income? (C.S.)

6. On one day as many hundreds of oranges could be bought for half a sovereign as dozens could be bought for a shilling on the previous day. A woman bought 75 on each day, and paid altogether for them 5*s.* 6*d.* How many oranges could be bought for sixpence on the first day? (C.P.)

7. I have a certain sum of money wherewith to buy a certain number of nuts, and I find that if I buy at the rate of 40 a penny I shall spend 5*d.* too much, if 50 a penny 10*d.* too little. How much have I to spend? (C.PR.)

8. A man is 72 yrs. old, and his wife is 63. When they were married he was half as old again as his wife. How long have they been married? (K.S.)

9. The highest score in an innings was $\frac{2}{9}$ of the total, and the next highest was $\frac{3}{5}$ of the remainder. These scores differ by 8 runs. What was the total? (C.W.)

10. In a workshop a certain number of men receive £1 a week each, three times as many receive £1 6*s.* 9*d.* each, and seven times as many receive 15*s.* 3*d.* each. The total amount of their wages for a week is £31 1*s.* How many workmen are there? (C.W.)

11. The elder of two brothers inherited from his father $\frac{2}{5}$ more money than the other did. Since their father's death the elder brother has increased his capital to the extent of $\frac{3}{4}$ of its original value, and the younger brother has lost £707. He finds that he has now only £43 for every £100 that his elder brother has. How much had each of them at first? (C.S.L.)

12. A man adds to his capital annually £50 more than the eighth part of the original amount, and at the end of 6 yrs. finds that the whole amounts to £11675. What was the original capital? (C.S.)

13. Two brigands, A and B, robbed a traveller of £370; if A had taken £25 more, and B £25 less, A would have had $\frac{3}{5}$ of what B took. How much did each take? (P.T.)

14. On Monday a score of eggs cost as many pence as half a dozen chickens cost shillings. On Tuesday the price of eggs had fallen $12\frac{1}{2}$ per cent., and the price of chickens had risen 25 per cent.; and on that day a chicken and a score of eggs could be bought for 4*s.* 6*d.* What was the price of a chicken on Monday? (C.S.L.)

15. 700 eggs are sold partly at 16 for the shilling and partly at 18 for the shilling, and the total price is £2. Show that, if the numbers sold at each price had been interchanged, there would have been an increase of $6\frac{4}{17}$ per cent. in the total. (C.P.)

16. A farm consists of arable land let at 20*s.* per acre, and pasture let at 30*s.* per acre, the total rent being £350. When the rent of the arable land is reduced 5*s.* per acre, and that of the pasture 8*s.* per acre, the total rent is reduced by £90. Find how many acres are in the farm. (L.M.)

17. Of boys returning to a school at the beginning of a term, the number of upper-school boys was to the number of lower-school boys as 2 to 3. This ratio was maintained by promoting $\frac{1}{4}$ of the lower-school boys to the upper school, and 105 new boys joining the lower school. How many boys returned to the school? (C.P.)

18. A man 51 yrs. old has a son and a daughter. Three years ago his age was half as great again as that of his wife. The wife is five times as old as the daughter, and two years ago was three times as old as the son. What are the ages of the son and the daughter? (C.P.)

19. A, B, and C bought bacon at the same price per pound. A sold 105 lbs. for the same sum that B charged for 1 cwt., and the difference between their rates of profit was 8 per cent. How many pounds did C sell for the same sum if his rate of profit was double B's rate of profit? (C.P.)

20. During the shooting season a sportsman used 370 cartridges to kill 261 hares and partridges. He used on an average 3 cartridges to kill 2 hares, and 7 cartridges to kill 5 partridges. How many hares did he kill? (C.P.)

21. I pay £212 10*s.* for a carriage and a pair of horses, one of the horses costing half as much again as the other. If, however, each horse had cost 15 per cent. more, and the carriage 19 per cent. less, the whole amount to be paid would have been unaltered. Find the respective prices of each horse and of the carriage. (C.S.)

22. A house is offered for sale at an increase of 12 per cent. upon its cost price, but, not being disposed of, is offered again for £80 less than before. It being still unsold, this second offer is still further reduced by $2\frac{1}{2}$ per cent. of itself, and the house is then sold. If a profit of 4 per cent. is thus made, determine the original cost of the house. (O.S.L.)

23. A tax having been increased from 5 per cent. to 7 per cent., the revenue derived from it was found to be diminished by one-third. Find the diminution in the consumption of the article. (L.M.)

24. The uniform daily cost of dining alone is the same for each one of three men; but if two of them or three of them dine together, the cost is diminished at the rate of 4 per cent., or $11\frac{1}{9}$ per cent. for each man. A certain sum will pay the cost, either of three dining together for a certain time, or of two dining together for a week longer. For how many days will it pay the cost of one dining alone? (C.P.)

CHAPTER XLIII.

SQUARE ROOT.

Involution.—If we take a number and multiply it by itself any number of times, the operation is known as Involution; *e.g.* $2 \times 2 \times 2 \times 2 \times 2 = 32$, or, more briefly, $2^5 = 32$. It will thus be seen that *Involution is a form of continued multiplication, in which each factor is the same.*

Evolution.—The reverse of this operation is called Evolution. We are said to extract or *evolve* the root of the number, and the operation is one of finding a number which is the *root* of the given quantity; *e.g.* 8 is a root of 64, because $8 \times 8 = 64$. (See also p. 34.)

Square Root.—*The square root of a given number is either of two similar factors whose product is the given number; e.g.* 8 is the square root of 64, because $8 \times 8 = 64$. The square root of a given number, then, is such a number that, when multiplied by itself, produces the given number, and for any given number the square of its square root is the given number; *e.g.*—

Given number = 36; square root = 6; and $6^2 = 36$

Perfect Square.—By a perfect square is meant a number whose square root can be expressed *exactly* by an integer (or integers) or by a fraction; *e.g.* 121 is a perfect square, because its square root is 11; $\frac{9}{49}$ is a perfect square because its square root is $\frac{3}{7}$. But the majority of natural numbers are *not* exact or perfect squares, for an examination of the numbers from 1 to 100 gives only 10, and from 1 to 1000 only 31 perfect squares. *But every number may be a root,* because it may be raised or involved to any power; *i.e.* it may be multiplied by itself any number of times.

Some Properties of Perfect Squares.

1. Square roots: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 100
   Perfect squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 10,000

Note that *no perfect square ends in* 2, 3, 7, 8, for not one of the 9 digits, multiplied by itself, will give a product whose units' digit is either 2, 3, 7, or 8. Note also that *no perfect square ends* in an *odd* number of ciphers.

2. The number of integers in the root of a perfect square may be seen from a study of the following:—

$1^2 = 1$; $\sqrt{1} = 1$
$9^2 = 81$; $\sqrt{81} = 9$
$10^2 = 100$; $\sqrt{100} = 10$
$100^2 = 10000$; $\sqrt{10000} = 100$
$1000^2 = 1000000$; $\sqrt{1000000} = 1000$

(*a*) Squares consisting of 1 or 2 digits have 1 digit in the root.
(*b*) Squares consisting of 3 or 4 digits have 2 digits in the root.
(*c*) Squares of 5 or 6 digits have 3 digits in the root; and so on.

Thus perfect squares consisting of $(n-1)$ or n digits have $\frac{n}{2}$ digits in their root.

3. *Every perfect square of an even number must be divisible by* 4.

For since every even number is divisible by 2, any even number may be represented by $2n$, and the square of $2n$ is $4n^2$. Such a number will, therefore, always be divisible by 4.

4. *Every perfect square of an odd number must be divisible by* 4 *when the square is reduced by* 1.

5. *Every perfect square ending in* 5 *has a root also ending in* 5. *Furthermore, no number ending in* 5 *can be a perfect square unless the tens' digit is* 2.

Let a represent the tens' digit, and 5 the units' digit.

Then the number $= 10a + 5$

$$(10a+5)^2 = 100a^2 + (2 \times 5 \times 10a) + 5^2$$
$$= 100a^2 + 100a + 5^2$$
$$= \text{a certain number of hundreds} + 5^2$$
$$= \quad ,, \quad ,, \quad ,, \quad +25$$

$\therefore$ the tens' digit must always be 2, and the last digit must be 5.

Pointing in Square Root.—From what has been said (p. 385), it will have been noticed that—

a number having 1 digit has a square of 1 or 2 digits
,, ,, 2 digits ,, ,, 3 or 4 ,,
,, ,, 3 ,, ,, ,, 5 or 6 ,, etc.

A glance at the digits 1, 3, 5 or 2, 4, 6 shows that the common difference is 2; hence, if a point or dot be placed over every alternate digit (beginning at the right-hand digit) in the number whose square root is to be found, the number of points or dots will give the number of digits in the root; *e.g.*—

$\dot{2}\dot{9}16$ has 2 points, hence the root has 2 digits
$\dot{3}4\dot{9}9\dot{0}2\dot{5}$,, 4 ,, ,, ,, 4 ,,

It will be noticed in the last example that the odd digit (3) counts as a pair.

Method of extracting the Square Root.—The principle involved is not new. In the method of long division the remainder is diminished step by step by subtracting the several products of the divisor and the respective figures of the quotient. Similarly in square root the remainder is diminished by subtracting successive portions of the square. There is one important difference, however; *in square root the remainder may be greater than the divisor.*

The rule for the extraction of the square root is derived from the algebraical formula—

$$(a+b)^2 = a^2 + (2a+b)b$$

when written in this form—

$$(a+b)^2 - a^2 = (2a+b)b$$

Example.—*Find the square root of* 144.

Pointing $\dot{1}44$, we see that the root will contain 2 digits, *i.e.* the root must lie between 10 and 99.

Now, $10a^2 = 100$ and $20^2 = 400$; *i.e.* the square root of 144 must lie between 10 and 20, and 10 must be the tens' value of the root.

Now, let a = the tens' and b = the units' value of the root.

$$\text{Then } 144 = (a+b)^2 = a^2 + (2a+b)b$$
$$\text{Now, } a = 10 \text{ (proved above)}$$
$$\text{Then if } 144 = a^2 + (2a+b)b$$
$$\text{then } 100 = a^2$$

$$\text{And by subtraction, } 44 = (2a+b)b$$
$$\text{Then } \frac{44}{2a+b} = b$$

Now, b represents *units*, whereas a represents *tens*, i.e. b will be much smaller than a, and may be disregarded.

$$\text{Thus } \frac{44}{2a+b} = \frac{44}{2a} = \frac{44}{2 \times 10} = 2 \text{ approximately}$$

Hence the unit is probably 2.

Substituting in the equation $(2a+b)b = 44$ (given above)
we have $(2 \times 10 + 2)2 = 44$
i.e. $44 = 44$

∴ the unit *is* 2, and 10 has been found to be the tens' value.

Hence the square root = 10 + 2, or 12 *Ans.*

Example.—*Find the square root of* $10\dot{3}0\dot{4}\dot{1}$.

Pointing, we see that the root will contain 3 digits, *i.e.* the root lies between 100 and 999.

Now, $(300)^2 = 90000$ and $(400)^2 = 160000$ Hence the root lies between 300 and 400.

$$\text{Let } 103041 = a^2 + (2a+b)b$$
$$\text{and } (300)^2, \text{ or } 90000 = a^2$$

$$\text{then, subtracting, } 13041 = (2a+b)b$$

We have found a (300) the square root of a^2 (90,000), and we now require to find b. Thus 300 is the *first approximation*.

$$\text{Since } (2a+b)b = 13041$$
$$b = \frac{13041}{2a+b} = \frac{13041}{600+b}$$

Now, in the denominator b is always small compared with $2a$ (600). Then, if we divide the remainder by 600, we shall approximate to the correct number for the next digit in the root. Hence the *next approximation* $= \frac{13041}{600} = 20$, with a remainder.

(Note, as this approximation is for the tens' digit, the quotient must be a multiple of 10.)

$$\text{Now, } (2a+b)b = 13041$$
$$\text{But } (2 \times 300 + 20)20, \text{ or } 620 \times 20 = 12400$$

subtracting, we have 641

Now let $641 = (2a + b)b$, but note that both a and b have different values from those of the last approximation; a is now the whole quotient (320); b has to be found. We proceed as in the last step.

$$(2a + b)b = 641$$
$$\therefore b = \frac{641}{2a + b} = \frac{641}{2 \times 320 + b}$$

Now, disregarding b, $\frac{641}{640} = 1$ approximately; hence $b = 1$
then, since $(2a + b)b = 641$
$(2 \times 320 + 1)1 = 641$
i.e. $641 = 641$

And since there is no remainder, 1 is the *last approximation*. The three approximations (300 + 20 + 1) give the root, viz. 321. The process may be set out as follows :—

1. Long Form.

Remember that $103041 = a^2 + (2a + b)b = (a + b)^2$

$$\begin{array}{rl} & \dot{1}0\dot{3}0\dot{4}\dot{1}\ (300 + 20 + 1) \\ (300)^2 = & 90000 = a^2 \\ \hline & 13041 = 103041 - (300)^2 \\ (2 \times 300 + 20)20 = & 12400 = (2a + b)b \\ \hline & 641 = 103041 - (320)^2 \\ (2 \times 320 + 1)1 = & 641 = (2a + b)b \\ \hline \end{array}$$

Note the varying values of a and b.

2. Ordinary Form.

$$\begin{array}{r|l} & \dot{1}0\dot{3}0\dot{4}\dot{1}(321 \\ & 9 \\ \hline 62 & 130 \\ & 124 \\ \hline 641 & 641 \\ & 641 \\ \hline \end{array}$$

It will be noted that—

1. All unnecessary ciphers are omitted.
2. Periods are brought down as required.
3. The face value of the remainders and divisors is taken, and not their real values.

Method of Work.—Deduced from the "Long Form" above.

First Approximation.

1. Point the quantity. There are 3 dots; hence the root will contain 3 digits, and 3 approximations will be necessary.

2. The first period is 10, which lies between 9 and 16, the respective squares of 3 and 4. The first approximation, therefore, lies between 300 and 400.

3. Place 300 in the quotient as one part of the required root, and subtract its square (90000) from 103041. The remainder is 13041.

Second Approximation.

1. Divide 13041 by 2×300 for a new trial divisor, to find the second or tens' digit of the root.

2. The quotient is 20, with a remainder. Hence the remaining part of the root is less than 30.

3. Place 20 in the quotient as the second part of the root, and subtract $(2 \times 300 + 20)20$ or 12400 from 13041. The remainder is 641.

Third Approximation.

1. Divide 641 by 2×320 for a new trial divisor, to find the third or units' digit of the root.
2. The quotient is 1, with remainder 1.
3. Place 1 in the quotient as the third part of the root, and subtract $(2 \times 320 + 1)1$ from 641. There is no remainder. Hence 321 is the required root.

General Rule.—Deduced from the "Ordinary Form" of extracting a root.

1. Point the quantity.
2. Find the greatest square number contained in the first period (counting from the left), and subtract it from that period. Place the root of this greatest square number in the quotient as in ordinary division.
3. Bring down the next period and affix it to the remainder to form a new dividend.
4. For a new divisor double the root in the quotient, and divide this quantity into the whole of the dividend except the last figure. The result will be a digit. Affix this digit to both the quotient and the divisor.
5. Multiply the divisor thus formed by the digit last placed in the quotient.
6. Subtract the product from the last dividend, and proceed as before.
7. Repeat these operations until all the periods are exhausted.

To find the Square Root of a Decimal Fraction.

Since $\cdot1^2 = (\frac{1}{10})^2 = \frac{1}{100} = \cdot01 \quad \therefore \sqrt{\cdot\overline{01}} = \cdot1$

„ $(\cdot01)^2 = (\frac{1}{100})^2 = \frac{1}{10000} = \cdot0001 \quad \therefore \sqrt{\cdot\overline{0001}} = \cdot01$

„ $(\cdot001)^2 = (\frac{1}{1000})^2 = \frac{1}{1000000} = \cdot000001 \quad \therefore \sqrt{\cdot\overline{000001}} = \cdot001$, etc.

The Pointing of Decimal Places.—A study of the above figures will show that there are twice as many places in the square of a decimal number as in the root itself; hence *no decimal quantity ending in a significant digit can be the square of a decimal number, unless that decimal quantity contains an even number of places.* If the number of decimal places is odd, a cipher must be affixed to make an even number. The decimal can then be pointed from *left to right*—the reverse of the process for integers. The method of extracting the root is the same in either case.

Care should be exercised in the pointing of decimal places; *e.g.* $\sqrt{\cdot9}$ is not ·3. The rule for pointing will show this, ·9 being ·90, 00, 00, etc. The first period, therefore, is ·90, not ·9, and the first figure of the root will consequently be ·9.

Examples.—*Extract the square root of* ·03295 *correct to three places of decimals.*

$$\dot{0}3\dot{2}9\dot{5}\dot{0}\dot{0}(\cdot 181$$

1

28 | 229
224

361 | 550
361

189

Example.—$\sqrt{\cdot 0\dot{0}0\dot{6}\dot{3}}$.

$\cdot 0\dot{0}0\dot{6}3\dot{0}\dot{0}(\cdot 02509\ldots$

4

45 | 230
225

5009 | 50000
45081

4919

The first period is composed of 2 ciphers, which give a cipher in the root. The work then proceeds as usual from the first significant digit (6).

Example.—$\sqrt{3\cdot\dot{5}}$ *to three places of decimals.*

$3\cdot\dot{5}\dot{5}\dot{5}\dot{5}\dot{5}\dot{5}(1\cdot 885\ldots$

1

28 | 255
224

368 | 3155
2944

3765 | 21155
18825

2330

In extracting the root of a circulating decimal it is advisable to write the decimal in full, and to bring down the periods (55) as required.

Irrational Numbers.—Irrational or incommensurable quantities, or surds, are such as have not an exact root; that is, such numbers are not perfect squares. See the chapter on *Surds*.

To find the Square Root of an Irrational Number.

By the addition of ciphers the root may be carried out to any number of decimal places, and the farther the root is carried the greater its accuracy.

Example.—*Find the square root of* 3 *to* 5 *places of decimals.*

$3\cdot 0000000000(1\cdot 73205\ldots$

1

27 | 200
189

343 | 1100
1029

3462 | 7100
6924

346405 | 1760000
1732025

27975

	(*a*)	(*b*)
$\sqrt{3}$ lies between	1·0	and 2·0
it also ,,	1·7	,, 1·8
,,	1·73	,, 1·74
,, ,,	1·732	,, 1·733
,, ,,	1·7320	,, 1·7321
,, ,,	1·73205	,, 1·73206, etc.

Ans. 1·73205 . . .

Note.—1. The square of any one of the quantities in the right-hand column (b) would be > 3.

2. The square of any one of the quantities in the left-hand column (a) would be < 3.

3. The difference becomes less at each step. In the last case it is only $\frac{1}{1000000}$.

To find the Square Root of a Vulgar Fraction (see the chapter on *Surds*).

Some Useful Roots to remember. A knowledge of them will often save the labour of extracting the square root.

$\sqrt{2} = 1{\cdot}414213\ldots$ $\quad$ $\sqrt{3} = 1{\cdot}732050\ldots$

$\sqrt{5} = 2{\cdot}236068\ldots$ $\quad$ $\sqrt{6} = 2{\cdot}449490\ldots$

$\sqrt{10} = 3{\cdot}162278\ldots$

Square Root by Simple Division.—This is a labour-saving process for the extraction of the square root in long examples. If more than half the number of the digits in the root are found by the ordinary method, the remaining digits may be found by simple division.

Example.—$\sqrt{\cdot 4115}$ *to five places of decimals.*

Ordinary Method.

$\cdot\dot{4}1\dot{1}\dot{5}(\cdot 64148$

36

124 | 515

| 496

1281 | 1900

| 1281

12824 | 61900

| 51296

128288 | 1060400

| 1026304

34096

Method of Division.

$\cdot\dot{4}1\dot{1}\dot{5}(641$

36

124 | 515

| 496

1281 | 1900

| 1281

128) 619(48

512

107

102

4

Ans. ·64148.

Note.—1. The first 3 digits in the root are obtained in the ordinary way.

2. The quotient 641 is doubled, which gives 1282.

3. The last digit (2) is cast away. If this digit is greater than 5, the next digit is raised one unit; *e.g.* the divisor then would have been 129, and not 128.

4. The last remainder (619) is then divided by 128 in the ordinary way for the next two digits (48).

5. After each division one digit is cut off from the divisor, and no cyphers are added to the dividend.

Rule.—1. Double the part of the root already found.

2. Reject the units' digit.

3. If the rejected digit is greater than 5, raise the next digit to the next unit.

4. Divide the result into the remainder until the required number of digits is obtained, recollecting to cut off one digit from the divisor, or after each division.

Limit of the Division Rule.—When $(n + 1)$ digits have been found by the ordinary process, then n and sometimes $(n + 1)$ may be found by the division process, where n equals half the number of the digits in the required root. If $(n + 1)$ be greater than 5, then $(n + 1)$ more digits may be found by division; but if $(n + 1)$ be equal to or less than 5, then n only can be depended on.

Example.—*Find* $\sqrt{3}$ *to eight significant figures.*

		Contracted Method.
1	3)1·732050807	3)1·732050807
1	1	
27	200	200
7	189	
343	1100	1100
3	1029	
3462	7100	7100
2	6924	
34640[5]	1760000	1760000
	1732025	
	27975	27975
	27712	
	263	263
	242	
	21	21

Ans. 1·73205081.

Note.—1. Five digits are found by the ordinary method.
2. The remaining digits up to 4 (1 less than 5) can be found by long division.
3. In the short form the subtrahends are omitted.

Given the square of one number, to find the square of another number, without going through the operation of squaring.

The digits of the first number must be contained in the second.

Example.—*Given that the square of* 54 *is* 2916, *find the square of* 543. (C.W.)

Given $(54)^2 = 2916$
then $(540)^2 = 291600$
and $(543)^2 = (540 + 3)^2 = (540)^2 + (2 \times 540 \times 3) + 3^2$
$= 291600 + 3240 + 9 = 294849$

This solution is based on the algebraic formula $(a + b)^2 = a^2 + 2ab + b^2$.

Given the square root of a number, to find the square root of another number without going through the ordinary operation of finding the root.

Example.—*Given that the square root of* 15625 *is* 125, *find the square root of* 1582564.

$1582564 - 1562500 = 20064$
Given the root of $15625 = 125$
then by *inspection* the root of 64 is 8
The required root, therefore, is 1258

Note that $20064 = 20000 + 64$
$= 2(1250 \times 8) + 8^2$
where 1250 and 8 are noted by inspection.

To find the smallest number by which a given number must be multiplied or divided to make it a perfect square; *e.g.* 3240.

Resolve 3240 into its prime factors.

2	3240
2	1620
2	810
5	405
3	81
3	27
3	9
3	3
	1

The prime factors are $2 \times 2 \times 2 \times 5 \times 3 \times 3 \times 3 \times 3$. Re-arranging, we get $(2 \times 2) \times (3 \times 3) \times (3 \times 3) \times 5 \times 2$. The last two factors are dissimilar. Take them out. This is equivalent to dividing the whole quantity by 10. Then, if 3240 be divided by 10, the quotient (324) is a perfect square. Or, if 3240 be multiplied by 10, the product (32400) is a perfect square. Hence 3240 must be multiplied by 10 to make it a perfect square.

To find the smallest number which must be added to or subtracted from a given number to make a perfect square; *e.g.* 7294.

	7294	(85
	64	
165	894	
	825	
	69	

There is a remainder of 69. The given number 7294 is, therefore, too great by 69, and 69 is the smallest number which can be *subtracted* to convert the given number into a perfect square.

Since $(86)^2 = 7396$, then the smallest number to be *added* to convert 7294 into a perfect square is $7396 - 7294 = 102$.

THEORY.

1. (*a*) Define the "square root" of a number; (*b*) explain why, in extracting the square root of 961, you *double* the first figure obtained. (K.S.)

2. Show on what principles the extraction of the square root depends. (L.M.)

3. Give algebraically the rationale of the process of finding the square root in arithmetic, and illustrate clearly by reference to the numbers 289 and 10609. (P.T.)

4. Explain the rule for pointing in the extraction of the square roots of whole numbers and decimals. (L.M., C.W., R.M.A.)

5. How can you determine by inspection the number of digits in the square root of any integer? Give full reasons for your answer. (P.E.A.)

6. Prove that no whole number which is a perfect square can end in any one of the digits 2, 3, 7, and 8. (P.E.A.)

7. Explain briefly why we cannot find the exact square root of 62·5. (C.M.)

8. State the rule for extracting the square root of a vulgar fraction. (L.M.)

9. Enunciate the several principles relating to the involution of numbers, which ought to be taught before proceeding to explain or work the rule for the extraction of the square root. Give a numerical illustration of each of these principles; say in what order they should be taught, and why. (C.W.)

10. Give with examples rules for shortening the process of squaring a number. (C.W.)

11. If the square root consists of 7 digits, show how the process can be shortened after 4 of these have been obtained. (C.S.)

12. If a decimal be a perfect square the number of decimal places must be even. Show this. (L.M.)

EXERCISE LX.

Extract the square root of each of the following *whole numbers* :—

1. 549081. (C.J.)
2. 546121. (C.PR.)
3. 63409369. (R.M.A.)
4. 13104400. (C.S.)
5. 231496225. (C.S.)
6. 3915380329. (L.M.)
7. 3344771556. (C.S.)
8. 10074538384. (L.M.)
9. 657836167041. (O.J.)
10. 196540602241. (L.M.)

Find the square root of the following *decimals* :—

11. ·0001. (C.S.)
12. ·000625. (P.E.A.)
13. 0·571 (*4*).* (S.K.)
14. ·007 (*4*). (C.S.)
15. ·000056 (*5*). (O.R.)
16. ·101321183 (*5*). (C.PR.)
17. ·0029929 (*6*). (L.M.)
18. ·081 (*6*). (L.M.)
19. 363·201 (*2*). (P.T.)
20. 49381·7284 (*2*). (C.P.)
21. 5·2 (*3*). (S.K.)
22. 2·420064 × 312·481 (*4*). (L.M.)
23. 0·051 (*4*). (S.K.)
24. 14·4 (*5*). (L.M.)
25. ·00005120459 (*6*). (C.P.)
26. 12·1 (*7*). (L.M.)
27. 197·96492 (*7*). (R.M.A.)
28. ·00005477 (*8*). (C.S.)

* The digit in italics after the decimal indicates the number of places to which the square root is to be obtained correctly.

Find the square root of each of the following *fractions* :—

29. $\frac{7}{5}$ (to 7 places). (P.P.)

30. $\frac{22}{7}$ (to 4 places). (C.J.)

31. $\frac{3757}{65533}$. (L.M.)

32. $\frac{151129}{1823929}$. (L.M.)

33. $\frac{8934}{176886}$. (C.M.)

34. $\sqrt{\frac{926}{117}} \div \sqrt{\frac{691}{248}}$. (I.I.)

Find the square root of each of the following *mixed numbers* :—

35. $10\frac{1}{36}$. (O.R.)

36. $27\frac{1}{25}$. (C.S.)

37. $14\frac{21}{44} \times 2\frac{7}{55} \times \frac{9}{20}$. (C.S.)

38. $\sqrt{62\frac{169}{196}} - (2\frac{1}{7})^2$. (C.W.)

39. $3\frac{8934}{17689}$. (C.M.)

Extract the square root of each of the following *recurring decimals* :—

40. $\cdot\dot{4}$ (to 4 places). (I.I.)

41. $\cdot00\dot{6}7\dot{5}$ (to 5 places). (C.S.)

42. $\cdot1\dot{6}\dot{3}$ (to 4 places). (O.S.L.)

43. $4\cdot69\dot{4}$. (C.S.)

44. $\cdot2\dot{8}\dot{7}$ (to 3 places). (C.PR.)

45. Calculate, to 7 places of decimals, the difference between $\sqrt{\frac{1}{2}}$ and $\frac{7}{10} + \frac{7}{985}$. (L.M.)

46. Find the difference, correct to 4 decimal places, between the square root of $\cfrac{5}{6 + \cfrac{7}{8\frac{10}{11}}}$ and twice the square root of ·2. (K.S.S.)

47. If 2·23 be taken as the square root of 5, show that the error must be less than ·01. (K.S.)

48. The first four figures of a square root are 52·82, and the remainder is ·478. Continue the extraction for three figures more (*a*) by the contracted method, (*b*) by the ordinary method. (I.I.)

CHAPTER XLIV.

MEASUREMENT OF AREA.

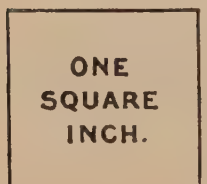
THE term **dimension** means *measurement.*

A **Line** has *one* dimension only—length; *i.e.* it can be measured in only one direction.

A **Surface** has *two* dimensions—length and breadth.

The **Area** of a figure is the surface enclosed within the lines which bound it.

The area of a figure is expressed by the number of times it contains a given unit of measure. This unit of measure is a square, the side of which measures one unit—one inch, *or* one foot, *or* one yard, etc. Thus a square which measures one inch in its side is called a *square inch.*



A **Square Inch** must be carefully distinguished from an *inch square.* A *square inch* is not a shape, but an amount of surface. An *inch square* is a shape and an amount of surface. The above diagram is both a square inch and an inch square.

The rectangle is the commonest plane figure treated of in arithmetic.

A Rectangle is a plane figure bounded by four sides having all its angles right angles.

The *opposite* sides of a rectangle are equal to one another.

If *all* the sides of a rectangle are equal, then the figure is said to be a square. Thus a square is a particular kind of rectangle.

To find the Area of a Rectangle.

Multiply the number of linear units in the length by the number of units of the *same* denomination in the breadth; the product is the number of *corresponding* square units in the area.

A common error is to say:—

$$3 \text{ in.} \times 2 \text{ in.} = 6 \text{ sq. in.}$$

This is wrong, since we cannot multiply an inch by an inch any more than we can multiply a pint by a pint.

It is, of course, permissible to say—

$$3 \text{ sq. in.} \times 2 = 6 \text{ sq. in.}$$

Long custom has practically sanctioned the following: "To find the area of a rectangle, multiply the length by the breadth."

This must be regarded as an abbreviated form of the statement above.

To find the Area of a Square.—Multiply the length of a side by the length of a side, or, since the sides are equal, *square a side.*

Example.—*The length of a rectangular blackboard is* 5 *ft.* 6 *in., and its breadth is* 3 *ft.* 8 *in. Find its area.*

Taking a *foot* as the unit of length, and the *square foot* as the corresponding unit of area, we have—

$$\begin{aligned} \text{Number of units in the length} &= 5\tfrac{1}{2} \\ \text{,, ,, breadth} &= 3\tfrac{2}{3} \\ \text{,, ,, area} &= 5\tfrac{1}{2} \times 3\tfrac{2}{3} \\ &= \tfrac{11}{2} \times \tfrac{11}{3} = \tfrac{121}{6} = 20\tfrac{1}{6} \end{aligned}$$

But each of the units of area is a *square foot.*

$$\therefore \text{ the area} = 20\tfrac{1}{6} \text{ sq. ft.} \quad \textit{Ans.}$$

Given the Area and the Length or Breadth of a Rectangle, to find its Breadth or Length.

Let the number of units in the area = A
and let the number of corresponding units in the length } = L
also let the number of corresponding units in the breadth } = B

$$\text{Then } L \times B = A$$

$$\text{Consequently } L = \frac{A}{B}$$

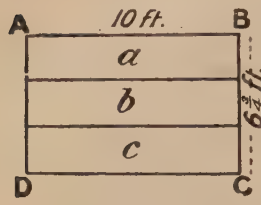
$$\text{and } B = \frac{A}{L}$$

Thus we get the **Rule**—

Divide the number of units in the area by the corresponding number of units in the breadth (or length), and the quotient is the number of units in the length (or breadth).

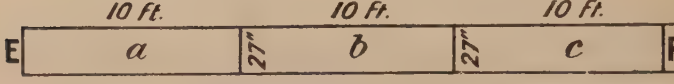
As a corollary to this, the rule follows that to find the length of the side of a square we must find the square root of the area of the square.

Carpeting a Room.



Carpets are generally sold by the *linear* and not by the *square* yard; they are usually 27 in. wide.

Suppose a small rectangle 10 ft. long and 6¾ ft. broad is to be covered with carpet 27 in. wide. We could lay down strips *a*, *b*, and *c*, each 10 ft. long and 27 in. wide. The total length of carpet used would be 30 ft. The rectangle EF (30 ft. long and 27 in. wide) would be equal in area to the rectangle ABCD (10 ft. long and 6¾ ft. wide).



Example.—*How much carpet, 27 in. wide, would be required to cover a room 18 ft. long and 12 ft. broad?*

Taking a *foot* as the unit of length, and consequently the square foot as the unit of area—

Number of units in the length = 18
" " breadth = 12
" " area = 18 × 12

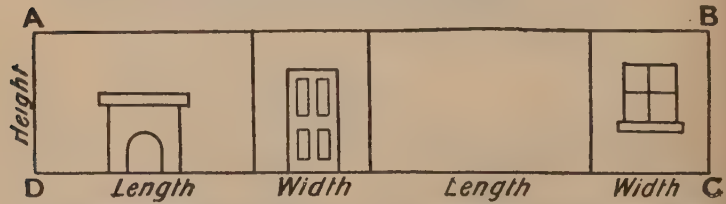
The problem now becomes: If a rectangle has an area of 18 × 12 sq. ft. and a breadth of 27 in. (or 2¼ ft.), what is its length? And the length of such a rectangle will be the length of the carpet.

$$\text{Number or units in the length} = \frac{\text{number of units in the area}}{\text{number of units in the breadth}}$$

$$= \frac{18 \times 12}{2\frac{1}{4}} = 96$$

Length of carpet = 96 ft. = 32 yds. *Ans.*

To find the Area of the Walls of a Room.



Suppose that the walls of the room are spread out flat, so as to form a large rectangle as shown above. The area of this rectangle will be the product of the length AB and the breadth AD.

The length AB is the perimeter of the room; *i.e.* the length round the room.

From the diagram the perimeter = length + width + length + width
= (length + width) × 2, or 2(L + W)

The breadth of the rectangle is in this case the height of the room.

Hence we arrive at the following :—

$$\text{Area of walls of a room} = \text{perimeter} \times \text{height}$$
$$= (\text{L} + \text{W}) \times 2 \times \text{H}$$

Hence—

To find the Perimeter of a Room, multiply the sum of the length and width by 2.

To find the Area of the Walls of a Room, multiply the perimeter by the height.

To find the Amount of Paper required to cover the Walls of a Room.

Having found the area of the walls of the room in accordance with the preceding paragraph, we next adopt exactly the same method as that employed in estimating the length of carpet required to cover a room.

Example.—*What would be the cost of papering a room* 17 *ft. long,* 13 *ft. wide, and* 14 *ft. high with paper* 2 *ft.* 6 *in. wide, costing* $4\frac{1}{2}d.$ *a yard?* (O.J.)

$$\text{Perimeter of room} = (17 \text{ ft.} + 13 \text{ ft.}) \times 2$$
$$\text{Area of walls} = (30 \times 2 \times 14) \text{ sq. ft.}$$
$$\text{Length of paper required} = \frac{30 \times 2 \times 14}{2\frac{1}{2}} \text{ ft.}$$
$$\text{Cost of paper at } 4\tfrac{1}{2}d. \text{ per yard (or } 1\tfrac{1}{2}d. \text{ per foot)} = \frac{30 \times 2 \times 14 \times 1\frac{1}{2}d.}{2\frac{1}{2}}$$
$$= £2\ 2s. \quad \textit{Ans.}$$

Papering the Walls of a Room, and allowing for Windows, Doors, etc.

In *theory*, the area covered by windows, doors, etc., should be subtracted from the total area of the walls; in *practice*, windows and doors save the paper little, if at all. The pattern has to be matched, and the presence of doors, etc., only increases the waste.

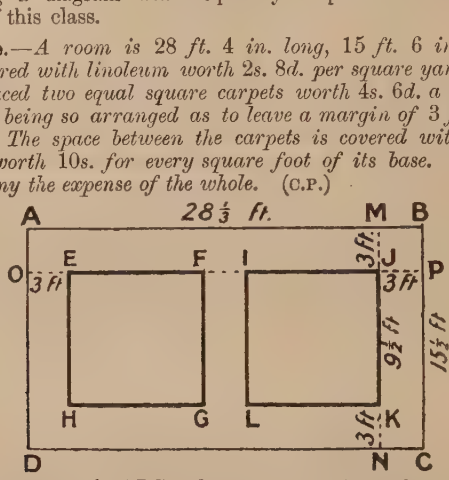
Example.—*A room* 25 *ft.* 7 *in. long,* 16 ft. 9 *in. broad, and* $15\frac{69}{254}$ *ft. high, has* 3 *windows each* 5 *ft. by* 3 *ft.* 6 *in., a door* 7 *ft. by* 4 *ft., and a fireplace* 5 *ft. by* 6 *ft. Find the cost of papering the walls with paper* $\frac{5}{8}$ *yd. wide at* $4\frac{1}{2}d.$ *per yard.* (C.S.)

$$\textit{Total} \text{ area of walls} = 2 \times (25\tfrac{7}{12} + 61\tfrac{3}{4}) \times 15\tfrac{69}{254} \text{ sq. ft.}$$
$$= 2 \times \tfrac{187}{3} \times \tfrac{3879}{254} \text{ sq. ft.} = 1293 \text{ sq. ft.}$$
$$\text{Area of 3 windows, door, and fireplace} = (3 \times 5 \times 3\tfrac{1}{2}) + (7 \times 4) + (5 \times 6) \text{ sq. ft.}$$
$$= 52\tfrac{1}{2} + 28 + 30 = 110\tfrac{1}{2} \text{ sq. ft.}$$
$$\text{Area to be covered with paper} = (1293 - 110\tfrac{1}{2}) \text{ sq. ft.} = 1182\tfrac{1}{2} \text{ sq. ft.}$$
$$= \tfrac{2365}{2} \text{ sq. ft.} = \tfrac{2365}{18} \text{ sq. yds.}$$
$$\text{Number of yards of paper required} = \tfrac{2365}{18} \div \tfrac{5}{8}$$
$$\text{Cost of paper at } 4\tfrac{1}{2}d. \text{ per yard} = \frac{2365 \times 8 \times 4\frac{1}{2}d.}{18 \times 5}$$
$$= £3\ 18s.\ 10d. \quad \textit{Ans.}$$

Covering the Middle of a Room with Carpet, making Paths round Gardens, etc.

Drawing a diagram will frequently help the student to solve problems of this class.

Example.—*A room is* 28 *ft.* 4 *in. long,* 15 *ft.* 6 *in. wide. Its floor is covered with linoleum worth* 2*s.* 8*d. per square yard, and upon this are placed two equal square carpets worth* 4*s.* 6*d. a square yard, the carpets being so arranged as to leave a margin of* 3 *ft. all round the room. The space between the carpets is covered with a piece of furniture worth* 10*s. for every square foot of its base. Find to the nearest penny the expense of the whole.* (C.P.)



Draw the rectangle ABCD showing dimensions of room.

$$\text{Area of linoleum} = (28\tfrac{1}{3} \times 15\tfrac{1}{2}) \text{ sq. ft.}$$

$$\text{Cost of linoleum in shillings} = \frac{28\frac{1}{3} \times 15\frac{1}{2}}{9} \times 2\tfrac{2}{3} = £6\ 10s.\ 1\tfrac{13}{27}d.$$

Next draw the carpets EFGH, and IJKL in position.

Since the carpet must be 3 ft. from the wall, MJ and KN are each 3 ft. long. Therefore the length (JK) of the carpet must be $15\frac{1}{2}$ ft. − 6 ft., or $9\frac{1}{2}$ ft.

$$\text{Area of one carpet} = (9\tfrac{1}{2} \times 9\tfrac{1}{2}) \text{ sq. ft.}$$

$$\text{,, two carpets} = (9\tfrac{1}{2} \times 9\tfrac{1}{2}) \text{ sq. ft.} \times 2$$

$$\text{Cost of the two carpets in shillings} = \frac{9\frac{1}{2} \times 9\frac{1}{2} \times 2 \times 4\frac{1}{2}}{9} = \tfrac{361}{4}s. = £4\ 10s.\ 3d.$$

The next problem is to find the area of the rectangle FILG between the carpets.

FG (*i.e.* JK) has already been determined to be $9\frac{1}{2}$ ft. long.

$$\text{FI} = \text{AB} - \text{OE} - \text{EF} - \text{IJ} - \text{JP}$$

$$= (28\tfrac{1}{3} - 3 - 9\tfrac{1}{2} - 9\tfrac{1}{2} - 3) \text{ ft.} = 3\tfrac{1}{3} \text{ ft.}$$

$$\text{Area of rectangle FILG} = (9\tfrac{1}{2} \times 3\tfrac{1}{3}) \text{ sq. ft.}$$

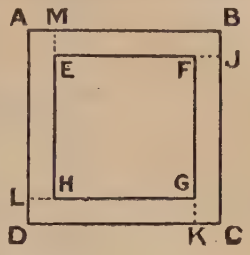
$$\text{Cost of the furniture in pounds} = 9\tfrac{1}{2} \times 3\tfrac{1}{3} \times \tfrac{1}{2} = \frac{19 \times 10}{2 \times 3 \times 2}$$

$$= \tfrac{190}{12} = £15\ 16s.\ 8d.$$

$$\text{Total cost to nearest penny} = £6\ 10s.\ 1d. + £4\ 10s.\ 3d. + £15\ 16s.\ 8d.$$

$$= £26\ 17s. \quad Ans.$$

Example.—*A path 9 ft. wide, running all round a square park, has an area of exactly 3 ac. Find the area of that part of the park enclosed by the path.* (C.S.L.)



Draw the square ABCD, and the square EFGH within it.

The area of the path = area of ABCD − area of EFGH = 3 acres. Produce the sides of the inner square as shown in the diagram.

Then the rectangle MEJB is $\frac{1}{4}$ of the area of the path, and has an area of $\frac{3}{4}$ ac.

The breadth (ME) of the path is 9 ft. (or 3 yds.).

$$\therefore \text{ the length (MB) of the path} = \frac{\frac{3}{4} \text{ ac.}}{9 \text{ ft.}} = \frac{3 \times 4840 \times 9}{4 \times 9} \text{ ft.}$$

$$= 3630 \text{ ft.} = 1210 \text{ yds.}$$

$$\text{The length (EF) of the side of the square EFGH} = (1210 - 3) \text{ yds.} = 1207 \text{ yds.}$$

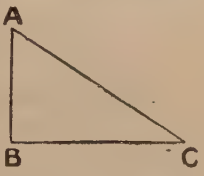
$$\text{Area of square EFGH} = 1207 \times 1207 \text{ sq. yds.}$$

$$= 1456849 \text{ sq. yds.}$$

$$= \frac{1456849 \text{ sq. yds.}}{4840} \text{ acres}$$

$$= 301 \text{ ac. } 9 \text{ sq. yds.} \quad \textit{Ans.}$$

Frequently problems in areas require a knowledge of square root and of Euclid I. 47. This proposition states that in a right-angled triangle, the sum of the squares upon the sides containing the right angle is equal in area to the square upon the side opposite to the right angle.



Thus in the right-angled triangle, ABC, the square on AB + the square on BC = the square on AC.

If AB be 3 ft. long, and BC 4 ft. long, then the length of AC can be found.

$$\text{For } AC^2 = AB^2 + BC^2 \text{ (I. 47)}$$

$$\text{Substituting, } AC^2 = 3^2 + 4^2 = 25$$

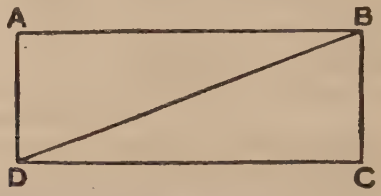
$$\text{and } AC = \sqrt{25} = 5$$

$$\text{length of AC} = 5 \text{ ft.}$$

The student should notice that if the sides of a triangle are as 3 : 4 : 5, the triangle is a right-angled triangle. Similarly, a triangle with sides as 5 : 12 : 13 is right-angled.

To find the Diagonal of a Rectangle, the Length and Breadth of the Rectangle being given.

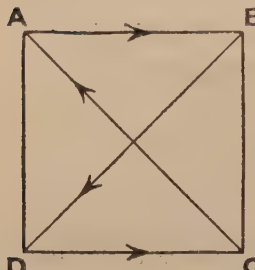
$$BD^2 = BC^2 + DC^2$$
$$\therefore BD = \sqrt{BC^2 + DC^2}$$



Rule.—To find the diagonal of a rectangle, find the square root of the sum of the squares of two adjacent sides.

From this follows as a corollary—

To find the diagonal of a square, find the square root of twice the square of a side.



Example.—*There is a square enclosure of* 10 *ac.; a man walks at the rate of* 3 *mi. an hour along one side, along a diagonal, along another side, and so returns along the other diagonal to the starting point. How many minutes does it take him to walk the distance?*
(S.K.)

Let the square enclosure be represented by ABCD.

Length of side AB $= \sqrt{10}$ ac. $= \sqrt{10 \times 4840}$ yds.
$= \sqrt{48400}$ yds. $= 220$ yds.
Length of diagonal AC $= \sqrt{AB^2 + BC^2}$ (I. 47) $= \sqrt{2AB^2}$
$= \sqrt{2 \times 48400}$ yds. $= \sqrt{96800}$ yds.
$= 311$ yds. (approximately)

If he starts from the corner A, he walks along AB, BD, DC, CA, *i.e.* he walks $2(AB + AC) = 2(220 + 311)$ yds. $= 2 \times 531$ yds.

The problem now becomes: If he walks 3×1760 yds. in 60 min., in how many minutes will he walk 2×531 yds.?

$$\text{Number of minutes} = \frac{60 \times 2 \times 531}{3 \times 1760} = \frac{531}{44} = 12\tfrac{3}{44}.$$

Ans. $12\frac{3}{44}$ min.

THEORY.

1. Distinguish between a *square* and a *rectangle.*

2. Show by means of diagrams the difference between—
(*a*) A square inch and an inch square.
(*b*) Three square feet and three feet square.

3. What is the result obtained on multiplying—
(*a*) 3 by 3. (*b*) 3 ft. by 3. (*c*) 3 by 3 ft.
(*d*) 3 ft. by 3 ft. (*e*) 3 sq. ft. by 3. (*f*) 3 by 3 sq. ft.
Give reasons for your answer in each case. (C.W.)

4. Distinguish between the area of a room and the area of the walls of a room.

5. What is there incorrect in the following? $\frac{54 \text{ sq. ft.}}{8 \text{ ft.}} = 6 \text{ ft.} + 6.$

6. Can feet be multiplied by feet? If not, what is meant by the formula for rectangles, area = length × breadth? (C.S.)

7. What is meant by the *perimeter* of a room?

EXERCISE LXI.

AREAS.

Given the Length and Breadth, to calculate the Area.

1. Find the area (in yards, etc.) of the floor of a room 20 ft. 6 in. long by 25 ft. 9 in. broad.

2. Find the area of a rectangle 2·38 in. in breadth and 3·14 in. in length.

3. Find the area of a rectangle 976 lks. by 638 lks.

4. Required the area of a square, the side being 23 ft. 9 in.

5. Find the area of a square field, the side of which measures 9 ch. 45 lks. (C.S.)

6. Find the area of a court 59 ft. 4 in. long and 48 ft. 9 in. wide, and also the cost of paving it at 4*s.* 6*d.* per square yard. (C.PR.)

7. Each side of a square common measures 28½ ch. What would be its value, estimated at the fourth part of a farthing per square inch? (Chain = 4 po.) (C.P.)

8. A parish contains 6 mi. of roads of an average breadth of 44 ft. Find the extent of these roads in acres. (C.S.L.)

9. A railway covers ground to the breadth of 22 yds. How many acres will be covered by a mile of the railway? (S.L.C.)

10. The daily issue of a newspaper is said to be 30000 copies. Three days of the week it consists of 2 sheets, and the remaining three of 3 sheets. If a sheet be 2½ ft. long and 1⅔ ft. broad, find the number of acres, etc., which the weekly issue of the newspaper would cover. (C.P.)

11. A farmer pays £4 16*s.* 8*d.* a year as rent for a square tillage field, 200 yds. broad. A quarter of the surface consists of bare rock, and 1000 sq. yds. of the rest is occupied by a lake. What rent per acre does the farmer really pay for the tilled portion of the field? (P.T.)

12. A rectangular field is twice as long as it is broad, and it would

cost £92 8*s.* to put a fence round it at 3*s.* 6*d.* per yard of fence. Find the area of the field in acres. (C.S.L.)

13. A rectangular field is $\frac{3}{4}$ of a mile round, and the length is twice the breadth. Determine the area in acres. (O.S.L.)

Given the Area and the Length or Breadth, to find the Breadth or Length.

14. The area of the floor of a room is 485 sq. ft.; the length is 30 ft. Find the breadth.

15. Find the length of a room 11 ft. 11 in. wide, the floor of which is just covered by a carpet 17 sq. yds. 2 sq. ft. 131 sq. in.

16. The area of a rectangular piece of land is $5\frac{1}{2}$ ac.; one side measures 250 yds. What is the length of the other side? (P.T.)

17. If a roll of paper one mile long will cover 1210 sq. yds., how wide is the paper? (O.R.)

18. A rectangular court, 44 ft. long, is paved with 1936 tiles, each being 6 in. square. What is the width of the court? (C.P.)

19. What must be the breadth of a piece of ground whose length is $40\frac{1}{2}$ yds., in order that it may be twice as great as another piece of ground whose length is $14\frac{5}{8}$ yds., and whose breadth is $13\frac{3}{13}$ yds.? (P.P.)

20. The floor of one room is 13 ft. 4 in. long and 12 ft. 3 in. wide.; the floor of a second room is 5 ft. 4 in. longer, and its area twice as great. How much wider is it? (C.P.)

EXERCISE LXII.

Carpeting a Room.

1. How many yards of carpet are required to cover a floor 12 yds. long and 5 yds. 1 ft. wide, the width of the carpet being $\frac{7}{8}$ of a yard? (P.T.)

2. If linoleum cost 2*s.* 3*d.* per square yard, what would it cost to cover a floor 18 ft. long by 10 ft. 6 in. wide? (C.P.)

3. A floor is 27·3 ft. long and 20·16 ft. broad. What will be the cost of covering it with matting 2·4 ft. broad at 2*s.* 6*d.* a yard? (C.S.)

4. Find the cost of carpet 2 ft. 3 in. wide for a room 20 ft. 3 in. long by 13 ft. 4 in. wide, at 5*s.* a yard. (R.M.A.)

5. Find the cost of the carpet, 27 in. wide, required to cover the floor of a rectangular room, 8 yds. long and 6 yds. wide, at 1*s.* 8*d.* a linear yard. (C.W.)

6. Find the number of yards of matting required to cover a room 29 ft. 6 in. long by 18 ft. 4 in. wide, the matting being 20 in. wide. (C.P.)

7. If it costs £16 to carpet a square room 24 ft. broad, what would it cost, at the same rate per square foot, to carpet an oblong room 18 ft. by 15 ft.? (P.T.)

8. If the carpet for a room 17 ft. 6 in. long and 12 ft. wide costs £4 1*s.* 8*d.*, what will be the cost of a carpet of the same material for a room 23 ft. 3 in. long and 16 ft. wide? (O.S.L.)

9. Find what fraction of the floor of a room 21 ft. 9 in. by 18 ft. 8 in. is covered by a rectangular carpet 17 ft. $1\frac{1}{4}$ in. by 16 ft. 11 in. (L.M.)

Area of Walls of a Room.

10. What is the area of the walls of a room 22 ft. long, 14 ft. wide, and 9 ft. high?

11. How many square yards of paper would be required to cover the walls of a room 23 ft. 5 in. long, 18 ft. 7 in. wide, and 9 ft. 6 in. high? (P.E.A.)

12. What would it cost to colour the walls of a room 10 ft. high, 16 ft. long, and 4 yds. wide, at $3\frac{3}{4}d.$ per square foot? (P.T.)

13. Find the expense of papering the walls of a room 22 ft. long, 17 ft. wide, and 12 ft. high with paper 2 ft. wide, costing $4\frac{1}{2}d.$ per yard. (O.S.L.)

14. Find the cost of papering a square room 17 ft. 4 in. long and 14 ft. 3 in. high with paper 1 ft. 7 in. in width, at 1*s.* 8*d.* per yard. (O.R.)

15. An oblong room 18 ft. in length contains the same area as a square room, whose side is 15 ft. Each room is 10 ft. high. Find by how much the surface of the walls of one room exceeds the surface of the walls of the other. (P.T.)

Papering or painting the Walls of a Room, with deductions for Windows, etc.

16. How many yards of wall paper $22\frac{1}{2}$ in. wide will be required for a room 20 ft. long, 16 ft. wide, and 10 ft. high? Sixty square feet are to be deducted for window and door space. (P.E.A.)

17. Find the number of yards of paper 2 ft. wide required for a room 35 ft. long, 25 ft. wide, and 10 ft. high, allowing for a door 7 ft. high and 6 ft. wide, two windows each 6 ft. high and 4 ft. wide, and a fireplace covering 18 sq. ft. (O.P.)

18. Find the expense of papering a room whose length is 28 ft., breadth 20 ft., and height 10 ft. with paper 16 in. wide, at 10*d.* per yard, allowing for a fireplace 5 ft. by 4 ft., a door 7 ft. by 4 ft., and two windows, each 5 ft. by 3 ft. (O.W.)

19. A house is, on three sides, re-lathed and re-plastered. It stands on a base 44 ft. square, and is 28 ft. high. Two of the sides have four windows each; the other only two. Each window measures 6 ft. by 4 ft. Supposing the cost to be at the rate of 2*s.* 4*d.* per square yard, what is the whole expense? (O.P.)

20. Calculate the cost of paper for a room 18 ft. long, 16 ft. broad, and 12 ft. high, allowing 40 sq. ft. for doors and windows, the price of the paper being 2*s.* 4*d.* per piece of 12 yds. long and 21 in. broad. (C.P.R.)

21. How much will it cost to paper a room 18 ft. 6 in. broad by 20 ft. long and 11 ft. high with paper costing 9*d.* per square yard? The room has a door 4 ft. broad by 7 ft. high, and a window 5 ft. broad by 8 ft. high. (O.R.)

22. A room 30 ft. long and 24 ft. wide has the lower part of its walls papered to a height of 15 ft., the upper part being covered by a painted frieze; the paper is 27 in. wide and cost 6*d.* a yard, and the painting cost 4*d.* a square foot. The total cost of papering and painting is £9 4*s.* $4\frac{1}{2}d.$ Assuming that the doors and windows occupy $\frac{1}{16}$ of the frieze and $\frac{1}{6}$ of the lower part of the walls, find the height of the room. (O.J.)

23. What length of paper 2 ft. 3 in. wide will be required to cover the walls of a room 13 ft. 3 in. high, 22 ft. 7 in. long, 17 ft. 5 in. broad, containing a door 10 ft. 6 in. high by 4 ft. broad, two windows, each 9 ft. 4 in. high by 5 ft. 3 in. broad, and two fireplaces, one 4 ft. high by 5 ft. broad, and the other 4½ ft. high by 6 ft. broad? (O.R.)

24. A room is 21 ft. 4 in. long and 15 ft. 9 in. broad; the door and windows together occupy 65 sq. ft., and the cost of papering the remaining part of the surface of the walls with paper 25 in. wide, at 3*s.* 9*d.* per piece of 12 yds., is £2 8*s.* 8*l.* Find the height of the room. (O.S.L.)

25. A room is 19 ft. 4 in. long, 12 ft. 2 in. wide, and 10 ft. 6 in. high. Find the difference of cost of papering the walls with English paper, which is 21 in. wide, and costs 7*d.* a yard, or with French paper, which is 18 in. wide, and costs 6½*d.* a yard, allowing 13 sq. yds. for windows, etc. (C.M.S.)

Papering the Walls of a Room and carpeting the Floor.

26. A room is 18 ft. long, 12 ft. wide, and 11 ft. high. What length of paper a yard wide would be required to line its four walls and its ceiling? (S.K.)

27. A square cistern is 6 ft. long and 4 ft. 6 in. deep. How much lead would be required to line the sides and bottom if a square foot weighs 6 lbs.?

28. The paper on the walls of a room 15 ft. long, 10 ft. 9 in. wide, and 9 ft. high is ¾ yd. wide, and worth 8½*d.* a yard. The carpet on the floor being 30 in. wide, is worth 4*s.* 4*d.* a yard. What was the whole expense for the paper and the carpet? (C.P.)

29. A room is 10 ft. long by 12½ ft. broad. How much carpet 1 ft. 8 in. in breadth will be wanted, and how much will the painting of the ceiling cost at 1½*d.* per square foot? (O.R.)

30. The three walls of a glass-roofed building, one end of which is closed by curtains, are decorated at a cost of 3*s.* 9*d.* a square foot; and the floor is carpeted at a cost of £2 7*s.* 6*d.* a square yard. The dimensions being 25 ft. long, 12 ft. wide, and 11 ft. high, determine the whole expense. (C.P.)

Covering Courtyards with Tiles, etc.

31. How many planks, each 15 ft. long and 10 in. wide, will be required to floor a room 30 ft. by 22½ ft.?

32. A courtyard is paved with 1539 tiles, each 6 in. square. How many bricks, measuring 9 in. by 4½ in., will be required in paving another yard seven times as large? (C.P.)

33. Find the number of children accommodated, at 8 sq. ft. of floor per child, in an oblong room 23 ft. 9 in. by 16 ft. 2 in., neglecting the fraction of a child. (C.W.)

34. A rectangular field of two acres is divided into allotment gardens amongst a number of labourers. Supposing each portion is 121 ft. by 36 ft., how many men can have portions? (C.W.)

35. How many rectangular tiles 4½ in. by 6 in. will be required to

cover a floor 18 ft. long by 13 ft. 6 in. broad, and what will be the cost at 3*s.* 4*d.* per dozen tiles? (L.M.)

36. The floor of a room 17 ft. 6 in. long by 16 ft. 3 in. wide is to be covered with tiles at £2 5*s.* per thousand, the size of each being 6 in. by 3 in. What is the cost of the necessary tiles? (S.L.C.)

37. How many meshes are there in a square foot of wire netting, each mesh being $\frac{3}{16}$ of an inch long and $\frac{1}{8}$ of an inch wide? (C.P.)

38. How many allotments of 4 ac. 2 ro. 36 rds. each may possibly be taken out of a stretch of land 56 mi. long by $20\frac{1}{4}$ mi.? (C.P.)

39. Find how many postage stamps ·95 in. long by ·78 in. broad will cover the ends of a room 17 ft. long, 13 ft. wide, and 9 ft. 6 in. high. (C.P.)

40. A man endeavoured to paper a wall with postage stamps, of which he had 38080, measuring $\frac{13}{16}$ of an inch by $\frac{13}{17}$ of an inch. If the wall was 17 ft. 2 in. by 9 ft. 2 in., what fraction of its surface would the stamps cover? (S.L.C.)

41. A postage stamp is $\frac{23}{24}$ of an inch long and $\frac{4}{5}$ of an inch wide. How many will be required to cover a board 1 ft. 11 in. long and 1 ft. wide? (K.S.)

42. A square court, of which each side is $25\frac{1}{4}$ ft., is paved with rectangular unmortared bricks. Each brick is 10 in. by 5 in. by 4 in., and lies with its longest edge parallel to one side of the square, and with its shortest edge parallel to another (these two edges being horizontal). No brick is cut, but when as many bricks have been laid as the court will hold, the interstices are filled with asphalte, at a cost of 2*s.* 8*d.* per cubic foot of asphalte used. Find (1) the number of bricks used; (2) the cost of the asphalte. (C.M.)

Covering middle of Room with Carpet, making Paths round a Garden, etc.

43. A rectangular grass plot 45 ft. long and 15 ft. broad is surrounded by a path 5 ft. wide. What is the area of the path? (O.S.L.)

44. A room is 29 ft. long by $18\frac{1}{2}$ ft. broad. A Turkey carpet laid in it is $2\frac{1}{4}$ ft. from each long side, and 4 ft. from each short side. Draw a rough plan, and then find the cost of this carpet at £1 10*s.* per square yard. (P.T.)

45. 32 yds. of carpet 2 ft. 8 in. wide are used in making a square which is laid evenly in the middle of a rectangular room, the dimensions of which are 24 ft. by $18\frac{1}{2}$ ft. Determine the area of the uncovered portion of the floor and the distance of the carpet from either side of the room. (C.P.)

46. How much carpet 27 in. wide would be required for a room 22 ft. long and 18 ft. wide, a space round the carpet everywhere 2 ft. wide being left uncovered, and what would be the total cost of the carpet at 2*s.* 9*d.* per linear yard? (C.H.L.)

47. Find the saving in carpeting a room 18 ft. long and 15 ft. wide, if, instead of carpeting the whole floor, a width of 3 ft. from each wall is stained. The carpet, 18 in. wide, costs 3*s.* 9*d* per yard, and staining costs 4*d.* per square foot. (C.P.)

48. A square carpet is composed of strips of plain material $\frac{3}{4}$ yd. wide, costing 6*s.* 6*d.* a yard, and is surrounded by a border $\frac{1}{2}$ yd. wide,

costing 12*s.* a square yard. If the cost of the plain material is twice that of the border, determine the cost of the two together. (O.S.L.)

49. A rectangular carpet 21 ft. 6 in. long and 16 ft. 10½ in. wide is put down in a room, leaving a space 16 in. wide along each side and 24 in. wide along each end uncarpeted. If the carpet cost 5*s.* 4*d.* per square yard, and the remaining space is covered with inlaid woodwork costing 8*d.* per square foot, find the whole cost. (C.S.L.)

50. A carpet which covered the central portion of a room cost £11 17*s.* 8*d.* The room was 25½ ft. long and 19½ ft. broad, and between the edge of the carpet and the wall was a distance all round of 2½ ft. What was the cost of the carpet per square yard? (C.P.)

51. A garden whose length is 67 ft. 9 in. has a path 4 ft. wide on the two sides and at one end. If it costs £4 10*s.* 3¾*d.* to turf the remainder at 6*d.* a square yard, what is the width of the garden? (R.M.A.)

52. The floor of a room is to be covered by a rectangle of carpet in the middle, and matting on all that part of the floor which is outside the carpet. Find the difference in the whole cost according as the rectangle of carpet is 12 ft. by 13 ft. 6 in. or 11 ft. 3 in. by 13 ft., the price of the carpet, 27 in. wide, being 5*s.* 9*d.* per yard, and that of the matting 2*s.* per square yard. (C.H.L.)

53. Find the cost of gravelling a 5-ft. path running along two consecutive sides of a garden which is 172 ft. long and 54 ft. wide, a load of gravel worth 12*s.* 9*d.* being required for every 40 sq. yds. of path, and the work occupying a man and a boy for 4½ days, at 3*s.* 9*d.* and 2*s.* a day respectively. (C.S.)

54. A rectangular garden 60 ft. long by 50 ft. wide has a gravelled path 3 ft. wide running all round it. The rest is turfed, with the exception of three circular flower-beds whose diameters are 8 ft., 16 ft., and 24 ft. respectively. Find the number of square yards of turf in the garden. (Area of a circle $= \frac{22}{7}$ (radius)^2.) (O.J.)

55. Within a rectangular court of length 200 ft. a gravelled path 10 ft. wide is laid down along the four sides, the cost of the gravel being 3*d.* per square foot. If the path had been twice as wide, the gravel would have cost £81 more. Find the breadth of the court. (O.S.L.)

EXERCISE LXIII.

Some of these Problems require a Knowledge of Square Root.

Given the area of a Square, to find the Length of the Side.

1. Find, to the nearest integer, how many inches there are in the length of one of the sides of a square field, whose area is 2 ac. (S.K.)

2. Find number of yards in a side of a square field, the area of which is 11 ac. and 36 per. (I.I.)

3. A square field contains 6 ac. 2 ro. 19 po. 6¼ sq. yds. Find the length of one side in yards. (S.K.)

4. The sides of two squares contain 77 yds. 1 ft. 9 in. and 7 yds. 2 ft. 4 in. respectively. Find the side of a square whose area equals the sum of the areas of the two squares. (C.S.L.)

5. If the cost of making a square lawn be £351 18*s.* 4½*d.*, and at the

rate of 3*s.* 4½*d.* per square yard, how many feet long is the side of the lawn? (C.P.)

6. The area of a square field is 1 ac. 785 sq. yds. Find the cost of erecting a fence round the field, the price of the fencing being 2*s.* 7*d.* per yard. (C.S.L.)

7. Determine in miles the length of barbed wire required to go five times round a square field of 90 acres. (O.S.L.)

8. The cost of levelling and turfing a square cricket-field at £175 9*s.* 4*d.* per acre is £987. Find the cost of surrounding it with a railing costing 3*s.* 2*d.* per yard. (C.S.L.)

Finding Diagonal of Rectangles, Perimeters, etc.

9. A square field contains 57 ac. 2 ro. 16 po. What is (i.) its perimeter, and (ii.) the rent at 25*s.* per acre? (C.P.)

10. Find, to the nearest inch, the length of the diagonal of a square field whose area is 1 ac. (L.M.)

11. The sides of a rectangle are 16 ft. and 10 ft. long respectively. Find to four places of decimals the length of a diagonal of a square whose area equals that of the rectangle. (S.K.)

12. There is a square enclosure of 10 ac. A man walks at the rate of 4 mi. an hour along one side, along a diagonal, along another side, and so returns along the other diagonal to the starting-point. How many minutes does it take him to walk the distance? (S.K.)

13. The length of a rectangular field of 7 ac. is $1\frac{1}{2}$ times its breadth. Find to the nearest foot the length of each side of the field. (O.S.L.)

14. A rectangular field, four times as long as it is broad, contains $2\frac{1}{4}$ ac. Find length and breadth of field. (P.E.A.)

15. A railing encloses a rectangular field of 15 ac. The length of the field is to its breadth as 3 is to 2. What is the whole length of the railing? (C.J.)

16. A bowling green, whose length is ten times its breadth, covers an area of 5 ac. 3 ro. 8 per. 18 sq. yds. Find its length. (I.I.)

17. A rectangular field, whose length is three times its breadth, contains 6 ac. 900 yds. Find its length and breadth. (C.S.)

18. The area of a rectangular field is 8 ac. 2 ro. 16 po. 30 yds. 108 in., and the length of each of the longer sides is 1 fur. 2 po. 4 yds. 2 ft. Find the length (1) of the other sides, (2) of the diagonal. (C.S.)

19. How long will it take a man to walk round a square field whose area is 160 ac., at the rate of 4 mi. an hour? (C.PR.)

20. The area of a square field is 4·9 ac. How long will it take a man to walk round the outside of it at the rate of 3 mi. an hour? (O.S.L.)

21. How often must I run round a square field of 10 ac. to run a mile? (C.W.)

22. How long will it take to describe the boundary of a square field, containing 13 ac. 81 sq. yds., at the rate of $3\frac{3}{4}$ mi. an hour? (L.M.)

23. A square field contains 13767 sq. yds. 1 sq. ft. How long will it take to walk round it at the rate of $4\frac{1}{2}$ mi. per hour? (C.W.)

Miscellaneous.

24. In a row of 25 houses, each house has 17 windows, each window 4 panes, and each pane measures 18 in. by 9 in. What will be the cost of glazing all these windows at 6*d.* per square foot? (O.S.L.)

25. What will it cost to cover with gravel a court 31 ft. 6 in. long and 18 ft. 9 in. broad, at the rate of 1*s.* 4*d.* per square yard? (O.R.)

26. If land is worth £16 5*s.* 6*d.* an acre, determine the value of a rectangular field, the sides of which are respectively 27 po. 1 yd. 1 ft. 6 in. and 14 po. 3 yds. 2 ft. in length. (O.S.L.)

27. A length of carpet 36 in. wide, worth 5*s.* 4*d.* per yard, is just sufficient to cover the floor of a room; so also is another length of carpet 27 in. wide, worth 4*s.* 4*d.* a yard. Compare the values of the two lengths of carpet. Also, if the difference between the values is 18*s.*, what is the area of the floor of the room? (C.P.)

28. The length of a side of the base of the Great Pyramid, which is square, is 500 Egyptian cubits. (*a*) Find the area covered by it in acres, roods, etc., knowing that an Egyptian cubit is equal to 18·24 English inches; (*b*) find the height of the Pyramid in cubits, having given that the height of a model of the Pyramid, the area of whose base is 8 sq. ft. 73 sq. in., is 22·225 in. (C.PR.)

29. An oblong playground is 112 yds. long, by 63 yds. broad. It is converted into a square one, without altering the area, by shortening the length and increasing the breadth. By how much is the length shortened, and by how much is the breadth increased, and what fraction of the old playground belongs to the new, if the two have a common corner? (C.W.)

30. A room is 18 ft. long and 16 ft. wide. In order to increase the area by a sixth part, a rectangular bay window 12 ft. wide is thrown out from the middle of one of the longer sides. Find (*a*) the depth of the bay; (*b*) the length of the longest straight line that can be drawn on the floor of the room after the alteration. (S.K.)

31. An oblong room is 20 yds. long by 13 yds. broad. It is to be covered, as far as possible, with carpet 2 ft. 3 in. wide, which may not be cut lengthwise. Robinson proposes to lay all the lengths of carpet along the room; Jones proposes to lay all of them across the room. The difference in carpet-value between these two methods being 6*s.*, at what price is the carpet sold per yard of length? (K.S.)

32. The expense of carpeting a room was £11 5*s.*; but if the breadth had been 3 ft. less than it was, the expense would have been £9. What was the breadth? (C.S.)

33. A lawn tennis ground is half as long again as it is wide. The cost of levelling it at 9*d.* per square yard is £176 8*s.* Find the cost of enclosing it with an iron railing at 7*s.* 6*d.* per yard. (C.PR.)

34. The area of the floor of a room is 651 sq. ft., of the larger walls together 806 sq. ft., of the two shorter walls together 546 sq. ft. Find the length, breadth, and height of the room. (C.S.)

35. Two squares of ground, the sides of which are 24 yds. and 33 yds. long respectively, have to be levelled and fenced round. The cost of fencing per linear foot is the same as that of levelling per square yard. Find what fraction (in its lowest terms) the cost of levelling and fencing the former is of the cost of levelling and fencing the latter. (C.P.)

36. The first floor of a house is 12 ft. higher than the ground floor,

and is reached by a flight of steps, each step being 6 in. high and 10 in. broad. The staircase is 3 ft. wide, and is broken halfway up by a landing 7 ft. by 3. Find what length of stair-carpet, 3 ft. wide, will be required to carpet the stairs and landing, and project 16 in. on each floor-level. (C.H.L.)

37. A person measures the area of a square field incorrectly, taking the length of a side to be 1 po. more than it is, and consequently makes the area to be 1 ro. 19 po. too much. What is the true area of the field? (S.A. and C.S.)

38. A pedestal for a statue consists of 3 square slabs. The top slab, 1 yd. broad, rests on the next, which is $1\frac{1}{4}$ yd. broad; this rests on a slab $1\frac{1}{2}$ yd. broad, which rests on the ground. These slabs form a four-sided staircase, each step being 6 in. high. What will it cost to gild this pedestal at 5*s.* a square foot, omitting the upper surface of the uppermost slab? (P.T.)

39. A wall surrounding a square courtyard is built of 12000 blocks of stone, each 12 in. long, 8 in. wide, and 6 in. thick, fitted together without mortar. If 2400 more blocks had been used it would have been possible either to make the courtyard 10 ft. larger each way, or to make the wall 2 ft. higher, the thickness of the wall being the same in every case. Determine the internal dimensions of the courtyard. (O.S.L.)

40. If it cost 25*s.* to cover the walls of a room with paper 21 in. wide at $1\frac{3}{4}d.$ a yard, what would it have cost if the paper had been an inch narrower and a farthing less per yard? (P.E.A.)

CHAPTER XLV.

CUBE AND HIGHER ROOTS.

Definition.—If three equal numbers be multiplied together, the product is called the Cube of the number, and any one of the equal factors is called the *Cube Root* of the product.

Since $4 \times 4 \times 4 = 64$, $\therefore$ the cube of 4 is 64, and the *cube root* of 64 is 4.

Table of Cube Roots.

The following table should be committed to memory:—

$0^3 = 0$; $1^3 = 1$; $2^3 = 8$; $3^3 = 27$; $4^3 = 64$; $5^3 = 125$; $6^3 = 216$; $7^3 = 343$; $8^3 = 512$; $9^3 = 729$; $10^3 = 1000$; $11^3 = 1331$; $12^3 = 1728$.

Rule for the Pointing of Integers.

$1^3 = 1$
and $(10)^3 = 1000$
and $(100)^3 = 1000000$
and $(1000)^3 = 1000000000$, etc.

From a consideration of this table and the preceding one, we deduce that—

(1) Numbers of 1 digit have cubes of 1, 2, or 3 digits; *i.e.*—

The cube contains not more than 3 and not less than $3 - 2$ digits.

(2) Numbers of 2 digits have cubes of 4, 5, or 6 digits; *i.e.*—

The cube contains not more than 3×2 and not less than $3 \times 2 - 2$ digits.

(3) Numbers of 3 digits have cubes of 7, 8, or 9 digits; *i.e.*—

The cube contains not more than 3×3 and not less than $3 \times 3 - 2$ digits.

(4) Generally, numbers of n digits have cubes of not more than $3n$ and not less than $(3n - 2)$ digits; *i.e.* a number composed of n digits will have in its cube root—

$\frac{n+2}{3}$ digits if $(n + 2)$ is divisible by 3

$\frac{n+1}{3}$ „ $(n + 1)$ „ 3

n „ n „ 3

Thus, a number consisting of 7 digits will have $\frac{7+2}{3}$ digits in its cube root

" " 8 " $\frac{8+1}{3}$ " "

" " 12 " $\frac{12}{3}$ " "

Hence the given cube is marked off into periods of three. A commencement is made with the units' digit, and the number of dots indicates the number of digits in the required root.

Cube Root by Inspection.—The cube roots of perfect cubes, if less than one million, can be determined by inspection.

A quantity is said to be a *perfect cube* when it has an exact cube root, either integral or fractional; *e.g.* 27 and $\frac{1}{64}$ are perfect cubes, for $\sqrt[3]{27} = 3$, and $\sqrt[3]{\frac{1}{64}} = \frac{1}{4}$.

1. From the Table of Cube Roots (p. 412), we learn that if the last digit of a perfect cube is—

1, 8, 7, 4, 5, 6, 3, 2, or 9

then the last digit in the cube root is—

1, 2, 3, 4, 5, 6, 7, 8, or 9

2. From the Rule for Pointing we learn that for—
   (*a*) *Numbers* less *than* 1000, the cube root can be seen at once, as it consists of one digit.
   (*b*) *Numbers* more *than* 1000 *and less than* 1000000, the cube root consists of two digits.
      (1) The tens' digit of the root can be determined at once, as it is the greatest number the cube of which is contained in the first period.
      (2) The units' digit of the root can be seen by the aid of the Table of Roots.

Example.—*Find by inspection the cube root of* 941192.

(*a*) Nearest cube root of 941 is 9.
(*b*) Cube root of . . . 192 must be some digit which, when cubed, gives 2 as the last digit of the cube; 8 is the only digit.

$$\therefore \sqrt[3]{941192} = 98 \quad \textit{Ans.}$$

Cube Root by Prime Factors.—Only perfect cubes can be dealt with by this method (see p. 35).

The Ordinary Method for the Extraction of Cube Root depends upon the following Truths.

1. If a number be divided into any two parts, the cube of the whole number is equal to the cube of the first part, plus three times the square of the first part into the second part, plus three times the square of the second part into the first, plus the cube of the last part.

This truth is represented algebraically thus—

Let the whole number N be divided into two parts, a and b.

Then $N^3 = (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

2. If a number be divided into any two parts, and if from the cube of the whole number we subtract the cube of one of the parts, the remainder is the product of two factors: one factor is formed by the sum of three times the square of the subtracted part, plus three times the product of the two parts, plus the square of the remaining part; the other factor is the remaining part.

This truth is represented algebraically thus—

Let the whole number N be again divided into two equal parts, a and b.

Then $N^3 - a^3 = (a+b)^3 - a^3 = 3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)b$

Arithmetical Illustrations.

Truth 1.

Since $34 = (30 + 4)$

then $(30+4)^3 = (30)^3 + 3(30^2 \times 4) + 3(30 \times 4^2) + (4)^3$

An analysis of the number (34) and the multiplication of its parts will show this.

$$\begin{array}{l} 30 + 4 \\ 30 + 4 \\ \hline 30^2 + 4 \times 30 \\ \quad + 4 \times 30 + 4^2 \\ \hline 30^2 + 2 \times 4 \times 30 + 4^2 \\ 30 + 4 \\ \hline 30^3 + 2 \times 4 \times 30^2 + 4^2 \times 30 \\ \quad + 4 \times 30^2 + 2 \times 4^2 \times 30 + 4^3 \\ \hline 30^3 + 3 \times 30^2 \times 4 + 3 \times 30 \times 4^2 + 4^3 \end{array}$$

Truth 2.

$$\begin{aligned} (34)^3 - (30)^3 &= \{(3 \times 30^2 \times 4) + (3 \times 30 \times 4^2) + 4^3\} \\ &= \{(3 \times 30^2) + (3 \times 30 \times 4) + 4^2\} \times 4 \end{aligned}$$

Practical Application of the Two Truths.

Example.—*Find the cube root of* 39304. (C.S.L.)

(*a*) **Full Form.**—From the pointing, we know that the cube root will consist of two digits.

Let a = the tens' digit, and b = the units' digit.

			$\dot{3}9\dot{3}0\dot{4}(30+4$
$a^2 \times a$	$= 30^2 \times 30$	$=$	$27000 = 30^3$
$3a^2$	$= 3 \times 30^2$	$=2700$	$12304 = 34^3 - 30^3$
$3ab$	$= 3 \times 30 \times 4$	$= 360$	
b^2	$= 4^2$	$= 16$	
$(3a^2+3ab+b^2)$	$=(3\times30^2+3\times30\times4+4^2)$	$=3076$	
and $(3a^2+3ab+b^2)b$	$=(3\times30^2+3\times30\times4+4^2)4$	$=3076\times4$	12304

Ans. 34.

The Method explained.

1. Point the number. The pointing gives two periods, the first of which is 39000.

2. Find the nearest perfect cube below 39000; the number is 27000, which is the cube of 30.

3. Place the 30 in the quotient, and subtract its cube, 27000, from 39304; the remainder is 12304. This is the *first approximation*.

4. This remainder $= (3a^2+3ab+b^2)b = (3\times30^2+3\times30\times4+4^2)4$.

5. Now, of the quantities in the brackets, 3×30^2 is the largest, and may be used as a *trial divisor* for the next digit in the quotient. Then 12304 divided by 3×30^2, or 2700, gives 4 as an approximate quotient.

6. Place 4 in the quotient. We can now find the remaining quantities within the brackets; for $3ab = 3 \times 30 \times 4$, and $b^2 = 4^2$.

7. Add the quantities within the brackets, and multiply the sum by 4, and the product = 12304. This is the *second approximation*, and as there is no remainder, it is also the last.

(*b*) **Ordinary Form.**

$\dot{3}9\dot{3}0\dot{4}(34$ *Ans.*
27

$3 \times 30^2 =$	2700	12304
$3 \times 30 \times 4 =$	360	
$4^2 =$	16	
	3076	
	4	
	12304	12304

It will be noted that—

1. All superfluous ciphers are omitted.

2. The second period is affixed to the remainder to form a new dividend.

The cube root of larger numbers is found in the same way.

Ciphers in the Root.

Example.—*Find the cube root of* 8036054027. (C.S.)

$\dot{8}0\dot{3}6\dot{0}5\dot{4}0\dot{2}\dot{7}(2003$ *Ans.*

$2^3 =$	8		
$3 \times 20^2 =$	1200	36	= 1st remainder
$3 \times 200^2 =$	120000	36054	= 2nd „
$3 \times 2000^2 =$	12000000	36054027	= 3rd „
$3 \times 2000 \times 3 =$	18000		
$3^2 =$	9		
	12018009		
	3		
	36054027	36054027	

Note that on bringing down the second period (036) 1200 is not divisible into 36. We therefore affix a cipher to the root. Similarly, we find 120000 is not divisible into 36054, so another cipher is affixed to the root. We then see how many times 12000000 is contained in 36054027. We get 3 for the units' digit of the quotient, and then proceed as in the ordinary form.

To find the Cube Root of a Decimal Fraction.

Consider the following :—

$\cdot 1^3 = (\frac{1}{10})^3 = \frac{1}{1000} = \cdot 001$; $\therefore \sqrt[3]{\cdot 001} = \cdot 1$

$\cdot 01^3 = (\frac{1}{100})^3 = \frac{1}{1000000} = \cdot 000001$; $\therefore \sqrt[3]{\cdot 000001} = \cdot 01$

$\cdot 001^3 = (\frac{1}{1000})^3 = \frac{1}{1000000000} = \cdot 000000001$; $\therefore \sqrt[3]{\cdot 000000001} = \cdot 001$

Note.—1. The cube of a decimal fraction contains three times as many decimal places as the fraction itself.

2. Hence in every decimal fraction the cube root of which is sought, the number of places must be a multiple of 3. If they are not, they must be made so by adding ciphers.

3. The decimal fraction can then be pointed into periods, beginning at the tenths' digit, and placing a dot over every third digit to the *right*. (Note that in pointing whole numbers we work to the *left*.)

4. The method of extracting the root is the same both for decimal and integral quantities, since the system of notation is a decimal one in each case.

Example.—*Find cube root of* ·094818816.

$\cdot 094\dot{8}1\dot{8}816 (\cdot 456$ *Ans.*

	$4^3 =$ 64	
$3 \times 40^2 =$	4800	30818
$3 \times 40 \times 5 =$	600	
$5^2 =$	25	
	5425	
	5	
	27125	27125
$3 \times 450^2 =$	607500	3693816
$3 \times 450 \times 6 =$	8100	
$6^2 =$	36	
	615636	
	6	
	3693816	3693816

Example.—*Find cube root of* 12487·168.

$$
\begin{array}{rr|l}
 & & 1\dot{2}48\dot{7}\cdot16\dot{8}(23\cdot2 \quad \textit{Ans.} \\
2^3 = & 8 & \\
\hline
3 \times 20^2 = & 1200 & 4487 \\
3 \times 20 \times 3 = & 180 & \\
3^2 = & 9 & \\
\hline
 & 1389 & \\
 & 3 & \\
\hline
 & 4167 & 4167 \\
\hline
3 \times (230)^2 = & 158700 & 320168 \\
3 \times 230 \times 2 = & 1380 & \\
2^2 = & 4 & \\
\hline
 & 160084 & \\
 & 2 & \\
\hline
 & 320168 & 320168
\end{array}
$$

To find the Cube Root of a Vulgar Fraction.

1. When both Numerator and Denominator are Perfect Cubes.

Extract the cube root of both numerator and denominator; *e.g.*—

$$\sqrt[3]{\tfrac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \tfrac{2}{3}$$

2. When the Numerator only is a Perfect Cube; *e.g.*—

$$\sqrt[3]{14\tfrac{43}{49}} = \sqrt[3]{\tfrac{729}{49}} = \frac{\sqrt[3]{729}}{\sqrt[3]{49}} = \frac{\sqrt[3]{7 \times 729}}{\sqrt[3]{7 \times 49}} = \frac{\sqrt[3]{7}}{7}\,^{*}$$

3. When the Denominator only is a Perfect Cube.

$$\sqrt[3]{\tfrac{87}{64}} = \frac{\sqrt[3]{87}}{\sqrt[3]{64}} = \frac{\sqrt[3]{87}}{4} = \tfrac{1}{4}\sqrt[3]{87}$$

$$\sqrt[3]{\tfrac{6}{125}} = \frac{\sqrt[3]{6}}{5} = \tfrac{1}{5}\sqrt[3]{6}$$

4. When the Denominator is not a Perfect Cube (see the chapter on Surds).

To find the Cube Root of a Recurring Decimal.

1. When the Recurring Decimal is a Cube.

Example.—*Find the cube root of* $151\cdot\dot{7}0\dot{3}$.

$$\sqrt[3]{151\cdot70\dot{3}} = \sqrt[3]{151\tfrac{703}{999}} = \sqrt[3]{151\tfrac{19}{27}} = \sqrt[3]{\tfrac{4096}{27}} = \tfrac{16}{3} = 5\tfrac{1}{3} \quad \textit{Ans.}$$

* For further solution, see the chapter on Surds.

2. When the Recurring Decimal is not a Cube.

Example.—*Find the cube root of* $27\cdot\dot{4}2857\dot{1}$.

$$\sqrt[3]{27\cdot\dot{4}2857\dot{1}} = \sqrt[3]{27\tfrac{3}{7}} = \sqrt[3]{1\tfrac{92}{7}} = \sqrt[3]{\frac{192 \times 7 \times 7}{7 \times 7 \times 7}}$$

$$= \sqrt[3]{\frac{9408}{7^3}} = \tfrac{1}{7}\sqrt[3]{9408}$$

Horner's Method of Evolution.—Some authorities urge the use of this method in preference to all others. It is asserted that the work is easier, and the process both more gradual and more apparent. Furthermore, it can be used for the extraction of higher roots. Like most other methods, it is algebraic in its nature, depending on the truth of the following identity: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

The Method analyzed.—Let N be the number whose root it is required to find. Then $N - (a + b + c)^3 = 0$.

a a } sum	a^2 $2a^2$ } sum	$N(a + b + c$ a^3
$2a$	$2a^2$ (trial divisor)	$N - a^3$
a	$+ (3a + b)b$	$+ 3a^2b + 3ab^2 + b^3$
$3a + b$	$3a^2 + 3ab + b^2$ (real divisor)	$N - (a + b)^3$
b	$+ 3ab + 2b^2$	$+ 3(a + b)^2c + 3(a + b)c^2 + c^3$
$3a + 2b$	$3(a + b)^2$ (trial divisor)	$N - (a + b + c)^3 = 0$
b	$+ \{3(a + b) + c\}c$	
$3(a + b) + c$	$3(a + b)^2 + 3(a + b)c + c^2$ (real divisor)	

Take a numerical example; *e.g.* $\sqrt[3]{77308776}$, which equals 426.

400 400	400^2 2.400^2	$77\dot{3}0\dot{8}7\dot{6}(426$ 64000000
800	3.400^2	$77308776 - 400^3$
400	$+3.400.20+20^2$	$+3.400^2.20+3.400.20^2+20^3$
$1200+20$	$3.400^2+3.400.20+20^2$	$77308776-(400+20)^2$
20	$+3.400.20+2.20^2$	$+3(400+20)^2\times6+3(400+20)6^2+6^3$
1240	$3(400+20)^2$	$77308776-(400+20+6)^3=0$
20	$+3(400+20)6+6^2$	
$1260+6$	$3(400+20)^2+3(400+20)6+6^2$	

Ordinary Method; e.g. $\sqrt[3]{77308776}$.

4	16	$7\dot{7}3\dot{0}8\dot{7}7\dot{6}$(426	*Ans.*
4	32	64	
8	4800	13308	
4	244		
122	5044	10088	
2	248		
124	529200	3220776	
2	7596		
1266	536796	3220776	

Note.—

1. There are 3 columns for cube root.
2. The first column is built up by additions.
3. The second column is built up by additions and multiplications from (1).
4. The third column is built up by additions and multiplications from (2).
5. For each digit in the root there are—

3 additions in the first column;
2 ,, ,, second ,,
1 subtraction ,, third ,,

6. For each step, to the third addition in the first column the next digit in the root is affixed, and the quantity is multiplied by 10 at the same time; *e.g.* 12 becomes 122, *i.e.* $(12 \times 10) + 2$.
7. The second column is a column of "squares"; *e.g.* 16 is 4^2, and 32 is 2×4^2; 4800 is $(16 + 32) \times 100$, or $(16 + 32) \times 10^2$, and so on. The third column is one of cubes, and is similarly explained.
8. The trial divisors are found in the second column, and are respectively $3a^2$ (4800), $3(a + b)^2$ or 529200.
9. All unnecessary cyphers are omitted.
10. The remainders and divisors are taken for their apparent, and not their representative or real values.
11. Periods are brought down as required.

Contracted Method.—The work may be contracted by the use of the Italian method of division; *e.g.*—

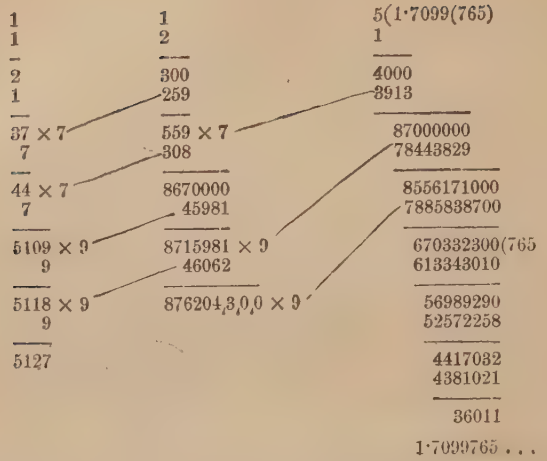
4	16	$7\dot{7}3\dot{0}8\dot{7}7\dot{6}$(426 *Ans.*
8	4800	13308
122	5044	3220776
124	529200	
1266	536796	

Cube Root by Division.

When the cube root is required to a large number of places, the work may be shortened by division. The answer is only approximate, but it is practically correct. The limit for division is that two less than the number of digits already found may be obtained without error.

Example.—*Find the cube root of* 5 *to* 7 *places of decimals.*

This will give 8 digits in the root, 5 of which will be found by the ordinary method, and 3 by division.



Higher Roots.

1. Special Methods for Special Roots.

(*a*) *Fourth Root.*—Since $2^4 = (2^2)^2 = 4 \times 4 = 16$

$$\therefore \sqrt[4]{16} = \sqrt[2]{\sqrt[2]{16}} = \sqrt[2]{4} = 2$$

Hence the fourth root of a number is found by extracting its square root, and then the square root of its square root.

(*b*) *Sixth Root.*—Since $5^6 = (5^2)^3$, find the square root of the number, and then the cube root of that square root.

(*c*) *Eighth Root.*—Since $3^8 = (3^4)^2 = \{(3^2)^2\}^2$, find the square root of the number, then the square root of the answer, and finally the square root of that answer.

(*d*) *Ninth Root.*—Since $5^9 = (5^3)^3$, find the cube root of the number, and then the cube root of the cube root.

And, generally, any root which is a multiple of 2 or 3, or both, may be extracted in a similar manner.

2. General Methods.

(*a*) *Horner's* method may be used generally to find the higher roots, but it is not recommended for the special cases given above. The headings for the various columns should be carefully noted; *e.g.* for the fifth root there are five columns, and their headings are respectively 1, $5a$, $10a^2$, $10a^3$, $5a^4$, a^5. For the fourth root there would be four columns, and the headings would be respectively 1, $4a$, $6a^2$, $4a^3$, a^4. The explanation is that $(1 + a)^4 = 1 + 4a + 6a^2 + 4a^3 + a^4$; and $(1 + a)^5 = 1 + 5a$

$+ 10a^2 + 10a^3 + 5a^4 + a^5$. Counting 1 as the first column, it should also be noted that the second column is multiplied by 10, the third by 100, the fourth by 1000, and the fifth by 10000.

Example.—*Find the fifth root of* 701583371424.

0	00	000	0000	$7\dot{0}1583\dot{3}7142\dot{4}$(234
2				
2×2	= 4			
2	4×2	= 8		
4×2	= 8	8×2	= 16	
2	12×2	= 24	16×2	= 32
6×2	= 12	32×2	= 64	
2	24×2	= 48		
8×2	= 16			
2				

10(0)	40(00)	80(000)	80(0000)	3815833
3				
103×3	= 309			
3	4309×3	= 12927		
106×3	= 316	92927×3	= 278781	
3	4627×3	= 13881	1078781×3	= 3236343
109×3	= 327	106808×3	= 320424	
3	4954×3	= 14862		
112×3	= 336			
3				

115(0)	5290(00)	121670(000)	1399205(0000)	57949071424
4				
1154×4 =	4616			
4	533616×4 =	2134464		
1158×4 =	4632	123804464×4 =	495217856	
4	538248×4 =	2152992	14487267856×4 =	57949071424
1162×4 =	4648	125957456×4 =	503829824	
4	542896×4 =	2171584		
1166×4 =	4664			
4				

Ans. 234

(*b*) The following method may also be used for the extraction of higher roots.

Example.—*Find the fifth root of* 701583371424.

$5a^4$	$5(200)^4$	8000000000	$\dot{7}0\dot{1}58\dot{3}3\dot{7}1\dot{4}2\dot{4}(234$
$10a^3b$	$10(200)^3(30)$	2400000000	32
$10a^2b^2$	$10(200)^2(30)^2$	360000000	381583371424
$5ab^3$	$5(200)(30)^3$	27000000	
b^4	$(30)^4$	810000	
		10787810000	323634300000
$5a^4$	$5(230)^4$	13992050000	57949071424
$10a^3b$	$10(230)^3(4)$	486680000	
$10a^2b^2$	$10(230)^2(4)^2$	8464000	
$5ab^3$	$5(230)(4)^3$	73600	
b^4	$(4)^4$	256	
		14487267856	57949071424

Note.—1. The reason for the rule is as follows:—

Since $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

then $(a+b)^5 - a^5 = (5a^4 + 10a^3b + 10a^2b^2 + 5ab^3 + b^4)b$

2. Other higher roots may be found in a similar way.
3. It involves one long multiplication $(230)^4$, but the total labour is less than in Horner's method.
3. The ciphers may be omitted, if preferred.

Dr. Hutton's Method for the Extraction of Cube Root.

The ordinary method of extracting the cube root is a cumbrous and complicated one. Hutton's method claims to be much simpler.

If it is required to extract the cube root of a given number, the nearest cube root of the number is found by inspection. If N be the number, r the nearest cube root, and R the cube of the number found by inspection, then the fraction $\frac{2N + R}{N + 2R} \times r$ gives a near approximation to the actual cube root.

Example.—*Find the cube root of* 77·308776.

The nearest cube root of 77 is 4, hence $r = 4$ and $R = 64$.

$$\text{Then } \frac{2N + R}{N + 2R} \times r = \frac{154{\cdot}617552 + 64}{77{\cdot}308776 + 128} \times 4 = \frac{218{\cdot}617552 \times 4}{205{\cdot}308776}$$

$$= 4{\cdot}259 \text{ or } 4{\cdot}26 \text{ (nearly).} \quad \textit{Ans.}$$

Example.—*Find the cube root of* 55.

The nearest cube root is 3, $\therefore r = 3$ and $R = 27$.

$$\text{Then } \frac{2N + R}{N + 2R} \times r = \frac{110 + 27}{55 + 54} \times 3 = \tfrac{137}{109} \times 3 = \tfrac{411}{109} = 3{\cdot}77 \ldots$$

r now $= 3{\cdot}77$, and $R = 53{\cdot}582633$

$$\therefore \frac{2N + R}{N + 2R} \times r = \frac{110 + 53{\cdot}582633}{55 + 107{\cdot}165266} \times 3{\cdot}77 = \frac{163{\cdot}582633}{162{\cdot}165266} \times 3{\cdot}77$$

$$= 1{\cdot}00874 \times 3{\cdot}77 = 3{\cdot}8029498 \ldots \quad \textit{Ans.}$$

Hutton's General Formula for High Roots.—This is an approximate method, in which the index of the given powers may be called n. A number nearly equal to the required root is then found by trial, and this is called the *assumed* root. The assumed root is then raised to the power whose index is n. If N be the given number, and x the assumed root, the rule may be expressed in the following formula, n being a first approximation :—

$$\frac{(n+1)\mathrm{N} + (n-1)x^n}{(n-1)\mathrm{N} + (n+1)x^n} \times x$$

The number thus found may be used as a new assumed root, and the operation may be repeated to find a still more accurate result.

Example.—*Find the 150th root of* 3·1416.

$$\frac{(n+1)\mathrm{N} + (n-1)x^n}{(n-1)\mathrm{N} + (n+1)x^n} \times x$$

Take 1 as the assumed root.

$$= \frac{(150+1)3{\cdot}1416 + (150-1)1^{150}}{(150-1)3{\cdot}1416 + (150+1)1^{150}} \times 1$$

$$= \frac{151 \times 3{\cdot}1416 + 149}{149 \times 3{\cdot}1416 + 151} \times 1 = \frac{474{\cdot}3816 + 149}{468{\cdot}0984 + 151}$$

$$= \frac{623{\cdot}3816}{619{\cdot}0984} = 1{\cdot}0069 \quad \textit{Ans.}$$

THEORY.

1. Define the cube root of any number. (C.P.)

2. Why are three figures brought down in the process of finding the cube root? (C.S.)

3. Show that the rule for extracting cube root in arithmetic is based on the algebraic method. (C.M.)

4. Explain the pointing in the extraction of a cube root when the number is partly decimal. Give a reason. (K.S.)

5. Is the cube root of a decimal fraction greater or less than the fraction? Give a reason. (K.S.)

6. What is a prime number? Explain how such numbers may be utilized to simplify the process of finding the square and the cube roots of numbers. (C.S.)

7. Can a number having 3, 6, or 8 as its units' figure be a cube number? (C.S.)

8. Without testing by the extraction of the roots, how are we able to say that 37028 is not an exact square, and that 5217400 is not an exact cube? (I.I.)

9. Show that the cube root of every perfect cube less than a million can be determined by inspection. (K.S.)

10. Deduce that the cube of the difference of two positive numbers is always less than the difference of their cubes. (L.M.)

11. If a whole number be added to its square, show that their sum can only end in 0, 2, or 6; and if a whole number be added to its cube, the sum can only end in 0, 2, or 8. (C.S.)

12. Prove that the sum of the cubes of any three consecutive numbers is divisible by the sum of the numbers themselves, and state in words the value of the remaining factor. (L.M.)

13. In the extraction of the cube root of a number, what is the method of procedure when the root contains ciphers?

14. What is the method of extracting the cube root of a vulgar fraction? of a recurring decimal?

15. State and explain the process called "cube root by division."

EXERCISE LXIV.

Find the cube root of each of the following:—

(*a*) *Whole numbers.*

1. 941192. (K.S.) **4.** 857375000. (C.P.) **7.** 112678587. (R.M.A.)
2. 2248091. (S.K.) **5.** 14172488. (I.I.) **8.** 83568086848. (C.S.)
3. 124251499. (I.I.) **6.** 160103007. (R.M.A.) **9.** 17053975224. (O.S.)
10. 10546683057. (C.S.L.)

(*b*) *Decimals.*

11. ·000001. (C.S.) **13.** 768575·296. (S.K.) **15.** 303464·448. (I.I.)
12. 2·370. (C.S.) **14.** 66430·125. (R.M.A.) **16.** 207995·797125. (O. and C.S.C.)

(*c*) *To 3 places of decimals.*

17. 7. (C.S.) **19.** 0·9. (S.K.) **21.** 6434·856. (S.K.)
18. 8701432. (I.I.) **20.** 3·00415. (C.P.)

(*d*) *To 4 places of decimals.*

22. 11. (C.M.) **23.** ·002. (C.S.)

(*e*) *Mixed numbers.*

24. $16\frac{503}{729}$. (K.S.) **26.** $1776\frac{325}{729}$. (C.S.) **28.** $1458640\frac{19}{37}$. (C.S.)
25. $2\frac{1371}{2744}$. (C.S.) **27.** $1277289\frac{1}{8}$. (C.M.)

Simplify—

29. $\frac{\sqrt[3]{9\cdot663597}}{\sqrt{4\cdot5369}}$. (I.I.) **30.** $\frac{\sqrt[3]{12568\frac{5}{64}}}{\sqrt{\cdot000625}}$. (K.S.S.)

31. $\sqrt[3]{\frac{91}{119}}$ of $\sqrt[3]{\frac{189}{441}}$ of $\sqrt[3]{\frac{225}{343}}$. (C.S.)

32. $\frac{11}{10}$ of $\sqrt{\frac{(1\cdot345)^2-(\cdot905)^2}{1\cdot\dot{2}+\cdot\dot{2}\dot{5}}} \times \sqrt[3]{\cdot\dot{0}3\dot{7}}$. (I.I.)

Find the fourth root of—

33. (*a*) 256; (*b*) 6561; (*c*) 14641.
34. (*a*) ·1296; (*b*) 3164·0625; (*c*) $7\frac{58}{81}$.
35. (*a*) 2·0736; (*b*) 2 (correct to 3 places). (C.H.L.)
36. (*a*) 38950081; (*b*) 1229637·0321. (C.H.L.) **37.** 1211·7361. (C.S.)
38. Find the sixth root of (*a*) 117749; (*b*) 432809; (*c*) 85·766121.
39. Find the eighth root of (*a*) 1679616; (*b*) 2562890625; (*c*) $\frac{256}{4304672}$.
40. Find the ninth root of (*a*) 262144; (*b*) 198359290368.
41. Find the thirtieth root of 2·08.
42. Find the 120th root of 1·23.
43. Find the fifth root of 59049; 16807; 7962624; 7 to 4 places of decimals. (S.A.)

CHAPTER XLVI.

SURDS.

A Surd is a quantity the root of which cannot be exactly obtained; that is, the root does not work out exactly to any integer (or integers), or to a finite or to a recurring decimal.

Conversely, surds may be defined as roots of numbers which are not exact powers.

Rational Quantities.—Quantities are said to be *rational* when their ratio to other numbers can be exactly represented.

Irrational Quantities.—Quantities are said to be *irrational* when their ratio to other numbers cannot be exactly represented.

Irrational quantities are better known as **Surds.**

Classification of Surds.—Surds are classified according to the roots they are under. The two chief classes are—

1. *Quadratic Surds:* those under the *square* root; *e.g.* $\sqrt{2}$.
2. *Cubic Surds:* those under the *cube* root; *e.g.* $\sqrt[3]{2}$.

Simplification of Surds.—A surd is said to be in its simplest form when its surd factor is integral and as small as possible. For the purpose of reduction or simplification, the following points should be noted:—

1. *To multiply a square root by itself, omit the root sign;* or, expressing the rule as a general formula—

$$\sqrt{n} \times \sqrt{n} = n; \quad \sqrt{2} \times \sqrt{2} = 2$$

Example.—$3\sqrt{7} \times 2\sqrt{7} = 3 \times \sqrt{7} \times 2 \times \sqrt{7} = 3 \times 2 \times \sqrt{7} \times \sqrt{7} = 6 \times 7 = 42$.

Caution.—Surds of different orders may *not* be multiplied in this way.

2. *The product of the square roots of different quantities is the square root of the product of those quantities;* e.g.—

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}; \quad \sqrt{3}\sqrt{5} = \sqrt{15}$$

Examples.—

(*a*) $3\sqrt{3} \times 5\sqrt{5} \times 2\sqrt{7} = 3 \times 5 \times 2 \times \sqrt{3} \times \sqrt{5} \times \sqrt{7}$
$= 30\sqrt{105}$ *Ans.*

(*b*) $\sqrt{32} \div \sqrt{8} = \sqrt{\frac{32}{8}} = \sqrt{4} = 2$ *Ans.*

Thus the product or quotient of any two (or more) surds can be obtained by taking the product or quotient of the rational or irrational factors respectively.

3. *A rational factor may be squared and placed under the root sign without altering the value of the surd*; e.g.—

$$4\sqrt{3} = \sqrt{16 \times 3} = \sqrt{48}$$

4. *Surds may be factorized*; e.g. $\sqrt{10} = \sqrt{2} \times \sqrt{5}$.

Example.—*Simplify* $5\sqrt{12} \times 6\sqrt{20}$.

$$\begin{aligned} 5\sqrt{12} \times 6\sqrt{20} &= 5\sqrt{2 \times 2 \times 3} \times 6\sqrt{2 \times 2 \times 5} \\ &= 5 \times \sqrt{2} \times \sqrt{2} \times \sqrt{3} \times 6 \times \sqrt{2} \times \sqrt{2} \times \sqrt{5} \\ &= 5 \times 2\sqrt{3} \times 6 \times 2 \times \sqrt{5} \\ &= 5 \times 6 \times 2 \times 2 \times \sqrt{3} \times \sqrt{5} = 120\sqrt{15} \quad \textit{Ans.} \end{aligned}$$

Thus, a surd can be reduced if one or more factors in the root are perfect squares.

A square factor under the radical sign may have its square root taken, and the root written as a rational factor; *e.g.*—

$$\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$$

5. *Most simplifications depend upon the application of the following formulæ*—

$$(a \pm b)^2 = a^2 \pm ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

An example will be found on p. 427.

The Comparison of Surds.

For purposes of comparison, surds should be converted into entire surds, *i.e.* the whole expression should be brought under the root. If the expressions are under different roots, they should be reduced to a common root. This can be done by taking the L.C.M. of the powers of the roots.

Examples.—*Compare* $\frac{4}{5}\sqrt{3}$ and $\frac{6}{7}\sqrt{2}$.

$$\frac{4}{5}\sqrt{3} = \frac{\sqrt{16 \times 3}}{\sqrt{25}} = \sqrt{\frac{48}{25}} = \sqrt{1\tfrac{23}{25}}$$

$$\frac{6}{7}\sqrt{2} = \frac{\sqrt{36 \times 2}}{\sqrt{49}} = \frac{\sqrt{72}}{\sqrt{49}} = \sqrt{1\tfrac{23}{49}}$$

Now, $\sqrt{1\tfrac{23}{25}} > \sqrt{1\tfrac{23}{49}}$

$\therefore \frac{4}{5}\sqrt{3} > \frac{6}{7}\sqrt{2}$ *Ans.*

Rationalizing the Denominator.—By this is meant the process of converting a surd denominator into a rational quantity. The object is to make the fraction easier to evaluate, as division by a long decimal is laborious.

To rationalize a Square Surd.

Case A. One Surd Term in the Denominator.—Multiply both numerator and denominator by the surd in the denominator.

Examples.—

$$(a)\quad \frac{6}{\sqrt{2}} = \frac{6 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2} \quad \textit{Ans.}$$

$$(b)\quad \frac{3\sqrt{7}}{4\sqrt{8}} = \frac{3\sqrt{7} \times \sqrt{8}}{4\sqrt{8} \times \sqrt{8}} = \frac{3\sqrt{56}}{4 \times 8} = \frac{3\sqrt{4 \times 14}}{32} = \frac{6\sqrt{14}}{32} = \frac{3\sqrt{14}}{16}$$

Ans.

In explanation of this process two facts should be noted—

1. The value of the fraction is unaltered, because both numerator and denominator are multiplied by the same quantity.
2. It is necessary to multiply by the *surd* portion of the denominator only.
3. The denominator is rationalized because a surd is multiplied by itself, and so no root sign is left.

Case B. Two Surd Terms in the Denominator.—The simplification depends on the application of the algebraic formula : $(a + b)(a - b) = a^2 - b^2$.

Example.—*Rationalize* $\dfrac{\sqrt{5}}{\sqrt{7} - \sqrt{3}}$.

Let $N = \sqrt{5}$, $a = 7$, and $b = 3$.

Then $$\frac{\sqrt{5}}{\sqrt{7} - \sqrt{3}} = \frac{N}{\sqrt{a} - \sqrt{b}} = \frac{N(\sqrt{a} + \sqrt{b})}{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})} = \frac{N(\sqrt{a} + \sqrt{b})}{a - b}$$

Here $(\sqrt{a} + \sqrt{b})$, *i.e.* $(\sqrt{7} + \sqrt{3})$, is the rationalizing factor.

Applying this principle—

$$\frac{\sqrt{5}}{\sqrt{7} - \sqrt{3}} = \frac{\sqrt{5}(\sqrt{7} + \sqrt{3})}{(\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3})} = \frac{\sqrt{5}(\sqrt{7} + \sqrt{3})}{7 - 3}$$

$$= \frac{\sqrt{5}(\sqrt{7} + \sqrt{3})}{4} \quad \textit{Ans.}$$

Case C. Three Surd Terms in the Denominator.

Example.—*Rationalize* $\dfrac{1}{\sqrt{3} + \sqrt{2} - \sqrt{5}}$. (C.S.)

$$\frac{1}{\sqrt{3} + \sqrt{2} - \sqrt{5}} = \frac{1}{(\sqrt{3} + \sqrt{2}) - \sqrt{5}} \times \frac{(\sqrt{3} + \sqrt{2}) + \sqrt{5}}{(\sqrt{3} + \sqrt{2}) + \sqrt{5}}$$

$$= \frac{\sqrt{3} + \sqrt{2} + \sqrt{5}}{5 + 2\sqrt{6} - 5} = \frac{\sqrt{3} + \sqrt{2} + \sqrt{5}}{2\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{\sqrt{18} + \sqrt{12} + \sqrt{30}}{12} = \frac{3\sqrt{2} + 2\sqrt{3} + \sqrt{30}}{12}$$

Ans.

Explanation.

1. Group the three terms into two terms; *e.g.* $(\sqrt{3}+\sqrt{2})-\sqrt{5}$
2. Rationalize the expression $(\sqrt{3}+\sqrt{2})-\sqrt{5}$ as if it were $\sqrt{a}-\sqrt{b}$.
3. Note that a surd still remains in the denominator; hence rationalize again to make the denominator rational.

The same principle may be applied to the rationalizing of four or more surds in the denominator.

To rationalize a Cubic Surd.

Example.—*Rationalize* $\dfrac{\sqrt[3]{5}-\sqrt[3]{3}}{\sqrt[3]{5}+\sqrt[3]{3}}$.

$$\frac{\sqrt[3]{5}-\sqrt[3]{3}}{\sqrt[3]{5}+\sqrt[3]{3}}=\frac{(\sqrt[3]{5}-\sqrt[3]{3})(\sqrt[3]{25}-\sqrt[3]{15}+\sqrt[3]{9})}{(\sqrt[3]{5}+\sqrt[3]{3})(\sqrt[3]{25}-\sqrt[3]{15}+\sqrt[3]{9})}$$

$$=\frac{5-2\sqrt[3]{75}+2\sqrt[3]{45}-3}{5+3}=\frac{2-2\sqrt[3]{75}+2\sqrt[3]{45}}{8}$$

$$=\frac{1-\sqrt[3]{75}+\sqrt[3]{45}}{4}\quad Ans.$$

Explanation.

$$(a+b)(a^2-ab+b^2)=a^3+b^3$$

$$\text{and } (a-b)(a^2+ab+b^2)=a^3-b^3$$

$$\therefore (\sqrt[3]{a}+\sqrt[3]{b})(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}=a+b$$

$$\text{and } (\sqrt[3]{a}-\sqrt[3]{b})(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})=a-b$$

$\therefore$ the rationalizing factors of $(a+b)$ are $(\sqrt[3]{a}+\sqrt[3]{b})(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2})$

and „ „ „ $(a-b)$ „ $(\sqrt[3]{a}-\sqrt[3]{b})(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})$

Denominators containing fourth roots can be rationalized in a similar manner.

To convert a Surd into a Continued Fraction.

Example.—*Convert* $\sqrt{17}$ *into a continued fraction.*

Observing that the greatest perfect square below 17 is 16, we proceed thus—

$$\sqrt{17}=4+\sqrt{17}-4=4+\frac{\sqrt{17}-4}{1}=4+\frac{(\sqrt{17}-4)\times(\sqrt{17}+4)}{1\times(\sqrt{17}+4)}$$

$$=4+\frac{1}{\sqrt{17}+4}=4+\frac{1}{8+\sqrt{17}-4}$$

$$=4+\frac{1}{8+\dfrac{(\sqrt{17}-4)\times(\sqrt{17}+4)}{(\sqrt{17}+4)}}=4+\frac{1}{8+\dfrac{1}{\sqrt{17}+4}}$$

and so on.

$$\text{Hence } \sqrt{17}=4+\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8+\text{etc.}}}}$$

Thus a surd produces a recurring fraction. Of course it could not produce a terminating fraction, since it is incommensurable. (See also p. 82.)

To find the Square Root of a Vulgar Fraction.

1. The Case in which both Numerator and Denominator are Perfect Squares.

Examples.—

(1) $\sqrt{\frac{121}{256}} = \frac{11}{16}$ *Ans.* (2) $\sqrt{1\frac{11}{567}} = \sqrt{\frac{16}{81}} = \frac{4}{9}$ *Ans.*

Rule.—Reduce the fraction to its lowest terms, and extract the root of the numerator for the numerator of the root, and the root of the denominator for the denominator of the root.

Note that the square root of a proper fraction is greater than that fraction. (See p. 66.)

2. The Case in which the Denominator only is a Perfect Square.

Examples.—

(1) $\sqrt{\frac{47}{49}} = \frac{\sqrt{47}}{\sqrt{49}} = \frac{6{\cdot}8555\ldots}{7} = {\cdot}9793\ldots$ *Ans.*

(2) $\sqrt{\frac{744}{968}} = \sqrt{\frac{93}{121}} = \frac{9{\cdot}6436\ldots}{11} = {\cdot}8766\ldots$ *Ans.*

Rule.—Reduce the fraction to its lowest terms; extract the root of the numerator to several places of decimals; divide the answer by the root of the denominator.

3. The Case in which neither is a Perfect Square.

Examples.—

$$\sqrt{\frac{13}{20}} = \sqrt{\frac{13 \times 20}{20 \times 20}} = \frac{\sqrt{260}}{20} = \frac{16{\cdot}1245}{20} = {\cdot}80622\ldots \text{ Ans.}$$

$$\text{or } \sqrt{\frac{13}{20}} = \sqrt{{\cdot}65} = {\cdot}80622\ldots$$

Rule.—Convert the fraction in its lowest terms into a decimal, and find the square root of the decimal, or rationalize the denominator.

4. The Case in which the Numerator only is a Perfect Square is dealt with as in (2) above.

5. The Case of Mixed Numbers.

Examples.—

$$\sqrt{13\tfrac{4}{9}} = \sqrt{\frac{121}{9}} = \frac{11}{3} = 3\tfrac{2}{3} \text{ Ans.}$$

$$\sqrt{\frac{14\frac{2}{5}}{16\frac{9}{10}}} = \sqrt{\frac{\frac{72}{5}}{\frac{169}{10}}} = \sqrt{\frac{72 \times 10}{169 \times 5}} = \sqrt{\frac{72 \times 2}{169}}$$

$$= \sqrt{\frac{144}{169}} = \frac{12}{13} \text{ Ans.}$$

Rule.—Convert the mixed number into an improper fraction in its lowest terms, and then proceed as in previous cases.

THEORY.

1. Define surd, rational and irrational quantities, with examples.

2. When is a surd said to be in its simplest form? On what rule or truth do most simplifications depend?

3. The exact value of a quadratic surd cannot be obtained. Why?

4. What is meant by rationalizing the denominator, and what is the object of it?

5. Give, with examples, the rule for rationalizing (*a*) a denominator of one term; (*b*) a denominator of two terms.

6. Explain the process of rationalizing a denominator of three or more terms.

7. Show the connection between surds and continued fractions.

EXERCISE LXV. 6

Simplify—

1. $2\sqrt{5} \times 5\sqrt{2}$.

2. $4\sqrt{2} \times 5\sqrt{3} \times 3\sqrt{7}$.

3. $3\sqrt{3} \times 2\sqrt{2} \times 7\sqrt{7} \times 5\sqrt{5}$.

4. $\sqrt{25} \div \sqrt{5}$.

5. $5\sqrt{15} \times 4\sqrt{18} \div 10\sqrt{3}$.

6. $3\sqrt[3]{2} \times 4\sqrt[3]{2} \times 5\sqrt[3]{2}$.

7. $5\sqrt[3]{2} \times 3\sqrt[3]{9} \times 7\sqrt[3]{3} \times 6\sqrt[3]{10}$.

8. $2\sqrt[4]{9} \times 3\sqrt[4]{12} \times 5\sqrt[4]{20} \div 15\sqrt[4]{45}$.

9. $\sqrt{40}$.

10. $3\sqrt{72}$.

11. $\sqrt{25200}$.

12. $\sqrt[3]{1944}$.

13. $\sqrt[3]{81} \times \sqrt[4]{256}$.

Show in each case which of the following is the greater :—

14. $2\sqrt{12}$ or $3\sqrt{15}$.

15. $4\sqrt{5}$ or $3\sqrt{9}$.

16. $7\sqrt{7}$ or $8\sqrt{6}$.

17. $115\sqrt{16}$ or $57\sqrt[3]{512}$.

18. $20\sqrt[4]{625}$ or $2\sqrt[3]{12}$.

Arrange in order of magnitude—

19. (*a*) $2\sqrt{5}$; (*b*) $\frac{1}{3}\sqrt{100}$; (*c*) $3\sqrt{3}$; (*d*) $4\sqrt{\frac{3}{4}}$.

20. (*a*) $5\sqrt{6}$; (*b*) $2\sqrt{38}$; (*c*) $4\sqrt{9}$.

Compare the following :—

21. (*a*) $\sqrt[3]{15}$ and $\sqrt[5]{226}$; (*b*) $\sqrt{256}$ and $\sqrt[3]{4913}$.

22. $\sqrt{4\sqrt{20}}$ and $\sqrt{18}$.

23. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ and $\frac{\sqrt{3}-1}{\sqrt{3}+1}$.

Rationalize the denominators of the following fractions :—

24. $\frac{5}{2\sqrt{3}}$.

25. $\frac{5\sqrt{12}}{6\sqrt{20}}$.

26. $\frac{4\sqrt{7}}{3\sqrt{3} \times 2\sqrt{2}}$.

27. $\frac{11\sqrt{12}}{5\sqrt{10} \times 4\sqrt{2}}$.

28. $\frac{3\sqrt{5} \times 4\sqrt{8}}{2\sqrt{3} \times 3\sqrt{5} \times 5\sqrt{7}}$.

29. $\frac{2\sqrt{2} \times 3\sqrt{3} \times 5\sqrt{5}}{3\sqrt{7} \times 4\sqrt{11} \times 6\sqrt{6}}$.

30. $\dfrac{1}{\sqrt{5}+\sqrt{3}+1}+\dfrac{1}{\sqrt{5}-\sqrt{3}+1}$.

31. $\dfrac{1}{\sqrt[3]{7}+1}+\dfrac{1}{\sqrt[3]{7}-1}$.

32. $\dfrac{5-\sqrt{-2}}{5+\sqrt{-2}}$.

33. $\dfrac{1}{\sqrt{8}+\sqrt{4}+\sqrt{2}+1}$

34. $\dfrac{\sqrt[4]{5}-1}{\sqrt[4]{5}+1}$.

35. $\dfrac{1}{\sqrt[4]{8}+\sqrt[4]{6}}+\dfrac{1}{\sqrt[4]{8}-\sqrt[4]{6}}$.

Find the value of—

36. $\{\sqrt{\cdot 6}+\sqrt{1\cdot 5}\}\sqrt{\cdot 24}$. (C.S.)

37. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}+\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$. (L.M.)

38. $\sqrt{\dfrac{(\cdot\dot{7}\dot{8})\times(\cdot 00\dot{0}0\dot{4})}{(\cdot 0\dot{1}\dot{3})\times(\cdot\dot{0}1\dot{5})}}$. (L.M.)

39. $(35\sqrt{10}+77\sqrt{2}+63\sqrt{3}+28\sqrt{15})\times(\sqrt{10}-\sqrt{2}-\sqrt{3})$. (R.M.A.)

Correct to 2 places of decimals—

40. $\sqrt{3+2\sqrt{2}}$. (C.P.)

41. $\dfrac{1}{\sqrt{3}+1}+\dfrac{1}{\sqrt{2}-1}$. (L.M.)

42. $16\sqrt{3}+10\sqrt[3]{4}-4\sqrt{12}-3\sqrt[3]{108}$. (R.M.A.)

Correct to 3 places of decimals—

43. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$. (L.M.)

44. $\dfrac{2\sqrt{18}+\sqrt{32}}{\sqrt{56}+3\sqrt{14}}$. (L.M.)

45. $\dfrac{\sqrt{5}+\sqrt{10}}{0\cdot 067}$. (S.K.)

46. $\dfrac{7+\sqrt{5}}{6-\sqrt{5}}-\dfrac{7-\sqrt{5}}{6+\sqrt{5}}$. (C.S.)

47. $2\sqrt{3}-\frac{1}{2}\sqrt{12}+4\sqrt{27}$. (L.M.)

Correct to 4 places of decimals—

48. $\dfrac{1}{\sqrt{\cdot 5}}$. (C.M.)

49. $\dfrac{4}{\sqrt{17}-3}$. (C.S.L.)

50. $\dfrac{3-\sqrt{3}}{2-\sqrt{2}}$. (L.M.)

51. $\dfrac{3-\sqrt{8}}{3+\sqrt{8}}$. (S.K.)

52. $\sqrt{\dfrac{1+\cdot 2375}{1-\cdot 2375}}$. (L.M.)

Correct to 5 places of decimals—

53. $\dfrac{1}{1-\sqrt{\cdot 9}}$. (L.M.)

54. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$. (L.M.)

55. $\dfrac{7-\sqrt{11}}{7+2\sqrt{11}}$. (L.M.)

Correct to 6 places of decimals—

56. $\sqrt{\cdot 4}-\sqrt{\cdot 000005}$. (C.S.)

57. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}-\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$. (L.M.)

Correct to 7 places of decimals—

58. $\dfrac{1}{\sqrt{8}}$. (L.M.)

Simplify—

59. $\sqrt{\dfrac{\sqrt{2}+1}{\sqrt{2}-1}}$. (I.I.)

60. $\sqrt{\dfrac{0{\cdot}0864 \times 7{\cdot}53}{0{\cdot}00391}}$. (L.M.)

61. $\sqrt{\dfrac{0{\cdot}\dot{4}28571\dot{4} \times 0{\cdot}7\dot{1}4285\dot{7}}{0{\cdot}2\dot{8}571\dot{4} \times 0{\cdot}0\dot{5}7142\dot{8}}}$. (L.M.)

62. $\dfrac{44}{3-\sqrt{5}} + \dfrac{44}{3+2\sqrt{5}}$. (S.K.)

63. $\frac{7}{15}\left(\dfrac{\sqrt{2-\frac{3}{5}} \times \sqrt{3+\frac{4}{7}}}{3\sqrt{2-\frac{3}{5}} \div \sqrt{3+\frac{4}{7}}}\right)$. (C.S.)

64. $\dfrac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}} - \dfrac{1-\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$. (S.K.)

65. $\dfrac{\sqrt{3}}{\sqrt{6}+\sqrt{5}} - \dfrac{3\sqrt{5}}{\sqrt{6}+\sqrt{3}} + \dfrac{2\sqrt{6}}{\sqrt{5}+\sqrt{3}}$. (L.M.)

66. $\dfrac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} + \dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$. (C.S.)

67. $(\sqrt{6}-\sqrt{2}) \times (\sqrt{2}+\sqrt{3}+\sqrt{\frac{2}{3}})$. (S.K.)

68. $\left(\dfrac{\sqrt{11}+\sqrt{7}}{\sqrt{11}-\sqrt{7}}\right)^2 - \left(\dfrac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}}\right)^2$. (S.K.)

69. $\dfrac{\sqrt[3]{5{\cdot}12} + \sqrt[3]{{\cdot}03375}}{\sqrt[3]{80} - \sqrt[3]{{\cdot}01}}$. (S.K.)

70. $\left(\dfrac{3^{\frac{2}{3}}+2-2^{\frac{3}{3}}3^{\frac{1}{3}}}{3^{\frac{2}{3}}+2+2^{\frac{3}{3}}3^{\frac{1}{3}}}\right)^{\frac{1}{2}} \times \dfrac{3^{\frac{2}{3}}+(72)^{\frac{1}{3}}+2}{3^{\frac{2}{3}}-(72)^{\frac{1}{3}}+2} \times \dfrac{3+2^{\frac{3}{2}}}{3-2^{\frac{3}{2}}}$. (C.M.)

CHAPTER XLVII.

VOLUMES.

A Solid is a body which has three dimensions—length, breadth (or width), thickness (or height).

A Rectangular Solid is a body bounded by *six* rectangular plane faces.

Its opposite pairs of faces are equal and parallel.

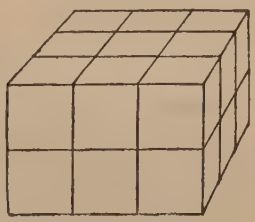
A Cube is a body bounded by six *equal* square faces.

Thus a cube is a special kind of rectangular solid.

The Volume (or solid contents) of a body is the space included within the surfaces which bound it. This space is expressed as so many times a given unit of measure.

The English units of measure are the cubic inch, cubic foot, and cubic yard.

To ascertain the Volume of a Rectangular Solid when its Length, Breadth, and Height are known.



If we take 18 cubes each 1 inch long and arrange them as shown in the diagram, they form a rectangular solid 3 in. long, 3 in. wide, and 2 in. deep. There are evidently 3 rows each containing 3 cubes in the upper layer, and there are 2 such layers.

Hence the total number of cubes is $3 \times 3 \times 2 = 18$. But each is a cubic inch, therefore the total cubic contents are 18 cub. in.

Hence the following rule :—

To find the Volume of a Cube.

(*a*) Express length, breadth, and height in units of the same name.

(*b*) Find the continued product of those units.

(*c*) The result is the number of cubic units in the volume.

It is customary to say—

Volume = length × breadth × height

Now, since *breadth* is a "short length," and *height* is an "upward length," also *depth* is a "downward length," any of the following products will give an *area* :—

$$\left.\begin{array}{l}\text{Length} \times \text{breadth}\\ \text{Length} \times \text{height}\\ \text{Breadth} \times \text{height}\end{array}\right\} = \text{area}$$

So we can re-write our formula thus—

$$\text{Volume} = \left\{\begin{array}{l}\text{area of any face} \times \text{that third dimension not included}\\ \quad\text{in the area of the face}\end{array}\right.$$

And from the two formulæ above, the following can be easily deduced :—

$$\frac{\text{Volume}}{\text{the product of any two dimensions}} = \text{the remaining dimension}$$

$$\frac{\text{Volume}}{\text{any dimension}} = \left\{\begin{array}{l}\text{the product of the remaining}\\ \quad\text{dimensions}\end{array}\right.$$

The sense in which length, breadth, height, and volume are used is the same as in p. 396.

Given the Volume of a Cube, to find the Side of the Cube.

Find the cube root of the number of units in the volume (see p. 412).

Example.—*Find the cubic content of a piece of timber* 15 *ft.* 5 *in. long,* 2 *ft.* 4 *in. broad, and* 2 *ft.* 1 *in. thick.*

Reducing to same unit, length $= 15\frac{5}{12}$ ft., breadth $= 2\frac{1}{3}$ ft., and thickness $= 2\frac{1}{12}$ ft.

Number of cubic feet in volume $= 15\frac{5}{12} \times 2\frac{1}{3} \times 2\frac{1}{12}$

$$= \frac{185}{12} \times \frac{7}{3} \times \frac{25}{12} = \frac{32375}{432} = 74\frac{407}{432}$$

Ans. $74\frac{407}{432}$ cub. ft., or 74 cub. ft. 1628 cub. in.

Example.—*A brick is* 9 *in. long,* $4\frac{1}{2}$ *in. wide, and* 3 *in. thick, and weighs* 6 *lbs. How many bricks would be required to build a wall* 1 *mi. long,* 7 *ft. high, and* 18 *in. thick, and how many tons would it weigh?* (P.E.A.)

Cubic contents of wall in cubic feet $= (1 \times 5280) \times 7 \times 1\frac{1}{2}$

" " brick " $= \frac{3}{4} \times \frac{3}{8} \times \frac{1}{4}$

$$\text{Number of bricks} = \frac{1 \times 5280 \times 7 \times 1\frac{1}{2}}{\frac{3}{4} \times \frac{3}{8} \times \frac{1}{4}}$$

$$= \frac{1 \times 5280 \times 7 \times 3 \times 4 \times 8 \times 4}{3 \times 3 \times 2}$$

$$= 788480 \text{ bricks}$$

$$\text{Weight in tons} = \frac{788480 \times 6 \text{ lbs.}}{2240} = 2112$$

Ans. 788480 bricks and 2112 tons.

Example.—*A box is made of wood* $\frac{1}{3}$ *in. thick. When closed its external measurements are* $9\frac{2}{3}$, 9, *and* $7\frac{1}{3}$ *in. Find its weight, if* 9 *cub. in. of the same wood weigh* 5 *oz.* (C.P.)

Volume of box (*outside* measurement) in cubic inches $= 9\frac{2}{3} \times 9 \times 7\frac{1}{3}$

„ (*inside* measurement) „ $= 9 \times 8\frac{1}{3} \times 6\frac{2}{3}$

The internal dimensions are found by subtracting *twice* the thickness of the wood from each of the given external dimensions.

Volume of wood forming the box (in cubic inches) $=$ volume of box (outside measurement) − volume of box (inside measurement)

$$= 9\tfrac{2}{3} \times 9 \times 7\tfrac{1}{3} - 9 \times 8\tfrac{1}{3} \times 6\tfrac{2}{3}$$
$$= \tfrac{29}{3} \times \tfrac{9}{1} \times \tfrac{22}{3} - \tfrac{9}{1} \times \tfrac{25}{3} \times \tfrac{20}{3}$$
$$= 638 - 500 = 138$$

$$\text{Weight in ounces} = \frac{138 \times 5}{9} = 76\tfrac{2}{3} = 4 \text{ lbs. } 12\tfrac{2}{3} \text{ oz.}$$

Ans.

Example.—*Find in tons what weight of water falls upon* $85\frac{1}{3}$ *ac. of ground during a rainfall of* $1\frac{3}{4}$ *in., a cubic foot of water being taken to weigh* 1000 *oz. avoirdupois.* (C.S.L.)

Area of ground in sq. inches $= 85\frac{1}{3} \times 4840 \times 9 \times 144$

Cubic content of rainfall in cubic feet $= \dfrac{85\frac{1}{3} \times 4840 \times 9 \times 144 \times 1\frac{3}{4}}{1728}$

$$\text{Weight of water in ounces} = \frac{85\frac{1}{3} \times 4840 \times 9 \times 144 \times 1\frac{3}{4} \times 1000}{1728}$$

$$\text{tons} = \frac{256 \times 4840 \times 9 \times 144 \times 7 \times 1000}{3 \times 1728 \times 4 \times 16 \times 2240}$$
$$= 15125 \text{ tons.} \quad \textit{Ans.}$$

EXERCISE LXVI.

Volumes: no Cube Root involved.

1. How many planks 17 ft. long, 8 in. broad, and $3\frac{1}{4}$ in. deep can be stored in a place 51 yds. long by 6 yds. broad and 13 ft. deep? (K.S.)

2. A brick measures $8\frac{3}{4}$ in. by $4\frac{1}{4}$ in. by $2\frac{3}{4}$ in. How many bricks are contained in a stack 5 yds. 1 ft. 6 in. long, 11 ft. 3 in. broad, and 9 ft. 11 in. high? (O.J.)

3. How much will a log of timber be worth, 30 ft. long, with a uniform area across of 3 sq. ft., at 5*s.* 3*d.* the cubic foot? (C.P.)

4. How many more cubic inches of wood are there in a solid cube of mahogany, the length of whose edge is 16 in., than in a solid rectangular block of oak, 5 ft. 4 in. long, 1 ft. 3 in. wide, and 4 in. thick? The weight of a cubic inch of the oak is to that of a cubic inch of the

mahogany as 39 is to 35, and the oak weighs 130 lbs. What is the weight of the mahogany? (C.P.)

5. Find the number of bricks required to build a wall 26 yds. long, 7 ft. 8 in. high, and 14 in. thick, each brick and its mortar being 9·1 in. long, 4·6 in. broad, and 3·2 in. deep. (P.E.A.)

6. Find the cost of the bricks needed for building a wall 30 yds. long, 6 ft. high, and 13½ in. thick, having given that 1000 bricks cost 25*s.*, and that each brick fills up a space 9 in. long, 4½ in. wide, and 3 in. deep. (S.K.)

7. Two thousand packets each 4 in. long, 3 in. broad, and 3 in. deep, are required to be packed in a box of which the inside dimensions are 4 ft. 3 in. in length, 3 ft. 4 in. in breadth, and 2 ft. 6 in. in depth. How many must be left out? (O.R.)

8. How many blocks of wood, 8 in. by 4 in., will be required for a floor 53 ft. by 18 ft.? If the block be 7 in. deep, what will be the cubical content of the floor in (i.) cubic yards and cubic feet, (ii.) in cubic metres? (c.m. = 35⅓ cub. ft. nearly.) (C.P.)

9. The larger of two rooms is 47 ft. long, 30 ft. wide, and 25 ft. high; the smaller is 25 ft. long, 20 ft. wide, and 18 ft. high. Compare their cubic contents. If the four walls of the larger room are painted at a cost of 1*s.* 3*d.* a square yard, and the four walls and ceiling of the smaller room at a cost of 1*s.* 4½*d.* a square yard, compare the expenses of painting the rooms. (S.K.)

10. A schoolroom is 32 ft. long, 18 ft. wide, and 10½ ft. high. How many children will it accommodate, allowing 8 sq. ft. of floor for each; and how many cubic feet of space will there be for each? (C.P.)

11. If 1 cub. ft. of water is equal to 6·25 gals. or 62·321 lbs., find the number of gallons and the weight of water contained in a cistern 4 ft. long by 2½ ft. broad and 3 ft. deep. (O.R.)

12. Determine the number of cubic inches of wood in a box with a lid made of wood ¾ in. thick, and measuring externally 3 ft. by 2 ft. by 1 ft. 5½ in. (O.S.L.)

13. Find in tons, cwts., etc., the weight of water contained in a tank 6 yds. long, 2 yds. high, and 8 ft. broad, the tank being full, and the weight of a cubic foot of water being 1000 oz. (O.S.L.)

14. The inside measurements of a tank are: length 15 ft. 6 in., width 5 ft. 4 in., and depth 4½ ft. (i.) What weight of water will it hold, taking a cubic foot of water as weighing 1000 oz.? (ii.) How many gallons (each equal to 277·264 cub. in.) will it contain? (C.P.)

15. A cistern is 6 ft. long and 5 ft. broad, and it is of such a depth that it would take 1440 bricks, each 9 in. long, 3 in. deep, and 4 in. broad, to fill it. How many gallons would the cistern hold, if a pint of water contains 72 cub. in.? (R.M.A.)

16. A gallon measures 277·274 cub. in., and a litre may be represented by a cube whose edge is 3·937 in. Find, to 2 places of decimals, how many pints there are in a litre. (S.K.)

17. If a cubic foot of water weighs 1000 oz., find in tons, etc., the weight of water contained in a reservoir ¾ mi. long, 1 fur. broad, and 20½ ft. deep. (C.PR.)

18. A cistern 5 ft. long by 4 ft. wide, whose depth is 3 ft. 8 in. (inside measure), has 30 cubic feet of water in it. Porous bricks are placed in the water until the cistern is brim full, each brick absorbing $\frac{1}{17}$ of its own volume of water. How many bricks can be put in without the water overflowing, each brick being 9 in. by 3 in. by 2½ in.? (C.J.)

19. Find the weight of the air contained in a room 88 ft. long, 56 ft. broad, and 35 ft. high, assuming that a pint of water weighs 20 oz., that a gallon contains 277·2 cub. in., and that water is 800 times as heavy as air. (C.H.L.)

20. The Mississippi brings down annually 7459267200 cub. ft. of sediment, and deposits it on an area of 12300 sq. mi. Calculate as accurately as the data permit (the last significant figure being approximate), the mean rate of thickness of deposit estimated in feet per century. (S.A.)

21. It is required to construct a reservoir which will hold ten million gallons of water. What must be the depth of the reservoir, if its length is 400 ft. and its breadth 200 ft., supposing its sides to be vertical, and assuming that a cubic foot is equal to 50 pts. (C.H.L.)

22. What is the cubical content of a rectangular hole 21 ft. 4 in. long, 19 ft. 4 in. wide, and 6 ft. 9 in. deep? If the four sides of the hole are to be lined with concrete 16 in. thick, how many cubic feet of concrete will be required? (C.P.)

23 A box is made of wood $\frac{1}{3}$ in. thick. When closed its external measurements are $9\frac{2}{3}$, 9, and $7\frac{1}{3}$ in. Find its weight, if 9 cub. in. of the same wood weigh 6 oz.? (C.P.)

24. A box (without a lid) made of wood an inch thick, measures on the outside 30 in. long, 21 in. wide, and 16 in. deep. Find the cubic contents of the interior, and the cost of painting the outside at 7*d.* per square foot? (K.S.)

25. The Black Hole of Calcutta contained 146 prisoners for a night of 10 hrs.; it was 18 ft. square and 10 ft. high. How often would the whole air of the room have been breathed in and out in that time, assuming that each prisoner breathed 15 times a minute, and passed in and out 30 cub. in. of air each time? (C.P.)

26. A rectangular cistern 4 ft. 2 in. long, 2 ft. 8 in. wide, and 2 ft. deep, is $\frac{1}{3}$ full of water. Find the least number of heavy cubical blocks, each measuring 6 in. every way, that can be introduced into the cistern to make the water just overflow. (C.M.)

27. A quadrangle 120 ft. by 100 ft. has a grass plot 60 ft. by 50 ft. in the centre. Find the cost of gravelling the rest of it to a depth of 6 in. at 6*s.* a cubic yard. (O.S.L.)

28. What depth of trench 9 ft. wide must be dug round a plot of ground 60 yds. long and 80 yds. wide, in order that the earth removed may be sufficient to raise the level of the whole plot 3 in.? (P.T.)

29. Along the four sides of a square plot, and inside of it, I dig a trench 3 ft. wide and 8 in. deep, and fill it with good loam for a flower border. If the area of the plot be 784 sq. yds., what will be the cost of the loam at 4*s.* 6*d.* a cubical yard? (C.P.)

30. A gravel walk 6 ft. wide runs round a grass plot 60 ft. long and 40 ft. wide. If gravel is 3*s.* per cubic yard, find the cost of a coat of gravel on the path 3 in. deep. (R.M.A.)

31. A tank 18 ft. long and 6 ft. deep cost £10 16*s.* If it had been 2 ft. wider the cost would have been £2 14*s.* more. Find the cost per cubic foot and the width of the tank.

32. The area of the base of a tank which can hold 150 gals. is a square yard. Find (correctly to the nearest thousandth part of an inch) the depth of the tank, assuming that a gallon contains 277·274 cub. in. (P.E.A.)

Rainfall.

33. During a rainfall of an inch, 100 tons of water fall on an acre of ground. How many ounces are there in a cubic foot of water? (C.PR.)

34. If a cubic foot of water weighs 1000 oz., what weight of water will fall on an acre of ground during a rainfall of an inch? (C.PR.)

35. An inch of rain has fallen on 3 ac. of land. Find the total weight of the water, assuming that a cubic yard of water weighs 15 cwt. (O.S.L.)

36. A level tract of land, 20 mi. long and $\frac{3}{4}$ mi. broad, is flooded to a depth of 5 ft. Given that a cubic foot of water weighs 62 lbs., find in tons the weight of the water on the land. (C.PR.)

37. A water-tank 16 ft. 6 in. long, 8 ft. 4 in. wide, and 6 ft. 3 in. deep, is just sufficient to hold the water which runs into it during a rainfall of 1 in. If it be connected with an auxiliary tank 7 ft. 6 in. long, 5 ft. 6 in. wide, and 5 ft. deep, for what rainfall would the two tanks then suffice? (C.S.)

38. The rainfall in 1 min. during a recent storm was found to be 8 oz. per square foot. Taking the roof of an ordinary house to be 20 ft. by 30 ft., find the weight of water which fell upon it, and, again supposing the downpour uniform over an acre of roofs, calculate the weight which fell upon them in the same minute of time. (C.P.)

39. The annual rainfall at a place is 24 in. Find the weight in tons, and volume in gallons, of the annual rainfall there on an acre of land, taking the weight of water to be 1000 oz. to the cubic foot, and a gallon to weigh 10 lbs. (C.P.)

40. A rectangular tank, measuring internally 3 ft. in width by 6 ft. in length, receives all the rain that falls on a roof, whose area is 7200 sq. ft., and it was estimated that in one day a pint of water fell on each square foot of roof. If the tank was empty to begin with, what was the depth of water in it at the end of the day, if a pint of water fills 12 cub. in.? (C.M.)

EXERCISE LXVII. 67

Volumes: some necessitating Cube Root.

1. A plate of metal is 106·58 in. long, 14·6 in. wide, and 2 in. thick. Supposing it to be melted and cast into an exact cube, what would be the edge of the cube? (S.K.)

2. A cube contains 50·653 cub. yds. Find the length of an edge and the total area of all its faces. (C.M.)

3. Prove that the length of a side of a square, the area of which is 15876 sq. yds., is to that of an edge of a cube the contents of which are 94818816000 cub. in. as 189 : 190. (C.P.)

4. Find the length of the edge of a cubical block of coal which weighs 2000 tons. A cubic foot of coal weighs 1120 oz. (C.W.)

5. If the surface of a cube has an area of 305·8776 sq. in., find the length of one edge of the cube. Find also to three decimal places the volume of the cube in cubic inches. (S.K.)

6. If 648 boxes are stowed close together on board a ship, find the total cost of freightage at 1*s.* 2*d.* per cubic yard, when each box measures

$4\frac{1}{2}$ ft. long, 3 ft. wide, and $2\frac{1}{4}$ ft. deep. If all the boxes are piled in the form of a cube, what is the length of an edge of the cube? (S.K.)

7. The content of a cube is 650·962 cub. ft. Find the cost of covering it with material 2 ft. 2 in. wide, at 9*d.* per yard. (C.S.)

8. It is desired to put a cubical case, whose content is 4019·679 cub. ft., through a square hatchway whose area is 37791·36 sq. in. Show whether this can be done. (R.M.A.)

9. Two cubes of metal, whose edges are 1 ft. and 8 ft. respectively, are to be melted and recast into a single cube. Find the edge of the new cube to two places of decimals, allowing $\frac{1}{10}$ of the metal to be wasted in the process. (C.P.)

10. A cube of iron weighs 7 cwt. 101 lbs. 13·6 oz., and is 7·82 times as heavy as the water which would fill a cubical vessel of equal volume. Find, to the nearest tenth of an inch, a side of the cube of iron. (C.S.)

11. The faces of a rectangular block contain 768, 576, and 432 sq. in. By how much does the aggregate length of its edges exceed that of a cube of equal volume? (S.A.)

12. A rectangular tank with square bottom is required such that, when it contains 600 gals. of water, the depth of the water may be $3\frac{1}{4}$ ft. Determine the length inside of the square base, correct to $\frac{1}{10}$ inch, taking a gallon of water to contain $277\frac{1}{4}$ cub. in. (P.P.)

13. A salt lake is said to contain 4000000000 tons of salt. Its water is 1·168 of the weight of fresh water, the salt being 3·355 of the weight of fresh water. Find the area of the lake to the nearest square mile, assuming its average depth to be 10 fath. (S.A.)

14. An open tank measuring on the outside 6 yds. 2 ft. in length, 3 yds. 2 ft. in width, and 7 ft. deep, is made with sides and floor of brick 1 ft. thick, and is filled with water. Find the weight of the tank and its contents, it being given that 1 cub. ft. of water weighs 1000 oz., and that brick is $1\frac{1}{2}$ times as heavy as water. (R.M.A.)

15. A piece of metal, measuring 1 ft. 4 in. by $10\frac{1}{2}$ in. by $3\frac{3}{10}$ in., being melted down, how many cubes $2\frac{1}{2}$ in. each way can be cast out of it? And how many $\frac{1}{10}$ in. each way can be made of what is left? (C.S.)

16. A hollow rectangular vessel, without lid, formed of material 1 in. thick, whose external dimensions are 13 ft. 6 in., 6 ft. 8 in., and 6 ft. 11 in. respectively, weighs 1 cwt. 1 qr. 10 lbs. Find the weight of a solid mass of the same material and the same dimensions. (O. and C.S.C.)

17. A reservoir contains 3217428 cub. ft. of water, its depth is $\frac{1}{3}$ of its length, and its breadth is half the difference between the length and $\frac{1}{3}$ of the depth. Find the dimensions. (C.S.)

18. The volume of timber in the trunk of a tree is roughly estimated by multiplying the length by the square of the mean quarter-girth. The trunk of a certain tree, if cut so as to be 27 ft. in length, would have a mean girth of 10 ft. 6 in.; and if it were cut 10 ft. longer, the mean girth would be 9 ft. 9 in. In the former case the value would be estimated at £5 10*s.* 3*d.*; what would it be in the latter? (C.S.)

19. A rectangular cistern, 8 ft. long and 7 ft. wide, holds when full $6\frac{1}{4}$ tons of water. Find its depth, assuming that a cubic foot of water weighs 1000 oz. Find also within $\frac{1}{100}$ of a foot the depth of a cistern that holds when full the same quantity of water, and has its length, width, and depth all equal. (S.K.)

20. A tank 9 yds. long, $3{\cdot}0\dot{3}$ yds. wide, and $2{\cdot}05\dot{3}$ ft. deep, holds water enough for 27 men for 308 days. Find the depth of a tank, 6·2 yds. long and $2{\cdot}0\dot{2}$ yds. wide, which will hold enough for 158 men, 48 women, and 81 children for 45 days, if a woman use $\frac{3}{4}$, and a child $\frac{2}{3}$ as much as a man. (C.S.)

21. A cubical cistern, open at the top, costs 15 guineas to line with lead at 1*s.* 9*d.* per square foot. How many cubic feet will it contain? (C.PR.)

22. A hollow, closed cubical box is made of iron, and is 2 in. thick, the external edge being 4 ft. long. It being given that iron weighs 7·112 times as much as water, and that a cubic foot of water weighs 1000 oz., find in tons the weight of the box. (L.M.)

23. A cistern 12 ft. long, 8 ft. 10 in. broad, and 6 ft. 6 in. deep, is filled with water, and after a time 4000 gals. are drawn off, leaving a sediment which is sold for £2 3*s.* 11½*d.* per cubic foot. Find what it brings, a gallon being equal to 277·279 cub. in. (C.S.)

24. What must be the area, to the nearest rood, of the surface of the water in a reservoir with vertical boundaries, so that, when no water enters the reservoir, the withdrawal of one million gallons of water per day may lessen the depth of the water by 1 ft. in a week? [A gallon contains 277·274 cub. in.] (C.P.)

25. A rectangular block of wood is overlaid with metal of thickness $\frac{1}{8}$ in. The whole is now 2 ft. long, and the ends are square and 3 in. broad. The value of the metal used is £9 14*s.* 1*d.* What is its value per cubic inch? (P.T.)

26. A wall five times as high as it is broad, and eight times as long as it is high, contains 18225 cub. ft. Find the breadth of the wall. (C.S.)

27. The length, breadth, and height of a hall are proportional to 5, 4, and 3, and it contains 555660 cub. ft. Find the cost of covering the floor at 1*s.* 3*d.* per square yard. (O.S.L.)

28. The length, breadth, and height of a room are in the ratios $8 : 4 : 3\frac{1}{2}$. Find the ratio of the area of the walls to that of the floor. Also if the cubical content be 3024 cub. yds., what are the dimensions of the room? (S.A.)

29. The length, breadth, and height of a covered box (outside measure) are in the proportion of $5 : 4 : 3$. If the length and breadth together are 72 in., and the wood of which it is made is $\frac{1}{2}$ in. thick, what is the cubical capacity of the box? (C.P.)

30. The length, breadth, and depth of a rectangular tank are as $4\frac{1}{5} : 3\frac{1}{4} : 2\frac{1}{3}$. If half a cubic inch evaporate from an area of 1 sq. in. in a day, the tank dries up in 85 days. Find the dimensions of the tank. (S.A.)

31. A box without a lid measures externally 4 ft. long, 3 ft. wide, and 2 ft. deep; the material has a uniform width of $\frac{3}{4}$ in. If the wood cost 7*s.* 9*d.* the cubic foot, and the making cost $\frac{1}{10}$ of the material, what did the box cost to the nearest penny? (C.S.)

32. A cubic foot of copper weighs 560 lbs.; it is rolled into a square bar of 40 ft. long. An exact cube is cut from the bar. What is its weight to four decimals of a pound? (S.K.)

33. A thin rectangular lamina of metal, 3 ft. 2 in. long and 2 ft. 8 in. wide, has cut from its four corners four squares whose sides are 3 in. long.

The four projecting portions are turned up at right angles to the rest of the lamina, and thus form a box. Find its cubical contents. (C.P.)

34. A cask weighing 2 cwt. 12 lbs. 4 oz. floats in a square cistern of water whose side is 2 ft. 6 in. On the removal of the cask, find how much the water will sink in the cistern, supposing a cubic foot of water to weigh 63 lbs. (C.S.)

Running Water, etc.

35. The dimensions of a rectangular room are 25 ft. 4 in. by 18 ft. by 13 ft. 6 in. Find how many cubic feet of air must be supplied by the ventilating pipes in order that the whole atmosphere of the room may be renewed in an hour. Also find (in square inches) what must be the least total sectional area of the pipes in order that the current in them may not exceed a foot and a half per second. (L.M.)

36. A room 24 ft. long, 20 ft. wide, and 14 ft. high is ventilated by a rectangular opening 16 in. by 10 in. At what rate per minute must the air pass through the opening that $\frac{1}{5}$ of the air in the room may be changed every hour? (C.W.)

37. A section of a stream is 10 ft. wide and 10 in. deep; the mean flow of the water through the section is 3 mi. an hour. Taking 25 gals. to equal 4 cub. ft., find how many gallons of water flow through the section in 24 hrs. (S.K.)

38. A cistern is filled in $3\frac{1}{2}$ hrs. by a pipe 3 sq. in. in cross-section, through which water flows at the rate of 6·4 mi. an hour. What is the volume of the cistern? (R.M.A.)

39. The internal measurements of a tank are 3 ft. 8 in., 3 ft., and 3 ft. 4 in. It is filled in 16 min. by a pipe whose cross-section is $\frac{3}{4}$ sq. in. What amount of water leaves the pipe in each minute, and what length of the pipe did this amount of water occupy? (C.W.)

40. A river 30 ft. deep and 200 yds. wide flows 4 mi. an hour. Find the number of cubic feet of water that falls into the sea in a minute; also its weight in tons (1 cubic foot = 1000 oz.). (C.S.)

CHAPTER XLVIII.

SCALES OF NOTATION.

THERE is no doubt that man learned to count in *tens* because he has 10 fingers on his hands. But there is no reason why arithmetic should be limited to counting by tens; we can count by twos, sevens, twelves, and, indeed, any number.

The Base of a Scale of Notation.—The number by which we count is termed the base of the scale.

Thus, if we count by sixes, 6 becomes the base, and 7 units will become 1 six and 1 unit, and be written as 11.

By the **Base, Radix** or **Root** of any system is meant the number of one denomination fixed upon to equal one of the next higher denomination.

Scales of Notation are named from their bases.

BASE.	NAME.	Significant digits in each scale.
2	Binary	1
3	Ternary	1, 2
4	Quaternary	1, 2, 3
5	Quinary	1, 2, 3, 4
6	Senary	1, 2, 3, 4, 5
7	Septenary	1, 2, 3, 4, 5, 6
8	Octenary	1, 2, 3, 4, 5, 6, 7
9	Nonary	1, 2, 3, 4, 5, 6, 7, 8
10	Denary (decimal)	1, 2, 3, 4, 5, 6, 7, 8, 9
11	Undenary (undecimal)	1, 2, 3, 4, 5, 6, 7, 8, 9, *t*
12	Duodenary (duodecimal)	1, 2, 3, 4, 5, 6, 7, 8, 9, *t*, *e*

The names are derived from the Latin distributive numerals.

In order to familiarize ourselves with the various systems, we shall examine one or more systems in detail, and to do this we proceed to write a few numbers in the **Senary Scale.**

1 would be read as one
2 „ „ two
3 „ „ three
4 „ „ four
5 „ „ five
} just as in the ordinary or decimal system.

but 6 „ „ one six and no units, and would be written as 10
7 „ „ one six and one unit, „ „ 11
13 „ „ two sixes and one unit, „ „ 21

Avoid saying that *thirteen* in the decimal scale is written as twenty-one in the senary scale; it is written as 2 sixes and 1.

Again continuing our investigations, but this time in the **Duodecimal Scale**, we obtain—

1, 2, 3, 4, 5, 6, 7, 8, 9 in the decimal scale as 1, 2, 3, 4, 5, 6, 7, 8, 9 in the duodecimal scale.

But when we come to 10 in the decimal scale we find ourselves confronted with the difficulty that our digits are exhausted. Hence two new digits, *t* for ten and *e* for eleven, are called into requisition.

Hence	10	in the decimal scale becomes	*t*	in the duodecimal scale
Similarly	11	„ „	*e*	„ „
and	12	„ „	10	„ „
„	13	„ „	11	„ „
„	20	„ „	18	„ „
„	82	„ „	6*t*	„ „
„	100	„ „	84	„ „
„	130	„ „	*tt*	„ „
„	143	„ „	*ee*	„ „

From the above and other considerations, we see that in any scale of notation—

1. The number of significant digits is one less than the base.

2. The right-hand digit represents so many times the unit, the second digit so many times the base of the scale, the third digit so many times the second power of the base of the scale, and so on.

Thus—

			In decimal notation.
1.	With a base of 10,	243 represents	$(2 \times 10^2)+(4 \times 10)+3 = 243$
2.	„ „	5, 243 „	$(2 \times 5^2)+(4 \times 5)+3 = 73$
3.	„ „	8, 243 „	$(2 \times 8^2)+(4 \times 8)+3 = 163$
4.	„ „	12, 243 „	$(2 \times 12^2)+(4 \times 12)+3 = 339$

Addition.

Add together 28846, 72466, 23446, 20018 *in the scale of nine.* (C.S.)

28846		Sum of units' column = 26 = 2 nines and 8 "over." Write the 8 and "carry" the 2.
72466		
23446		Sum of the "nines'" column = 17 (nines) = 1 nine (to the second power) and 8 (nines) over. Write the 8 and "carry" the 1.
20018		
155888	*Ans.*	Sum of the next (*nines-to-the-second-power*) column = 17 (*nines to the second power*) = 1 (*nine to the third power*) and 8 (*nines to the second power*) over, and so on.

Note.—A common source of error is to put down 14 as the sum of the last column. This is wrong, because 14 is in the scale of 10, and not in that of 9.

Subtraction.—*From* 9*e*426 *take* 5*et*41 *in the scale of* 12. (C.S.)

9*e*426	1 from 6 leaves 5 ; put down 5.
5*et*41	4 from 2 + (12) leaves 10 ; put down *t*.
	(1) + *t* from 4 + (12) leaves 5 ; put down 5.
3*e*5*t*5 *Ans.*	(1) + *e* from *e* + (12) leaves 11 ; put down *e*.
	(1) + 5 from 9 leaves 3 ; put down 3.

Multiplication.—*Multiply* 43620 *by* 265 *in the scale of* 7. (C.M.)

43620	
265	
	5 × 0 = 0 ; put down the 0.
315330	5 × 2 = 10 = 1 seven and 3 over ; put down the
362250	3 and "carry" 1.
120540	5 × 6 = 30 ; 30 + 1 = 31 = 4 sevens and 3 over ;
	put down the 3 and "carry" the 4, and so on.
16325130 *Ans.*	

Division.—*Divide* 42404 *by* 871 *in the scale of* 11. (C.M.)

871)42404(54 *Ans.*
3*t*25
3164
3164

If the student has mastered the preceding explanations, this sum will present no difficulty.

To change a number from the decimal scale to any other scale.

We have seen that if we wish to express 7 in the senary scale, we write it as 11 (one six and one unit), *i.e.* we divide the given number in the scale of 10 by the base of the number to which we wish to reduce the given number. Apply this knowledge to the following :—

Example.—*Express* 386 *in the undenary scale.*

11 | 386
11 | 35 times 11 + 1 over.
3 times 11^2 + 2 times 11 over.

The answer is : $(3 \times 11^2) + (2 \times 11) + 1$; or, written more simply, 321 in the undenary scale. *Ans.*

Hence the following **Rule**—

(1) Divide the given number by the base of the proposed scale.

(2) Divide the quotient obtained from (1) by the same base.

(3) Continue the process until a quotient is obtained which is less than the given base.

(4) The last quotient, with the several remainders taken in order from last to first, will express the given number in the scale required.

To change a number from any scale to the decimal scale.

Example.—*Change* tt69*e*9 *in the duodenary scale to the common scale.* (C.S.)

(*a*) *By the method of decomposition.*

$ttt69e9$ in the scale of 12

$= t\times12^6+t\times12^5+t\times12^4+6\times12^3+9\times12^2+e\times12+9$

$= 29859840+2488320+207360+10368+1296+132+9$

$= 32567325$ in the scale of 10 *Ans.*

(*b*) *By the method of division.*

10	$ttt69e9$
10	$1110824 + 5$
10	$138575 + 2$
10	$16t1e + 3$
10	$1t74 + 7$
10	$231 + 6$
10	$28 + 5$
	$3 + 2$

The successive divisions must be performed in the duodenary scale. The difficulty of this method is that the student is so likely to revert to the scale of 10.

Ans. 32567325 in ordinary scale.

To change a number from any one scale (not the decimal) to any other scale (not decimal).

Example.—*What number in the undenary scale is represented by* 6520116 *in the septenary scale?* (C.M.)

(*a*) By the combined method of decomposition and reduction. First reduce the number to the scale of 10.

6520116 in scale of 7

$= (6\times7^6)+(5\times7^5)+(2\times7^4)+(1\times7^2)+(1\times7)+(6)$

$= 794793$ in scale of 10.

Next, reduce the number from the scale of 10 to the required scale—

11	794793
11	$72253 + t$
11	$6568 + 5$
11	$597 + 1$
11	$54 + 3$
	$4 + t$

Ans. $4t315t$.

(*b*) By the method of division.

11	6520116	
11	420436 + *t*	
11	25102 + 5	The division is performed in the scale of 7.
11	1512 + 1	
11	105 + 3	
	4 + *t*	

Ans. 4*t*315*t*.

To transform a Vulgar Fraction from one scale to another.

Express the numerator and denominator each in the new scale.

Scales of Notation and Fractions.

Quantities less than unity may be expressed in any scale of notation.

Fractions which have powers of the radix for their denominations are known as Radix Fractions. Thus, ·635 in the scale of 7 is a radix fraction, for it stands for—

$$\frac{6}{7}+\frac{3}{7^2}+\frac{5}{7^3}$$

In reality, decimals are radix fractions with a base 10, but it is usual to use the term "radix fraction" when some base other than 10 is used.

To express a Fraction in the decimal scale by a series of Radix Fractions in any other scale.

Example.—*Express* ·32421875 *in the scale of* 8.

Let $a, b, c, d \ldots$ be the digits when expressed in the scale of 8. Then the value of each of the digits $a, b, c, d \ldots$ is less than 8.

Then $\cdot 32421875 = \frac{a}{8}+\frac{b}{8^2}+\frac{c}{8^3}+\frac{d}{8^4}+, \text{etc.}$ { (in scale of 8, where each fraction is a proper fraction)

Multiply by 8—

$$2\cdot 59375 = a+\frac{b}{8}+\frac{c}{8^2}+\frac{d}{8^3}+, \text{etc.}$$

$\therefore$ integral part a = integral part 2

$$\text{and } \cdot 59375 = \frac{b}{8}+\frac{c}{8^2}+\frac{d}{8^3}+, \text{etc.}$$

Multiply by 8—

$$4\cdot 75 = b+\frac{c}{8}+\frac{d}{8^2}+, \text{etc.}$$

$$\therefore b = 4$$

$$\text{and } \cdot 75 = \frac{c}{8}+\frac{d}{8^2}+, \text{etc.}$$

Multiply by 8—

$$6 = c + \frac{d}{8} +, \text{ etc.}$$

$$\therefore c = 6$$

$$\text{and } \frac{d}{8} +, \text{ etc.} = 0$$

$$\therefore \frac{a}{8} + \frac{b}{8^2} + \frac{c}{8^3} + \frac{d}{8^4} +, \text{etc.} = \frac{2}{8} + \frac{4}{8^2} + \frac{6}{8^3} = \cdot 246 \text{ in scale of 8.} \quad \textit{Ans.}$$

To change a Radix Fraction from one scale to another.

(*a*) Multiply the given fraction by the radix. The integral part of the product is the first digit after the point.

(*b*) Multiply the fractional part obtained in (*a*) by the radix. The integral part is the second digit, and so on.

N.B.—If any one of the successive products obtained by multiplying by the base is integral, the given fraction can be expressed by a finite number of digits in the new scale. If none of the products is an integer, we may, by repeating the multiplication, obtain a series of fractions in the new scale approximating as nearly as we please to the given fraction. Or we may continue the process until one of the products recurs, when we shall obtain a recurring radix fraction analogous to a recurring decimal.

Example.—*Express* $\frac{5}{16}$ *by a radix fraction in the scale of* 2.

$$\tfrac{5}{16} \times 2 = \tfrac{10}{16} = \mathbf{0} + \tfrac{10}{16}$$

$$\tfrac{10}{16} \times 2 = \tfrac{20}{16} = \mathbf{1}\tfrac{1}{4}$$

$$\tfrac{1}{4} \times 2 = \tfrac{2}{4} = \mathbf{0} + \tfrac{1}{2}$$

$$\tfrac{1}{2} \times 2 = \mathbf{1}$$

Ans. ·0101.

Or, more briefly, Divide the numerator by the denominator (as in ordinary vulgar fractions), *but be careful to work in the scale to which the fraction is to be reduced.*

$$\begin{array}{r|l} 16 & 5 \\ \hline & \cdot 0101 \end{array}$$

Be careful not to read this as *decimal point* nought one, etc., but as binary radix point nought one, etc.

Example.—*Express* 645·345 *in the septenary scale.*

The principle for the decimal fraction is the same as above, but the integral part must be reduced separately.

Integral part.

$$\begin{array}{r|l} 7 & 645 \\ \hline 7 & 92 + 1 \\ \hline 7 & 13 + 1 \\ \hline & 1 + 6 \end{array}$$

Hence 645 in decimal scale = 1611 in the septenary.

Fractional part.

$$\begin{array}{r} \cdot 345 \\ 7 \\ \hline 2\cdot 415 \\ 7 \\ \hline 2\cdot 905 \\ 7 \\ \hline 6\cdot 335 \\ 7 \\ \hline 2\cdot 345 \end{array}$$

After which the digits recur.

Hence $\cdot 345$ in decimal scale $= \cdot\dot{2}26\dot{2}$ in septenary scale.

And $645\cdot 345$ in decimal scale $= 1611\cdot\dot{2}26\dot{2}$ in septenary scale. *Ans.*

To convert a Circulating Radix Fraction to a Vulgar Fraction in the same scale.

Example.—*Reduce* $\cdot 6\dot{3}4\dot{5}$ *in scale of* 7 *to a vulgar fraction in the same scale.*

Let x = the required fraction.

$$\begin{array}{rl} \text{Then } 10000x = & 6345\cdot 345345 + \\ 10x = & \quad 6\cdot 345345 + \end{array}$$

Subtract in scale of 7—

$$6660x = 6336$$

$$\therefore x = \frac{6336}{6660}$$

$$= \frac{\text{whole radix fraction} - \text{non-recurring part}}{\text{six for every recurrer} + \text{nought for every non-recurrer}}$$

Hence the process is the same as for reducing a circulating decimal to a vulgar fraction (see p. 115). Note that, as in the case of the circulating decimal, we place a nine (base − 1) in the denominator for every recurring figure, so here we place a six (base − 1) for every recurring figure. In the same way, we put (base − 1) in other scales.

Reduction of Circulating Radix Fractions to the Denary Scale.

The student of algebra who is familiar with geometric progression should note that this and similar operations may be performed by treating the given circulating radix fraction as a G.P. to infinity, and applying the formula—

$$s = \frac{a}{1 - r}$$

Example.—*Reduce* $\cdot\dot{4}\dot{2}\dot{3}$ *in scale of* 5 *to scale of* 10.

$$\begin{aligned}
\cdot\dot{4}\dot{2}\dot{3} &= \cdot 4232323 + \text{ to infinity} \\
&= \tfrac{4}{5} + \left(\frac{2}{5^2} + \frac{3}{5^3}\right) + \left(\frac{2}{5^4} + \frac{3}{5^5}\right) + \left(\frac{2}{5^6} + \frac{3}{5^7}\right) + \text{ to infinity (in scale of 10)} \\
&= \tfrac{4}{5} + \frac{13}{5^3} + \frac{13}{5^5} + \frac{13}{5^7} +, \text{ etc.} \\
&= \tfrac{4}{5} + \frac{13}{5^3}\left(1 + \frac{1}{5^2} + \frac{1}{5^4} +, \text{ etc.}\right) \\
&= \tfrac{4}{5} + \frac{13}{5^3}\left(\frac{1}{1 - \frac{1}{5^2}}\right) \\
&= \tfrac{4}{5} + \tfrac{13}{120} \\
&= \tfrac{109}{120} \\
&= \cdot 9083\dot{3} \text{ (in scale of 10)} \quad \textit{Ans.}
\end{aligned}$$

Given a number expressed in two scales, of which one is known, to find the other.

Example.—*In what scale is* 3937 *written as* 14323?

(i.) The required scale must be less than the denary, for there are 5 digits in the number expressed in the unknown scale, but only 4 in it when expressed in the denary.

(ii.) Since the highest digit in 14323 is 4, the unknown base must be greater than 4. Hence we have to choose from the bases 5, 6, 7, 8, and 9.

(iii.) $3937 = 3930 + 7$, where 3930 is the sum of multiples of powers of 10, and is therefore divisible by 10.

$14323 = 14320 + 3$, where 14320 is the sum of multiples of powers of the unknown base, and is therefore divisible by this base.

But 14320 in required scale = 3934 in the denary scale.

Hence 3934 must be divisible by the required base.

Trying 5, 6, 7, 8 and 9 in succession, we find that 7 alone divides 3934 exactly. Hence 7 is the required base. *Ans.* In the scale of 7.

Problems.

In every scale we express a number as the sum of successive powers of some fixed base, or of multiples of these powers. Hence problems which involve the sum of the powers of a given number may be solved by "Scales of Notation" methods.

Example.—*I undertake to do a piece of work on condition that I am paid* 1*d. for the* 1*st day,* 3*d. for the* 2*nd,* 9*d. for the* 3*rd,* 2*s.* 3*d. for the* 4*th, and so on. I receive altogether* £41 0*s.* 1*d. How many days did the work take?*

£41 0*s.* 1*d.* = 9841 pence

9841 in scale of 10 } = 111111111 in scale of 3

That is, 9841 pence } $= 1.3^8 + 1.3^7 + 1.3^6 + 1.3^5 + 1.3^4 + 1.3^3 + 1.3^2 + 1.3 + 1.1$

Therefore the number of days was 9. *Ans.*

Example.—*Find which of the weights* 1, 3, 9, 27 . . . *lbs. must be used to weigh* 101 *lbs., not more than one of each kind being used, but in either scale pan that is necessary.*

Transform 101 into the scale of 3.

$101 \div 3 = 33$ threes + 2 units. Since we may use only one of each kind, the remainder 2 is inadmissible, we therefore write 101 as $102 - 1$, and proceeding thus, we get—

3	102 − 1 unit
3	34 + 0 unit
3	11 + 1 three
3	4 − 1 nine
3	1 + 1 twenty-seven
	0 + 1 eighty-one

Hence $101 = 1.3^4 + 1.3^3 - 1.3^2 + 1.3 - 1$

$\therefore$ we put $1.3^4 + 1.3^3 + 1.3$ in one scale pan

and $1.3^2 + 1$ in the other, and the difference is 101 lbs. *Ans.*

Duodecimals.

For the calculation of areas and volumes when the dimensions are expressed in feet and inches, the method of duodecimals is sometimes employed by builders and others.

Duodecimal Table.—1 foot (linear, square, or cubic) is divided into 12 equal parts called *primes*; each prime is divided into 12 equal parts called *seconds*; each second into 12 equal parts called *thirds*. Hence—

12 thirds (‴) = 1 second (″)
12 seconds = 1 prime (′)
12 primes = 1 foot

Comparison of the Duodecimal Table with the Ordinary Tables of Length, Area, and Volume.

Linear measure—

1 prime = 1 in.
1 second = $\frac{1}{12}$ in.
1 third = $\frac{1}{144}$ in.

Square measure—

1 prime = 12 sq. in. (*i.e.* $\frac{1}{12}$ of 1 sq. ft.)
1 second = 1 ,,
1 third = $\frac{1}{12}$,,

Cubic measure—

1 prime = 144 cub. in. (*i.e.* $\frac{1}{12}$ of 1 cub. ft.)
1 second = 12 „
1 third = 1 „

The Place of Duodecimals in Arithmetic.

Duodecimals are frequently treated as another "rule" in arithmetic. In reality, duodecimals are the expression of the foot and the fraction of a foot in radix fractions in the scale of 12, at least in so far as the fractions of a foot are concerned. The builder, however, does not reduce the *feet* to the scale of 12; he works his *whole numbers* in the decimal scale and his *parts* in the duodecimal scale.

In working sums in duodecimals, the student is advised to reduce the whole of the expressions concerned to the scale of 12, and then convert the whole numbers (if any) to the scale of 10.

Example.—*Express* 3 *ft.* 7½ *in. in duodecimals.*

3 ft. 7½ in. = 3 ft. 7′ + ½ in. = 3 ft. 7′ + $\frac{6}{12}$ in. = 3 ft. 7′ 6″
or 3·76 in the scale of 12 *Ans.*

Example.—*Express* 16 *sq. ft.* 68½ *sq. in. in duodecimals.*

16 sq. ft. 68½ sq. in. = 16 sq. ft. 5′ 8½″ = 16 sq. ft. 5′ 8″ 6‴
or 14·586 in the scale of 12 *Ans.*

Example.—*Express* 6 *cub. ft.* 1′ 2″ 3‴ *in cubic feet and cubic inches.*

6 cub. ft. 1′ 2″ 3‴ = 6·123 cub. ft. in scale of 12
= 6 cub. ft. + 123 (in scale of 12)
solid thirds = 6 cub. ft. + 123 cub. in. (in a scale of 12),
for a solid third = 1 cub. in. (see above).
But 123 (in scale of 12) = 171 in scale of 10
Ans. 6 cub. ft. 171 cub. in.

Example.—*Find the area of a window* 1 *ft.* 2′ 3″ *by* 11′ 5″. (R.M.A.)

Reducing to scale of 12, we have—

1·23 × ·e5 = 1·1683 = 1 sq. ft. 1′ 6″ 8‴ 3ᵛ in duodecimals
Ans. 1 sq. ft. 1′ 6″ 8‴ 3ᵛ.

THEORY.

1. Explain clearly the meaning of the terms "scale of notation" and "radix."

2. Show how to transform a given number from the scale of 10 to another scale of notation. (C.S.)

3. What is meant by the common scale of notation? (C.W.)

4. Distinguish between a *number* and a *figure*. What figures do you use to represent the number two dozen? Can you express two dozen using no other figure than 3? (S.A.)

5. Discuss the advantages of the duodenary as compared with the decimal system of notation. How many additional symbols would be required if the former were adopted? (C.M.)

6. If a digit increased twelvefold (instead of tenfold) by every remove of one place to the left, what number would be represented by 183? Give your result in words. Supposing shillings to be retained in use, in what respect is it better that the shilling should be divided into 12 pence, than that it should be divided into 10 pence? (P.T.)

7. What is meant by $\cdot\dot{4}$ in the scale of 6? (S.A.)

8. What is the condition among the digits in a scale of notation of which the radix is an odd number, that a given number is (1) odd, (2) even? (C.S.)

9. Prove that by means of the series of weights, 1, 2, 4, 8, 16, etc., any weight not exceeding the sum of the weights can be weighed.

10. What is meant by duodecimal fractions? (R.M.A.)

11. Name the terms used in duodecimals, and compare their values with the linear, square, and cubic inch.

12. How far are duodecimals, as used by builders, radix fractions in the scale of 12?

EXERCISE LXVIII.

Scales of Notation.

Reduction from the Decimal Scale to some other Scale.

1. Transform the following numbers into the *duodenary* scale:—

(*a*) 17400. (C.M.) (*b*) 22161. (C.S.)
(*c*) 30763451. (C.S.) (*d*) 20958084. (C.S.)

2. Convert 65650 from the decimal scale into the scale of 7. (C.S.)

3. What number in the scale of 2 represents 28 in the decimal scale?

4. Reduce 105336, which is in the scale of 10, to the scale of 6, and write down all the prime factors in the latter scale. (C.S.)

5. If 357234 in the scale of 10 becomes 3015333 in another scale, find by a clear process of reasoning (not by trial transformation) what is the new scale. (C.S.)

To change from any Scale into the Decimal Scale.

6. Transform 101101101101 from the septenary scale. (C.S.)

7. What number (in the denary scale) is represented by the number 2345 in the scale of notation whose radix is 6? (C.S.L.)

8. Transform 66666 from the septenary to the decimal scale. (I.I.)

To change from any Scale (not 10) to another (not 10).

9. Transform 45678 from the scale of 9 to the scale of 11. (C.S.)

10. Transform 6520115 from the septenary scale to the undenary scale. (C.S.)

11. Reduce 16325130 from the scale of 7 to the scale of 12. (C.S.)

12. Convert $te4 \cdot te4$ from the scale of 12 to the scale of 8. (C.S.)

Vulgar Fractions, Decimals.

13. Reduce $\frac{1}{18}$ from the scale of 10 to a duodecimal fraction. (C.S.)

14. Find the value in decimals of $\cdot 1\dot{5}$ in the scale of 7. (C.S.)

15. Reduce $10\cdot 42497$ duodenary scale to the decimal scale.

16. Reduce $275\cdot 9375$ into the duodenary scale. (C.S.)

17. Reduce $67\cdot 75$ into the scale of 5. (C.M.)

18. Express $\cdot 4\dot{5}$, which is in the scale of 7 as a decimal fraction. (S.A.)

19. Transform $296\cdot 7$ from the duodenary into the decimal scale. (C.S.)

The Four Rules—any Scale.

20. Add together 6565 and 362254 in the scale of 7. (C.S.)

21. Add together 1010, 2201, 122, 221, 100, 11 in the scale of 3.

22. Subtract 89497 from 95862 in the scale of 8.

23. Multiply 43620 by 265 in the scale of 7. (C.S.)

24. Multiply 432 by 234 in the scale of 5. (C.S.)

25. Find the square of 17400 (denary scale) in the duodenary scale. (C.S.)

26. Square 123 in the scale of 4. (C.S.)

27. Multiply 1*te*7 by 2*t*6 in the duodenary scale, and prove the answer by reducing the multiplicand, multiplier, and product to the denary scale. (C.S.)

28. Find the number in the scale of 4 which, when divided by 3, gives the quotient 133. (C.S.)

29. The numbers 1184323 and 589 are expressed in the duodenary scale. Divide the former by the latter in that scale. (C.S.)

30. Divide 55974 by 67 in the scale of 11. (C.S.)

31. Divide $418\cdot 9666$ by $6\cdot 289$ in the duodenary scale. (C.S.)

32. Find the square root of—

(*a*) 25400544 in the scale of 6. (C.S.)

(*b*) $105404\cdot 440241$ in the scale of 6, and express the result in the duodenary scale. (C.S.)

(*c*) 553416204 in the scale of 8. (C.S.)

33. Find the cube root of 113140331 in the scale of 7. (C.S.)

34. Express in the scale of 10 the greatest and least numbers that can be written with four digits in the scale of 6; and extract the square root of 32*e*75721 in the scale of 12. (C.S.)

EXERCISE LXIX.

Duodecimals.

All examination questions in this exercise were required to be solved by duodecimals.

Square Measure.

1. Multiply 13 ft. 5 in. 7 pts. by 9 ft. 3 in. 11 pts. Express answer in square feet and inches. (C.S.)

2. Multiply 7 ft. 5 in. 8 pts. by 9 ft. 4 in. 11 pts. Express answer also in square inches. (C.S.)

3. Find the number of feet of glass in a window which measures 7 ft. $10\frac{3}{4}$ in. by 4 ft. $7\frac{1}{2}$ in. (R.M.A.)

4. Determine the area of a rectangle whose adjacent sides respectively measure 10 ft. $9\frac{1}{3}$ in. and 5 ft. $2\frac{1}{2}$ in. How many such rectangles are there in one which contains 12125 sq. ft.? (C.S.)

5. Find the cost of paving a court at 3*s.* 2*d.* a square foot, the length being 27 ft. 10 in. and the breadth 14 ft. 9 in. (C.S.)

6. Find the area of a rectangle 2 ft. 3 in. by 5 ft. 7 in., and express your answer in square feet and inches. And find the cost of glazing a window containing 60 panes, each of which is 1 ft. 2′ 3″ by 11′ 5″, at 3*s.* 8*d.* per square foot. (R.M.A.)

7. Find the exact cost of glazing 12 windows, each containing 60 panes 1 ft. 2′ 3″ long and 11′ 5″ wide, at 3*s.* 8*d.* per square foot. (C.S.)

8. Find the cost of the lead lining of a rectangular cistern 3 ft. 2 in. long, 2 ft. 8 in. wide, and 2 ft. 6 in. deep, at 3*d.* per pound; 7 lbs. of lead covering a foot of surface. (C.S.)

9. Find (1) the area of a rectangle 42 ft. $10\frac{1}{2}$ in. long and 21 ft. $10\frac{1}{2}$ in. wide; and (2) the length of a side of a square which is equal in area to the rectangle. (C.S.)

10. Find the area of a floor whose dimensions are 8 yds. $2\frac{1}{2}$ ft. by 6 yds. $1\frac{1}{4}$ ft. Find the cost of staining a border of 1 ft. all round it, at 6*d.* a square yard, and of laying the interior area with carpet at 4*s.* 6*d.* a square yard. (C.S.)

Cubic Measure.

11. Multiply 11 ft. 4 in. 3 pts. by 8 ft. 9 in. 7 pts., and the product by 10 ft. 11 in. What does the result become when expressed in cubic feet, cubic inches, and the fraction of a cubic inch? (C.S.)

12. Multiply 6 ft. 4 in. 7 pts. by 4 ft. 2 in. 11 pts., and the product by 3 ft. 9 in. 5 pts. What does the answer become when expressed in cubic feet, etc.? (C.S.)

13. Multiply 8 ft. 7 in. 3 pts. by 3 ft. 7 in. 5 pts., and the result by 4 ft. 3 in. Express the result in cubic feet, cubic inches, and the fraction of a cubic inch. (C.S.L.)

14. Find the volume of a cubical tank whose edge is 3 yds. 2 ft. $6\frac{3}{4}$ in. Express the result in cubic feet. (C.S.)

15. Find the solid content of a block of marble 3 ft. 8 in. 4′ long, 1 ft. 7 in. 6′ broad, and 1 ft. 4 in. 9′ thick. Express your result in cubic feet and inches. (R.M.A.)

16. Multiply 3 ft. 5 in. by 4 ft. 9 in., and the product by 8 ft. 7 in. What does the answer become when expressed in cubic feet, cubic inches, and the fraction of a cubic inch? (R.M.A.)

EXERCISE LXX.

Scales.

Miscellaneous.

1. Multiply $1te7$ by $2t6$ in the duodenary scale, and prove the answer by reducing the multiplicand, multiplier, and product to the denary scale. (C.S.)

2. Transform 275·9376 from the decimal to the duodenary scale of notation; and extract the square root of 148915 in the scale of 11. (I.I.)

3. Show how to express fractions such as ·46, $\cdot\dot{1}\dot{6}$, and $\cdot 4\dot{6}$ as vulgar fractions, showing that your method must be correct, supposing them (i.) in the scale of 10, (ii.) in the scale of 8. (S.A.)

4. Simplify $\frac{13}{1t} \times \frac{104}{47} \div \frac{31}{t1}$ in the scale of 12; multiply 4·302 by 2·104 in the scale of 5; and reduce $37{\cdot}22\dot{9}1\dot{6}$ from the scale of 10 to the scale of 12. (C.S.)

5. A block of silver 13 in. long, 8 in. broad, and 6 in. deep, is beaten into a bar which has a uniform section of 1·01 sq. in. in the scale of 5. Find its length in feet in the same scale. Express $\frac{139}{192}$ in the scale of 10 as a series of radix fractions in the scale of 12. (C.S.)

6. Express the cubes of the first 6 digits in the scale of 7. Show that every number which is a perfect cube either is divisible by 7, or, being divided by 7, leaves as remainder 1 or 6. Can you state and prove any analogous theorem for perfect cubes divided by 9? (C.S.)

7. Multiply $\cdot 3\dot{4}\dot{1}$ by $32\dot{4}$ in the scale of 5, bringing your result to a recurring radix fraction. (S.A.)

8. Transfer the number 355·41 from the scale of 6 to the scale of 10 correctly to five places of decimals; Find the square root of 360784 in the scale of 9. If the area of a rectangle expressed in the scale of 12 is 333·06 sq. ft., and the length of one side in the same scale is $22{\cdot}e$ ft., what is the length of the other side in feet and inches? (I.I. and C.S.)

9. Reduce 123454321 in the senary scale to the decimal and duodenary scales. (I.I.)

10. The number of square inches in a meadow is represented by $142t5090$ in the undenary scale, and the area of an adjoining field is expressed in square inches in the duodenary scale by 8044422. How many acres do they together contain? Give the answer in the ordinary scale. (C.S.)

11. The number of square inches in a square plot of ground is represented in the undenary scale by the sequence 222950135. How many acres, roods, perches, etc., does it contain? (C.S.)

12. Reduce to the decimal of a cubic yard the volume which is expressed in the duodenary scale by $1t538$ cub. in. (C.S.)

13. Determine the least number of weights which must be selected out of the series 1, 2, 4, 8, etc., lbs. in order to weigh 1317 lbs.

14. Supposing you had weights of 1, 3, 3^2, 3^3, 3^4, 3^5, etc., show the least number of them you would use in order to weigh 334 lbs. (C.S.)

15. Transfer 1885 from the scale of 9 to the ordinary scale of 10; divide 65502 by 45, both being in the scale of 7; and find which of a series of weights of 1 lb., 2 lbs., 4 lbs., 8 lbs., 16 lbs., etc., must be used in order to weigh 233 lbs., if only one weight of each kind be used. (C.S.)

CHAPTER XLIX.

LOGARITHMS.

BEFORE commencing the study of Logarithms, the student must thoroughly understand—

1. The meanings of the terms *Power* (p. 34), *Index* (p. 34), and *Base* (p. 442).

2. The following fundamental principles :—

(*a*) To multiply two quantities, *add* the indices.

Example.—

$$10^{14} \times 10^{12} = 10^{14+12} = 10^{26}$$

(*b*) To divide two quantities, *subtract* the indices.

Example.—

$$10^{26} \div 10^{14} = 10^{26-14} = 10^{12}$$

Note that in (*a*) and (*b*) the quantities are the same, but that their indices may vary. We cannot say that $10^{14} \times 12^{12} = 10^{26}$ or 12^{26}.

(*c*) To raise any quantity to any power, *multiply* the index of that quantity by the power to which it is to be raised.

Example.—

$$(10^2)^3 = 10^{2 \times 3} = 10^6$$

(*d*) To find the root of any quantity, *divide* the index of the quantity by the root which is to be found.

Example.—

$$\sqrt[3]{10^6} = 10^{6 \div 3} \div 10^2$$

3. A new notation sign : *e.g.* 10^{-5}, which is another and more convenient method of writing $\frac{1}{10^5}$.

For a general proof of 2 (*a*), (*b*), (*c*), (*d*), and 3 the student is referred to works on algebra

Examples.—

1. $10^6 \times 10^7 \times 10^{-4} = 10^{6+7-4} = 10^9$
2. $\sqrt{10^{-8}} = (10^{-8})^{\frac{1}{4}} = 10^{-\frac{8}{4}} = 10^{-2}$
3. $10^{4\cdot362} \times 10^{1\cdot081} = 10^{4\cdot362+1\cdot081} = 10^{5\cdot443}$.

The student already knows that—

The index of 10 to the base of 10 is 1
" " 100 " " 10 " 2
" " 1000 " " 10 " 3, etc.

The question now arises: What is the index of a number which is less than 10? Manifestly it must be a quantity less than 1; *i.e.* it is a decimal fraction. Similarly, the index of a number like 12 must be between 1 and 2, and must be some quantity consisting of 1 *and* a decimal fraction, *e.g.* a number like 144 must have an index between 2 and 3; *i.e.* the index will be 2 and a decimal fraction; and so on.

The indices of the powers (to the base of 10) of all the whole numbers from 1 to 1000000 have been calculated to 7 places of decimals, and such a list (called a List of Logarithms) may be advantageously consulted in such cases as the following :—

Example.—*Given* $324 = 10^{2\cdot51055}$, $422 = 10^{2\cdot62531}$, *prove that* $324 \times 422 = 10^{5\cdot13586}$.

$$\text{Now, } 324 \times 422 = 10^{2\cdot51055} \times 10^{2\cdot62531} = 10^{2\cdot51055+2\cdot62531} = 10^{5\cdot13586}$$

Example.—*Multiply* 1287 by 7983, *and divide the result by* 4136.

$$\left.\begin{aligned} \text{Now, } 1287 &= 10^{3\cdot1095785} \\ 7983 &= 10^{3\cdot9021661} \\ 4136 &= 10^{3\cdot6165805} \end{aligned}\right\} \therefore \frac{1287 \times 7983}{4136} = \frac{10^{3\cdot1095785} \times 10^{3\cdot9021661}}{10^{3\cdot6165805}}$$
$$= 10^{3\cdot1095785+3\cdot9021661-3\cdot6165805}$$
$$= 10^{3\cdot3951641}$$

and from the list of logarithms, or tables, as they are called, 3·3951641 will be seen to equal 2484·1 (nearly).

Example.—*Given* $2 = 10^{\cdot3010300}$, $3 = 10^{\cdot4771213}$, $7 = 10^{\cdot8450980}$, *find the value of* $\sqrt[5]{252}$.

$$\text{Now, } \sqrt[5]{252} = (252)^{\frac{1}{5}} = (2^2 \times 3^2 \times 7)^{\frac{1}{5}}$$
$$= \{(10^{\cdot3010300})^2 \times (10^{\cdot4771213})^2 \times (10^{\cdot8450980})\}^{\frac{1}{5}}$$
$$= (10^{\cdot6020600+\cdot9542426+\cdot8450980})^{\frac{1}{5}}$$
$$= (10^{2\cdot4014006})^{\frac{1}{5}} = 10^{\cdot4802801}$$

and from the tables ·4802801 is found to be equal to 3·0219.

This process of calculation by using the indices of the powers of numbers to some fixed base is called the process by logarithms. Hence *the logarithm of a number to a given base is the index of that power of the base which equals the number.*

Fundamental Rules of Logarithms.—Since logarithms are indices, the fundamental rules of logarithms will be the same as the fundamental principles of indices (p. 456).

1. The log of a *product* is equal to the *sum* of the logs of its factors, *e.g.* $\log (2 \times 3) = \log 2 + \log 3$.

2. The log of the *quotient* of two numbers is the log of the dividend *minus* the log of the divisor, *e.g.* $\log (\frac{5}{7}) = \log 5 - \log 7$.

3. The log of a number raised to a given power is the product of the index of that power and the log of the number, *e.g.* $\log (5^4)^3 = 3 \log 4$.

4. The log of the root of a number is a special case of Rule 3, since a root can be expressed as an index, *e.g.* $\log \sqrt[5]{252} = (252)^{\frac{1}{5}} = \frac{1}{5} \log 252$

Thus—

Ordinary arithmetic	Multiplication, Division, Involution, Evolution	becomes	Addition, Subtraction, Multiplication, Division in Logarithms.

It should be noted that, although logarithms may be calculated to any base, the common system is that of 10, and in future examples this base will always be understood.

Examples.—(1) *Given log* 2 = ·3010300 *and log* 7 = ·8450980, *find log* 14.

Now, $\log 14 = \log (2 \times 7) = \log 2 + \log 7$ (Rule I.)
$= ·3010300 + ·8450980 = 1·1461280$

(2) *Given log* 2, *find log* 5.

Now, $\log 5 = \log \frac{10}{2} = \log 10 - \log 2$ (Rule II.)
But log 10 to the base 10 is obviously equal to 1
$\therefore \log 5 = 1 - ·3010300 = ·6989700$

(3) *Given log* 3 = ·4771213, *find log* 81.

$\text{Log } 81 = \log 3^4 = 4 \log 3 = 4 \times ·4771213 = 1·9084852$

(4) *Given log* 2, *log* 3, *and log* 7, *find log* 4·2.

$\text{Log } 4·2 = \log \frac{42}{10} = \log 42 - \log 10 = \log (3 \times 2 \times 7) - \log 10$
$= ·4771213 + ·3010300 + ·8450980 - 1 = ·6232493$

(5) *Given log* 2 *and log* 3, *find log* $\sqrt[5]{240}$.

$\text{Log } \sqrt[5]{240} = \log (240)^{\frac{1}{5}} = \frac{1}{5} \log 240 = \frac{1}{5} \log (2^3 \times 3 \times 10)$
$= \frac{1}{5}(\log 2^3 + \log 3 + \log 10) = \frac{1}{5}(3 \log 2 + \log 3 + \log 10)$
$= \frac{1}{5}(·9030900 + ·4771213 + 1) = \frac{1}{5}(2·3802113)$
$= ·4760422$

The Rule of the Characteristic.—The integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

From what has already been said, it will be seen that—

Log 9 = 0 + a decimal fraction
Log 99 = 1 + ,, ,,
Log 99·367 = 1 + ,, ,,
Log 100 = 2
etc. etc.

Hence it appears that the characteristic is one less than the number of integers in the given quantity.

Again, since—

$1 = 10^0$ ∴ log 1 = 0
$\frac{1}{10} = 10^{-1}$ ∴ log $\frac{1}{10}$ or log ·1 = −1
$\frac{1}{100} = \frac{1}{10^2} = 10^{-2}$ ∴ log $\frac{1}{100}$ or log ·01 = −2
$\frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$ ∴ log $\frac{1}{1000}$ or log ·001 = −3
etc.

Hence, the log of a number between 1 and ·1 will be a negative decimal fraction, but as it is convenient to *keep the mantissa always positive*, it can be stated as follows:—

The log of a number between 1 and ·1 will be −1 + some dec. fraction
,, ,, ,, ·1 ,, ·01 ,, −2 + ,, ,,
,, ,, ,, ·01 ,, ·001 ,, −3 + ,, ,,

It can thus be seen that *in the logs of all numbers less than 1 the characteristic will be negative, and one more than the cyphers between the decimal point and the first significant figure.*

The student is advised to study the following examples:—

(*a*) The characteristic of log 913 is 2, but the characteristic of log ·913 is $\bar{1}$; for ·913 lies between 1 and ·1, and therefore log ·913 is −1 + ·9604708, and is written $\bar{1}$ ·9604708.

(*b*) Log ·0036 has a characteristic $\bar{3}$, for ·0036 lies between ·01 and ·001, and therefore log ·0036 is −3 + ·5563025, and is written 3·5563025.

(*c*) Conversely, given the characteristic to be $\bar{5}$, the number whose log is taken will be a decimal fraction with 4 cyphers between the decimal point and the first significant figure; *e.g.* ·000016 has $\bar{5}$ as a characteristic.

The necessity of always keeping the mantissa positive is felt in dividing such a number as $\bar{2}$·3468794 by 5. The process is to add (−3) to the characteristic (since −2 and −3 = −5), and (+3) to the mantissa, *i.e.* to add and subtract the same amount.

$$\frac{\bar{2}\cdot3468794}{5} = \frac{\bar{5} + 3\cdot3468794}{5} = \bar{1}\cdot6693758$$

The Advantages of 10 as a Base.

1. It is the base of our common system of arithmetical notation.
2. The characteristic can be told by inspection.
3. The mantissa of the logarithm of all numbers which differ only in the position of the decimal point is unchanged, *e.g.*—

Log 866 = 2·9375179
Log 8·66 = ·9375179
Log ·000866 = $\bar{4}$·9375179

From this it will be seen that if two numbers differ only in the position of the decimal point, it is evident that the one is equal to the other multiplied by 10 or some power of 10.

For suppose ·x to be the log of 8. Then $8 = 10^{\cdot x}$, and because $80 = 8 \times 10$
∴ log 80 = log 8 + log 10 = ·x + 1 = 1·x
Again, similarly, log 800 = log 100 + log 8 = 2 + ·x = 2·x
and log 8000 = log 1000 + log 8 = 3 + ·x = 3·x, etc.

Logarithmic Tables.

Although seven-figured tables are usual, for comparatively rough calculations a table of logarithms correct to four places is sufficient. Such a table is given at the end of this chapter. It is the Table of Logarithms and Anti-Logarithms supplied to candidates at the Board of Education Examinations in Mathematics, and it is reproduced with the permission of the Controller of His Majesty's Stationery Office.

Explanation of the Tables.—Consider a number of four significant figures.

The first two significant figures will be found in the first vertical column.

The third significant figure will be found in the first section of numbers 0 to 9 in the top horizontal row.

The fourth significant figure will be found in the second section of numbers 1 to 9 in the top horizontal row.

Example.—*Find the log of* 416.

This consists of three significant digits. The 41 must be found in the vertical column. The mantissa will be in the same row, but in the column headed **6** in the first section of the numbers 0 to 9 in the top row. This is ·6191, therefore log 416 = 2·6191.

Note.—Had log 41600 been required, the method adopted would have been the same, since the three significant figures are 416. Hence log 41600 = 4·6191. Similarly, log ·0000416 = $\bar{5}$·6191.

Example—*Find log* 2532.

The mantissa of the first three significant figures can be found as before. This is found to be ·4031. To this must be added the figure in the same row, but in the column headed **2** of the second section of numbers 1 to 9 in the top row. This figure is 3, and it really represents ·0003. Hence log 2532 = 3·4031 + ·0003 = 3·4034.

To find the number whose logarithm is given, the table of anti-logarithms is used, but the method is similar in both cases.

Example.—*Find the number whose logarithm is* $\bar{2}$·1276.

The series of digits found from ·1276 is 1342, and since the characteristic is $\bar{2}$, the required number is ·01342.

Some examples involving the use of the tables are now given.

Example 1.—*Find the numerical value of* $\sqrt[5]{\cdot 00865}$

$$\begin{aligned}\text{Log } \sqrt[5]{\cdot 00865} &= \text{log. } (\cdot 00865)^{\frac{1}{5}} = \tfrac{1}{5} \text{ log. } \cdot 00865 \\ &= \tfrac{1}{5} \times \bar{3}\cdot 9370 \text{ (given in tables)} \\ &= \tfrac{1}{5} \times (\bar{5} + 2\cdot 9370) \text{ (see p. 459)} \\ &= \bar{1} + \cdot 5874 = \bar{1}\cdot 5874 = \log \cdot 3868 \text{ (from tables)} \\ \therefore \sqrt[5]{\cdot 00865} &= \cdot 3868\end{aligned}$$

Example 2.—*Find the compound interest on* £2500 *for* 8 *years at* 5 %.

$$\left.\begin{array}{l}\text{If } A = \text{amount}\\ \quad P = \text{principal}\\ \quad r = \text{rate \%}\\ \quad n = \text{time}\end{array}\right\} \text{Then } A = P\left(1 + \frac{r}{100}\right)^n$$
$$\therefore \log A = \log P + \log\left(\frac{100 + r}{100}\right)^n$$

$$\begin{aligned}\text{Then } \log A &= \log 2500 + \log\left(\tfrac{105}{100}\right)^8 = \log 2500 + 8 \log 1{\cdot}05\\ &= 3{\cdot}3979 + 8 \times {\cdot}0212 \text{ (from Table of Logs)}\\ &= 3{\cdot}5675 = \log 3694 \text{ (from Table of Anti-logs)}\\ \therefore A &= £3694 \text{ (approx.)}\end{aligned}$$

and interest = £3694 − £2500 = £1194 *Ans.*

Note.—If worked to a table of seven-figured logs, the answer would be £1193 12*s.* 9*d.*

THEORY.

1. Define the term *logarithm.* What is meant by *mantissa*? Why in every system will the log of 1 be 0?

2. Explain why $\log_{10} 6{\cdot}25$, $\log_{10} {\cdot}000625$, $\log_{10} \frac{1}{16}$ have the same mantissa.

3. Explain what is meant by the characteristic of a logarithm, and state the rule for finding the characteristic of the logarithms to the base 10 of the numbers less than unity. (R.M.A.)

4. Show that in a system of logs to the base 10, the logs of all numbers between 1 and 10 must be included between 0 and 1. Point out the utility of logs in calculation.

5. Show that in the common system the characteristic of the log of any number can be determined by inspection.

6. Show that in a system of logs to the base 10, the logs of all numbers which are expressed by the same succession of significant digits may be found from one opening of the tables.

7. Show that in any system of logs the sum of the logs of two numbers is equal to the log of the product of the numbers. (S.K.)

8. How does it appear that by means of a table of logs *multiplication* may be performed by addition, *division* by subtraction, *involution* by multiplication, and *evolution* by division?

9. State and prove the rule for finding the log of the quotient when the logs of the divisor and dividend are given. (S.K.)

EXERCISE LXXI.

1. Simplify $12^7 \times 12^3$; $\sqrt[3]{15^{12} \div 15^6}$; $\sqrt[12]{10^{-8}}$; $\sqrt{16^6 \div 4^9}$.

2. Given $2 = 10^{\cdot 3010300}$; $3 = 10^{\cdot 4771213}$; $7 = 10^{\cdot 8450980}$

(*a*) Find what powers of 10 are equal to 14; 2·4; 1000; $11{\cdot}\dot{6}$; $7\frac{7}{9}$; 10^2_{25}.

(*b*) Between what powers of 10 do the following numbers lie: 2^{64}; 7^9; 3^{14}; 49^{10}?

3. Given $571 = 10^{2\cdot7566}$; $987 = 10^{2\cdot9943}$

(*a*) Find approximately the numerical value of $\sqrt[3]{571 \times 987}$, having given $10^{1\cdot9169} = 82\cdot587$.

(*b*) Find the area of a square field whose side is 571 ft., having given that $10^{5\cdot5132} = 326041$.

(*c*) Find the side of a square field whose area is 974169, having given $10^{5\cdot9886} = 974169$.

Examples not involving Negative Characteristics.

Given log 2, log 3, and log 7, find the logarithms of—

4. 80; 24; 45; 450; 4·5; 36; 3·6; 360; 7·2.

5. 2000; 3000; 216; 1·08; 3528; 1323; 1·323; 2·1.

6. $(147)^2$; $(392)^4$; $(224)^4$; $2^{\frac{1}{100}}$; $\dfrac{(1\cdot08)^4}{(\cdot0147)^3}$.

7. $\sqrt[7]{625}$; $\sqrt[3]{(270 \div 8)}$; $\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt{27}$; $\sqrt[3]{256} \times 7^{\frac{1}{2}}$.

Examples involving Negative Characteristics.

8. Give the characteristics of the following numbers: 4362·1; 10·362; ·1436; ·00018; 1·0001; ·00005.

9. What is the position of the first significant figure in the numbers whose logs are $\bar{3}\cdot8712$; $\bar{4}\cdot1376$; $1\cdot1897$; $6\cdot3128$; $\cdot4687$?

10. Given log 3 and log 7, find (*a*) the number of digits in the integral part of 9^8; $(\frac{7}{3})^5$; $(27)^4$; 7^{10}; (*b*) the number of cyphers between the decimal point and the first significant figure in $(\frac{1}{3})^{100}$; $\sqrt[7]{\frac{1}{7}}$; $(\frac{3}{7})^{10}$; $(\cdot07)^4$.

11. Divide $\bar{5}\cdot8763$ by 2, 4, 6, and 7 respectively.

12. In a table of logarithms, side by side with the number 377 is the mantissa ·5763414. Write down the logs of 377; 37·7; 3·77; ·0037; ·377; 377000.

13. Find from the table of logarithms the logs of 136; 4·09; ·00083; 73160; ·01001; 45000.

14. Find from the table of anti-logarithms the numbers whose logs are $1\cdot6403$; $\bar{5}\cdot3608$; $\bar{2}\cdot6871$; $\cdot0086$; $\bar{4}\cdot1106$; $2\cdot0103$.

15. By the aid of the given tables, evaluate $36\cdot29 \times \cdot7284$; $139200 \times \cdot01365$; $\cdot03721 \div \cdot00342$; $\cdot8139 \div (1628 \times 173)$.

16. Evaluate $\sqrt[7]{453}$; $\sqrt[10]{(\cdot0063)}$; $\sqrt[5]{(138)^2} \times \sqrt[20]{183000}$; $\dfrac{(2\cdot001)^{\frac{2}{3}}}{\sqrt[6]{(1\cdot083)}}$.

17. Evaluate—

$$(\cdot186)^{-5} \div (1\cdot86)^{-3};\ (40\cdot3)^{-\frac{3}{2}} \times (\cdot3614)^{-1};\ \frac{(\cdot41)^2 \times (78\cdot01)^5 \div (1\cdot031)^{\frac{1}{3}}}{\sqrt[15]{(87000)} \div \sqrt[3]{(413)}}.$$

18. A solid cube contains 312·5 cub. in. Find by logs the length of the side.

19. A solid cube of lead weighs 126·4 lbs.; 998 oz. of water occupy 1 cub. ft.; and 1 cub. ft. of lead is 11·35 times as heavy as 1 cub. ft. of water. Find by logs the length, in inches, of the side of the cube.

20. Find the difference between the simple and compound interest on £1375 for 10 yrs. at 3 per cent.

21. Find the present value of £10000 due 8 yrs. hence at 5 per cent. compound interest.

22. Find in how many years the population of a country will first become increased by more than half of its original amount through the sole effects of births and deaths, when the birth rate at the end of each year is $\frac{1}{30}$ and the death rate $\frac{1}{40}$ of the population at the beginning of that year.

23. A person borrowed £1100 for 2 mths. at 5 per cent. At the end of the time the interest was added on, and the debt renewed for another 2 mths. This was continually repeated until, at the end of a year, the debt and interest were paid. Find by logs what this debt and interest amounted to.

24. The population of a town is 15000. What was its population 6 yrs. ago, considering the increase solely due to the excess of birth rate over death rate? The birth rate at the end of each year is $\frac{1}{40}$ and the death rate $\frac{1}{50}$ of the population at the beginning of that year.

NOTE.—*Some of the problems given in the Chapter on Compound Interest, especially those involving many periods of time, and those in which it is required to find "Rate" or "Time," should be worked by logarithms. The same remark applies to the finding of the Higher Roots in the Chapter on Cube Root. Speaking generally, logarithms should be used anywhere where their use will save time. It should be noted, however, that, as the table of logarithms here given is only a four-figured one, the answers in some cases will be but approximately correct.*

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 8 12	17 21 25	29 33 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 11	15 19 23	26 30 34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 10	14 17 21	24 28 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10	13 16 19	23 26 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9	12 15 18	21 24 27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 8	11 14 17	20 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8	11 13 16	18 21 24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2 5 7	10 12 15	17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7	9 12 14	16 19 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7	9 11 13	16 18 20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9606	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

CHAPTER L.

STRAIGHT-LINED OBJECTS.

Mensuration is the method of estimating the magnitude of geometrical figures: it is an application of arithmetic to geometry.

The geometrical figures commonly dealt with in mensuration are: Rectangles (including, of course, the square), triangles, polygons, circles, cubes, cones, etc.

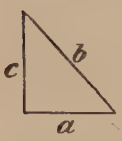
The Right-angled Triangle.

Given the lengths of the two sides containing the right angle, to find the length of the hypotenuse (or side opposite the right angle).

Rule.—Find the square root of the sum of the squares of the sides.

For explanation, see p. 402.

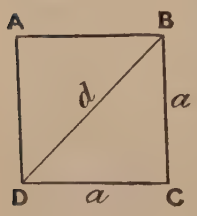
This is a very important theorem, and many riders and problems depend on it. The following cases are selected:—



(1) *Given the length of the hypotenuse and the length of one side of a right-angled triangle, to find the length of the other side.*

Since $b^2 = c^2 + a^2$ (Euc. I. 47)
Then $c^2 = b^2 - a^2$
And $c = \sqrt{b^2 - a^2}$

Rule.—Find the square root of the difference between the square of the hypotenuse and the square of the given side.



(2) *Given the length of the side of a square, to find the length of the diagonal.*

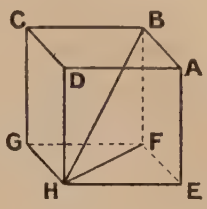
$d^2 = a^2 + a^2$ (Euc. I. 47) $= 2a^2$
$\therefore d = a\sqrt{2}$

Rule.—Multiply the length of a side by $\sqrt{2}$.*

* $\sqrt{2} = 1{\cdot}412 \ldots$

(3) *Given the length of the side of a cube, to find the length of the diagonal.*

Let S = number of units in the side of the cube.



Then $HF = S\sqrt{2}$ (see (2) above)

Now, $HB^2 = HF^2 + BF^2$

$= 2s^2 + s^2 = 3s^2$

$\therefore HB = S\sqrt{3}$

Rule.—Multiply the side by $\sqrt{3}$.*

Triangles.

To find the Area of any Triangle.

(*a*) *In terms of the base and the height.*

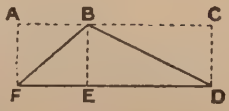
Area of parallelogram = AC . CD

= FD . BE

Now, area of triangle BFD = $\frac{1}{2}$ area of parallelogram ACDF (Euc. I. 41)

$\therefore$ area of triangle BFD = $\frac{1}{2}$FD . BE

= $\frac{1}{2}$base × height



Rule.—Multiply half the base by the height.

(*b*) *In terms of the sides.*

(*a*) Find half the perimeter of the triangle.
(*b*) From this in turn subtract each side separately.

Four quantities have now been obtained, one from (*a*), and three from (*b*).

(*c*) Multiply together the four quantities thus obtained.
(*d*) The square root of the product will be the area required.

For the proof of this somewhat complicated rule the student is referred to any work on trigonometry.

Let *a*, *b*, and *c* represent the three sides of a triangle; let *s* = half the sum of those sides, and let $\triangle$ = the area of the triangle.

$$\text{Then } \triangle = \sqrt{s(s-a)(s-b)(s-c)}$$

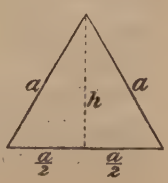
To find Area of an Equilateral Triangle in Terms of a Side.

The above formula may be employed. The following, however, is preferable in this special case:—

$$\text{Area} = \frac{h \cdot a}{2}$$

$$\text{Now, } h^2 = a^2 - \left(\frac{a}{2}\right)^2 = \tfrac{3}{4}a^2$$

$$\text{and } h = \frac{a\sqrt{3}}{2}$$



* $\sqrt{3} = 1{\cdot}7320\ldots$

Substituting the value of h—

$$\text{Area} = \frac{a\sqrt{3} \times a}{2 \times 2} = \frac{a^2}{4}\sqrt{3}$$

Rule.—Multiply a quarter of the square of the side by $\sqrt{3}$.

Quadrilaterals. (Four-sided figures.)

The chief quadrilateral figures are—

1. *The square*, which has *all* its sides equal, and all its angles right angles.
2. *The oblong*, which has its *opposite* sides equal, and all its angles right angles.
3. *The parallelogram*, which has its opposite sides equal, but its angles are not right angles.
4. *The rhombus*, which has *all* its sides equal, its opposite sides parallel, but none of its angles right angles.
5. *The rhomboid*, which has its *opposite* sides equal and parallel, but none of its angles right angles.
6. *The trapezium*, which has one pair of opposite sides equal, but none of its angles right angles.

Rectangles and Rectangular Figures.

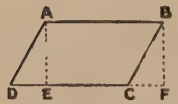
1. **To find the Area of a Rectangle** (including a square), see p. 396.
2. **To find the Length of the Side of a Rectangle** (or of a square), see p. 397.
3. **To find the Area of the Faces of a Cube.**—Since a cube has *six* equal faces, multiply the area of any one face by 6.
4. **To find the Area of the Faces of a Rectangular Prism.**

Since a rectangular prism has 4 "long" faces and 2 "short" faces, we get the following rule:—

(*a*) Find 4 times the area of one "long" face.
(*b*) Find twice the area of one "short" face.
(*c*) The sum of (*a*) and (*b*) is the area of the faces of the prism.

Parallelograms.

To find the Area of a Parallelogram.



Since ABCD = ABFE (Euc. I. 35)
and since ABFE is a rectangle
∴ area of ABCD = AB × AE
= length of side × perpendicular from it.

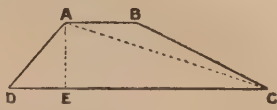
Rule.—Multiply the length of a side by the perpendicular height from that side.

To find the Area of a Rhombus.

Rule.—Divide the product of the diagonals by 2.

To find the Area of a Trapezium, *given the lengths of the two parallel sides and the perpendicular height between them.*

Join AC, and draw the perpendicular AE. Then the area of the trapezium equals the sum of the areas of the two triangles ACD, ABC.



$$\text{Area of } \triangle ACD = \frac{CD \times AE}{2}$$

$$\text{Area of } \triangle ABC = \frac{AB \times AE}{2}$$

$$\therefore \text{ area of trapezoid} = \frac{CD \times AE}{2} + \frac{AB \times AE}{2}$$

$$= \frac{AE}{2}(AB \times CD)$$

Rule.—Multiply half the height by the sum of the parallel sides.

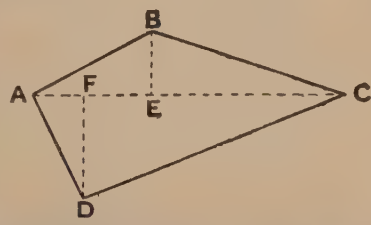
To find the Area of an Irregular Quadrilateral.

(1) *When the dimensions of the sides and of a diagonal are given.*

Draw the diagonal, and calculate the areas of the two triangles thus formed.

(2) *When the length of the diagonal joining one pair of vertices and the perpendiculars drawn to it from the other pair are given.*

Thus, when AC, BE, and FD are given, find the area of the irregular quadrilateral.

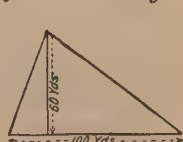


Area of quadrilateral ABCD
$= \triangle ABC + \triangle ACD$
$= \frac{1}{2}AC \cdot BE + \frac{1}{2}AC \cdot FD$
$= \frac{1}{2}AC(BE + FD)$

Rule.—Multiply half the diagonal by the sum of the perpendiculars.

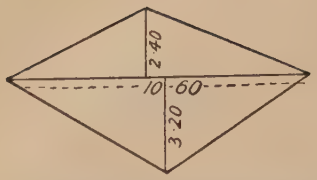
Some illustrative examples will now be given. The student is recommended to employ diagrams in his solutions.

Example.—*Find the area of a triangle, the base of which is* 100 *yds. long, and which has an altitude of* 60 *yds.*



Area of $\triangle$ in yards $= \frac{1}{2}$ base $\times$ height (p. 469)
$= \frac{1}{2} \times 100 \times 60$
$= 3000$ sq. yds. *Ans.*

Example.—*In a four-sided field the longest diameter is* 10 *chains* 60 *links, and the perpendiculars from the opposite angles let fall upon the diameter are respectively* 2 *chains* 40 *links and* 3 *chains* 20 *links. Find the area of the field.*



Area = $\frac{1}{2}$ × diameter × (sum of perpendiculars)
= {$\frac{1}{2}$ × 10·60(2·40 + 3·20)} sq. chains
= 5·30 × 5·60 sq. chains
= 29 sq. chains 68 links. *Ans.*

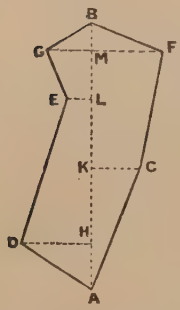
Example.—*Find the area of a triangle whose sides are respectively* 150 *yds.*, 120 *yds.*, *and* 200 *yds.*

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{235(235-150)(235-120)(235-200)} \text{ sq. yds.}$$
$$= \sqrt{235 \times 85 \times 115 \times 35} \text{ sq. yds.}$$
$$= \sqrt{(5 \times 47 \times 5 \times 17 \times 5 \times 23 \times 5 \times 7)} \text{ sq. yds.}$$
$$= 25\sqrt{47 \times 17 \times 23 \times 7} \text{ sq. yds.}$$
$$= 25 \times 358{\cdot}66 \ldots \text{ sq. yds.} = 8966{\cdot}5 \text{ sq. yds.} \quad \textit{Ans.}$$

The Field Book.

The mensuration of plane figures is put to practical use in land surveying and measuring. The surveyor records his measurements in his *Field Book.*

The Plan.



The Field Book.

	Links.	
	to B 550	
To G 100	500	150 to F
To E 50	400	
	250	100 to C
To D 150	100	
From	A	go North

The Field Book is read *upwards*, beginning at the lowest line. Thus, from A we are directed to go North for a distance of 100 links to H, when we are to draw a line 150 links long at right angles to the *left*, and so on.

The line AB is called the *base-line*, or the *chain-line*. The lines HD, CK are termed *offsets.*

The base-line is first drawn, then the offsets, and lastly the boundary is filled in.

Calculation of the area of the given figure, which is composed of three triangles AHD, AKC, GBF, and three right-angled trapeziums DHLE, MFCK, and GMLE.

Twice triangle AHD $= AH \times HD = 100 \times 150 = 15000$ sq. links
" " AKC $= AK \times KC = 250 \times 100 = 25000$ "
" " GBF $= GF \times BM = 250 \times 50 = 12500$ "
" trapezium DHLE $= HL(EL + DH) = 300(50 + 150) = 60000$ sq. links
" " MFCK $= KM(MF + KC) = 250(150 + 100) = 62500$ sq. links
" " GMLE $= ML(GM + EL) = 100(100 + 50) = 15000$ sq. links
" the whole figure $= (15000 + 25000 + 12500 + 60000 + 62500 + 15000)$ sq. links
$= 190000$ sq. links.
$\therefore$ the required area $= 190000$ sq. links $\div 2 = 95000$ sq. links
$= 9{\cdot}5$ sq. chains $= {\cdot}95$ acre
$= 3$ roods $3\frac{1}{5}$ poles. *Ans.*

EXERCISE LXXII.

Triangles.

1. Two large steamships travel respectively due north and due west from the same port, one at the rate of 16 and the other at the rate of 12 mi. an hour. Find how far they are apart at the end of 6 hrs. (P.P.)

2. One side of a triangular field is 369 lks. in length, and the perpendicular upon it from the opposite angle measures 582 lks. If the field is let for £6, find to the nearest penny the rent per acre.

3. Find the area of a triangle whose sides are 1840, 1336, and 1520 lks., giving the answer in acres, roods, and poles. (C.P.)

4. Find the difference between the area of a triangle whose sides are 16, 63, and 65, and the area of a square whose perimeter is the same as that of the triangle. (P.T.)

5. The sides of a triangle are respectively 17 ft., 15 ft., and 8 ft. What is the length of the perpendicular on the longest side from the opposite angle? (K.S.)

6. Show that the area of a triangle whose sides are 26, 35, and 51 is the same as the area of a right-angled triangle whose sides about the right angle are 7 and 120. (K.S.)

7. The sides of a triangle are in the ratio of 13, 13, and 10, and if its area be 3840 sq. yds., find the length of each side. (K.S.)

8. Find the area of a triangle whose sides are 13·6 in., 15 in., and 15·4 in. Also find (correct to the thousandth part of an inch) the length of one of the equal sides of an isosceles triangle, on a base of 14 in., having the same area. (R.M.A.)

9. The hypotenuse of a right-angled triangle is 6·5 ft., and one side is 5·2 ft. Find the area of this triangle, and the ratio of the parts into which it is divided by the perpendicular from the right angle upon the hypotenuse. (C.S.)

10. A plank 15 ft. long rests vertically against a perpendicular wall. How far must the bottom end be pulled out to lower the top end 3 ft.?
(O.S.L.)

Regular Quadrilaterals.

11. A rectangular field contains $1\frac{1}{2}$ ac.; the distance between opposite corners is $6\frac{1}{2}$ ch. Find the length and breadth of the field. (C.M.)

12. A square chess-board contains 64 equal squares, and the area of each square is 1·3225 sq. in.; the rim running round the board is $\frac{3}{4}$ in. wide. Find the length of the board's side. (K.S.)

13. Find the side of a square field containing 1 ac., and show, if two decimal places are found, the error cannot be more than ·004 in. (C.S.)

14. A farmer possesses 176 hurdles; he arranges them in an oblong with 4 hurdles at each of the shorter ends, and finds that he is able to enclose 1680 sheep. How many sheep could he enclose if he arranged the hurdles in a square, and how many more hurdles would he require to enclose 11520 sheep in a square? (C.M.)

15. A square hole $\frac{3}{4}$ in. broad is cut out of a pocket-handkerchief. A square piece of the same material, overlapping the hole by $\frac{1}{2}$ in. everywhere, is placed over it, and fastened on by stitches at the rate of 48 stitches per inch of length, made all round each square. How many square inches of handkerchief have now a double thickness, and how many stitches were used? (P.T.)

16. A square print is placed in a square frame, whose side wants 2 in. of being double the side of the print, and the number of square inches in the margin is less by 16 than 6 times the number of linear inches in the perimeter of the print. Find the side of the print. (K.S.)

17. Find the length of the side of a parallelogram, one angle being half a right angle, the altitude being equal to half the base, and the area containing 1250 sq. ft. (K.S.)

18. The diagonals of a rhombus being 88 and 234 ft., find the area; also find the length of a side and the height of the rhombus.

Irregular Quadrilaterals.

19. Find the area of a quadrilateral ABCD, in which the sides AB, BC, CD, DA and the diagonal AC are respectively 25, 60, 52, 39, and 65. (O.S.L.)

20. Find the area in acres of a quadrilateral field, of which the diagonal is 1274 lks., and the perpendiculars upon it from the opposite angles 550 and 583 lks.

21. Find the area of a quadrilateral figure in which the sides are 9, 40, 28, and 15 ft. respectively, and the angle between the first two is a right angle. (K.S.)

22. The lengths of the sides of a quadrilateral field are as follow: AB = 300 lks.; BC = 400 lks.; CD = 6 ch. 30 lks.; DA = 7 ch. 60 lks. The angle ABC is a right angle. Find the area of the field in acres, roods, and perches. (C.S.)

23. The area of a trapezoidal field is $4\frac{1}{2}$ ac., the perpendicular distance between the parallel sides is 120 yds., and one of the parallel sides is 10 ch. Find the other. (R.M.A.)

24. A field in the form of a trapezoid has its parallel sides equal to 5 ch. 15 lks. and 3 ch. 85 lks. respectively, and the perpendicular distance between them is 15 ch. Find the area; also draw a figure. (C.P.)

25. ABCD is a field in the form of a trapezium whose diagonal AC is 320 yds.; AD = 144, DC = 360, CB = 240, AB = 180 yds. Draw a fair plan of the field, and then find its area. (C.P.)

26. In a trapezium, two sides of which are parallel and the other two sides equal, the lengths of the parallel sides are 85·5 ft. and 9·5 ft. respectively, and the equal sides are each 47·5 ft. Find the area of the trapezium in square yards, feet, and inches.

27. The diagonal of a trapezium is 18 ft. 4 in., and the perpendiculars upon it from the opposite angles are 10 ft. 6 in. and 12 ft. Find the ratio of its area to that of a triangle whose sides are 20, 99, and 101 ft. (P.T.)

The Field-book.

28. Plan out and find the acreage of a field from the opposite notes. (O.S.L.)

	Links.	
	To E	
	1700	
	1500	400 to D
	900	800 to C
600 to B	600	
	From A	

29. Draw a plan and find the area of a level field with five straight hedges from the opposite notes. (All lengths in links.) (C.S.)

	To E	
	220	
To D 120	310	
	420	200 to C
To B 180	200	
	From A	

30. Describe (if possible by a Field-book) the measuring of a field ABCDE, F, H, K being the feet of perpendiculars drawn from B and E to AC, and from D to EC. Then find the area in acres, roods, and poles. (Given AC = 660 yds., CE = 550 yds., BF = 121 yds., EH = 363 yds., KD = 165 yds.) (C.P.)

31. Sketch the plan and calculate the area of a field ABEGFDC from the following notes:—

	Yards.	
	To G	
	204	
To F 94	198	
	122	10 to E
To D 64	117	
To C 14	88	
	63	70 to B
	From A	

32. Find the area and draw the plan of a field from the following Field-book. (C.P.) (All lengths in links.)

	To B	
	1200	To H 160
To G 280	1120	
To F 300	960	
	940	To E 400
To D 760	760	
	200	To C 30
	From A	

33. From the opposite dimensions, etc., draw out a plan of the irregular field; find the area in acres, roods, and poles. (The measurements are in links.) (C.P.)

Field-book.

	To G	
	2560	
To F 960	1880	
	1440	160 to E
To D 700	1200	
To C 250	850	
	600	912 to B
	From A	

34. Find the area of an irregular field, and draw a plan fairly to scale, from the following Field-book. (All lengths in links.)

	AB	
0	2628	31
58	2010	
	1960	13
	1570	31
78	1400	
	1110	38
54	920	
	670	25
84	520	
90	000	58
	From A go East	

CHAPTER LI.

THE CIRCLE.

If we measure the circumference of a penny with a piece of thread, and then measure the diameter, we shall find that the circumference is slightly more than *three times* the diameter. The ratio of the circumference of a circle to its diameter is usually given as $\frac{22}{7}$, but $\frac{355}{113}$ would be a nearer approximation. The real value of the ratio which the circumference of any circle bears to its diameter is *incommensurable;* expressed to *seven* places of decimals, its value is 3·1415926...; to *four* places, it is 3·1416. To save the trouble of writing a long decimal, this ratio is expressed by the Greek letter π.

$$\text{Circumference of a circle} : \text{diameter of same circle} :: \left\{ \begin{matrix} \pi \text{ or} \\ \frac{22}{7} \text{ or} \\ 3{\cdot}1416 \end{matrix} \right\} : 1$$

To find the Circumference of any Circle, having given the Diameter or the Radius.

Rule.—Multiply the diameter by $\frac{22}{7}$ or by 3·1416, or multiply twice the radius by the same constants.

$$\begin{aligned} \text{Circumference} &= \text{diameter} \times \pi \\ &= 2 \text{ radius} \times \pi \end{aligned}$$

$$\text{and } \pi = \frac{\text{circumference}}{\text{diameter}} = \frac{\text{circumference}}{2 \text{ radius}}$$

Given the Radius of a Circle, to find the Area of the Circle.

Rule.—Multiply the square of the radius by π.

$$\text{Area of circle} = \pi r^2$$

Given the Area of a Circle, to find the Radius.

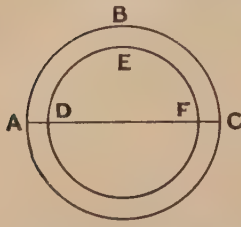
$$\text{Since } \pi r^2 = \text{area}$$

$$\therefore r^2 = \frac{\text{area}}{\pi}$$

$$\text{and } r = \sqrt{\frac{\text{area}}{\pi}}$$

Rule.—Divide the area by π, and find the square root of the result.

To find the Area of a Plane Circular Ring.



Rule.—(*a*) Find the area of the larger circle. (*b*) Find the area of the smaller circle. (*c*) The difference between (*a*) and (*b*) will give the area of the ring.

Or the following method may be employed:—

Let R be the radius of the larger ⊙ ABC;
and ,, r ,, ,, ,, smaller ⊙ DEF.

Then area of ⊙ ABC $= \pi R^2$
and ,, ⊙ DEF $= \pi r^2$

$\therefore$ area of ring $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$
$= \pi(R + r)(R - r)$

Rule.—Find the product of the three following: π, the sum of the radii, the difference of the radii.

Example.—*Find the area of a ring whose outer and inner radii are respectively* 20 *and* 15 *in.*

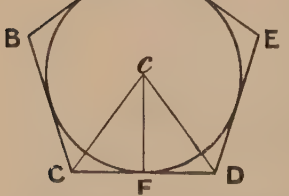
Area of ring in inches $= \pi(R + r)(R - r)$
$= \frac{22}{7}(20 + 15)(20 - 15) = \frac{22}{7} \times 35 \times 5$
$= 550$

Ans. 550 sq. in.

Polygons.

A Polygon is a plane figure bounded by more than four straight lines.

A Regular Polygon has all its sides equal and all its angles equal.



The commonest polygons are the *pentagon*, the *hexagon*, the *heptagon*, and the *octagon*, with *five*, *six*, *seven*, and *eight* sides respectively.

To find the Area of any Regular Polygon.

(*a*) *Given the side of the polygon and the radius of the inscribed circle.*

Let ABCDE be any regular polygon. Divide it into as many △'s as the figure has sides, by drawing straight lines from the centre of the ⊙ to the angular points of the polygon.

Then the △'s are all isosceles, and all equal. And if n be the number of sides, r the radius of the ⊙, and l the length of one side, then—

$$\text{Area of each } \triangle = \frac{CD \times FC}{2} = \frac{l \times r}{2}$$

$$\therefore \text{ area of polygon} = \frac{n \times l \times r}{2} = \frac{\textit{perimeter} \times r}{2}$$

Rule.—Multiply the radius by half the perimeter.

The radius (CF) of the inscribed circle is the perpendicular from the centre of the polygon on the side. Hence the next rule—

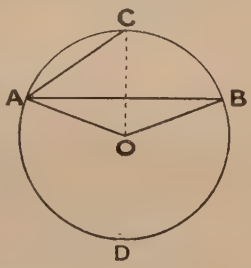
(*b*) *Given the side of any regular polygon and the perpendicular from the central point to the side.*

Multiply the perpendicular by half the perimeter.

The following method may also be employed. Divide the regular polygon into as many equal triangles as the polygon has sides by joining the centre of the polygon to its angular points. Find the area of one triangle, and hence the area of the whole polygon.

Chords, etc., of Circles.

The straight line AB in the figure is called the *chord of the circle*, and the parts of the circumference cut off by it are called *arcs of circles*.



The *chord* AB cuts the circumference of the circle into two *arcs*, ACB, ADB. The line AC is called the *chord of half the arc*.

The parts into which the area of the circle is divided by AB are called *segments of the circle*. ACB and ADB are segments of the circle.

The term *arc* has reference to the *circumference*, the term *segment* has reference to the *area* of the circle.

The parts OACB and OADB are termed *sectors* of the circle.

To find the Area of the Sector of a Circle.

Rule.—Multiply the radius of the circle by half the length of the arc, *or*—

State thus—

$$\text{As } 360° : \left\{\begin{matrix}\text{number of degrees}\\ \text{in angle of sector}\end{matrix}\right\} :: \text{area of circle : area of required sector.}$$

To find the Area of the Segment of a Circle.

Rule.—Find the area of the sector which has the same arc as the segment. From this deduct the area of the triangle contained by the radii and the chord.

To find the Chord of an Arc, the height of the Arc and the Diameter of the Circle being given.

Rule.—Multiply together the two parts into which the diameter is divided by the chord, and take twice the square root of the product.

To find the Chord of Half the Arc, the Height of the Arc and the Diameter of the Circle being given.

Rule.—Multiply the diameter by the height of the arc, and take the square root of the product.

To find the Length of the Arc of a Circle when the Chord of the whole Arc and the Chord of half the Arc are known.

Rule.—From eight times the chord of half the arc subtract the chord of the whole arc, and find one-third of the remainder.

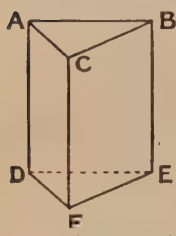
Solids bounded by Straight Lines.

A Rectangular Solid or Rectangular Parallelopiped is a body bounded by six rectangular faces, the opposite faces being equal and parallel.

When each of the six faces is a square, the rectangular solid is termed a *cube.*

For examples dealing with the cube, see Chapter XLVI.

A Prism is a solid having two of its opposite sides equal and parallel, and its remaining sides parallelograms.



If the ends of the prism are triangles, it is called a *triangular prism;* if hexagons, a *hexagonal prism;* and so on.

To find the Surface of a Triangular Prism.

Area of prism = area of sides + area of two ends.

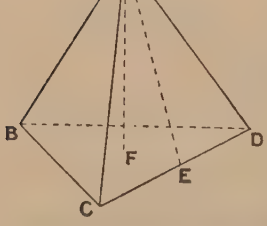
Now, if FC = p, DF = a, FE = b, DE = c, then the area of the three faces is ap, bp, and cp respectively.

And area of each end = $\sqrt{s(s-)(s-b)(s-c)}$

$\therefore$ total area of prism = $2\sqrt{s(s-a)(s-b)(s-c)} + p(a+b+c)$

Rule.—Add the areas of the two ends to the sum of the areas of the sides.

To find the Volume of a Triangular Prism.—The solid may be considered as composed of an infinite number of thin slices, each equal to the area of the base, placed the one over the other. Then—



Volume = area of base × height of prism.

Rule.—Multiply the area of the base by the height of the prism.

Pyramids.

A Pyramid is a solid with a rectilineal base, whose other sides are triangles, which meet at a point not in the plane of the base, called the vertex.

To find the Surface of a Triangular Pyramid.

1. *Lateral surface* or surface of sides.

$$\left.\begin{array}{l}\text{Area of } \triangle \text{ ACD} = \frac{1}{2}\text{AE . CD} \\ \text{,, } \quad \triangle \text{ ABC} = \frac{1}{2}\text{AE . CB} \\ \text{,, } \quad \triangle \text{ ABD} = \frac{1}{2}\text{AE . BD}\end{array}\right\} = \frac{1}{2}\text{AE(CD + CB + BD)}$$

i.e. the lateral surface = half the slant height × perimeter of base.

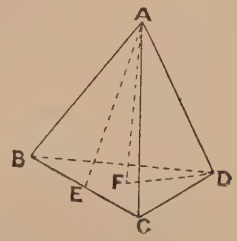
2. *Total surface* = base surface + lateral surface.

Rule.—Find the sum of the base and lateral surfaces.

To find the Volume of any Pyramid.

Rule.—Multiply the area of the base by a third of the perpendicular height of the pyramid.

Example.—*The base of a pyramid is an equilateral triangle. Each of the equal sides is* 20 *in., and the edges of the isosceles triangles forming the faces of the pyramid are each* 30 *in. Find the total area of its surface.* (C.S.)



Draw the slant height AE perpendicular to BC.

Then △'s ABE and AEC are equal in all respects (Euc. I. 26).

$\therefore$ BE = EC = $\frac{1}{2}$BC = $\frac{1}{2} \times 20$ in. = 10 in.

Again $AB^2 = BE^2 + AE^2$ (Euc. I. 47)

$\therefore AE^2 = AB^2 - BE^2 = (30^2 - 10^2)$ sq. in.
$= 800$ sq. in.

$\therefore$ AE = $\sqrt{800}$ sq. in. = $\sqrt{400 \times 2}$ sq. in.
= $20\sqrt{2}$ sq. in.

Then *lateral surface* = $\frac{1}{2}$AE × (perimeter of base)
= $(\frac{1}{2} \times 20\sqrt{2} \times 60)$ sq. in. = $600\sqrt{2}$ sq. in.

$$\textit{Area of base} = \left(\frac{(\text{side})^2\sqrt{3}}{4}\right) \text{sq. in.} = \frac{20^2\sqrt{3}}{4} \text{ sq. in.} = 100\sqrt{3} \text{ sq. in.}$$

$$\therefore \textit{total area of surface} = (600\sqrt{2} + 100\sqrt{3}) \text{ sq. in.}$$
$$= (848{\cdot}52 + 173{\cdot}2) \text{ sq. in.} = 1021{\cdot}72 \text{ sq. in.} \quad \textit{Ans.}$$

The Right Circular Cylinder.

A Right Circular Cylinder is a solid described by the revolution of a rectangle about one of its sides which remains fixed.

It may be regarded as a "round prism having a circle at each of its equal and parallel ends."

To find the Total Surface of a Cylinder.

1. *The Curved Surface.*—Multiply the circumference of the base by the height.

We may look upon the curved surface of a cylinder as capable of being unrolled so as to assume the shape of a rectangle.

2. *The Whole Surface.*—Add the area of the two ends to the area of the curved surface.

Let h = height of cylinder, and r = radius of base.

Then $2\pi r$ = circumference of base
and $2\pi r \times h$ = area of curved surface
Now, πr^2 = area of one end of cylinder
so $2\pi rh + 2\pi r^2$ = total surface of cylinder
or $2\pi r(h + r)$ = ,, ,, ,,

To find the Volume of a Cylinder.

Rule.—Multiply the area of the base by the height.

Regarding the cylinder as a "round prism," the reason for this rule is at once apparent.

Example.—*Find the total surface and volume of a right cylinder whose radius is 7 ft., and whose height is 13 ft.*

Total surface of cylinder $= 2\pi r(h + r)$

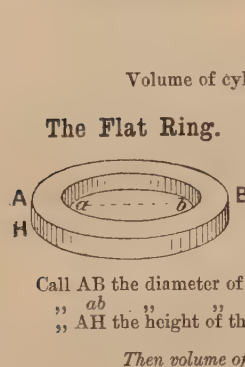
$$= \left(\frac{2 \times 22 \times 7}{7}(13 + 7)\right) \text{ sq. ft.}$$

$$= 2 \times 22 \times 20 \text{ sq. ft.} = 880 \text{ sq. ft.} \quad \textit{Ans.}$$

Volume of cylinder $= \pi r^2 h = (\frac{22}{7} \times 7^2 \times 13)$ cub. ft.

$= 2002$ cub. ft. *Ans.*

The Flat Ring.



The flat ring may be regarded as the difference between two cylinders.

To find the Volume of the Ring.

(*a*) *In terms of the diameter.*

Call AB the diameter of outer cylinder, D;
" *ab* " " inner " *d*;
" AH the height of the cylinder, *h*.

Then volume of outer cylinder $= \frac{\pi h D^2}{4}$

and " *inner* " $= \frac{\pi h d^2}{4}$

$\therefore$ *volume of ring* $= \frac{\pi h}{4}(D^2 - d^2)$

$$= \frac{\pi h}{4}(D + d)(D - d)$$

And re-written in terms of the radii (R and *r*), we have—

$$\textit{Volume of ring} = \pi h(R + r)(R - r)$$

(*b*) *In terms of the area of the cross-section and the mean circumference.*

Regarding the ring as a long rectangular prism bent round into a circle until the two ends meet, and recollecting that—

Volume of rectangular prism = area of base × height

we get—

Volume of ring = area of cross-section × mean circumference.

Example.—*Given that the radius of the outer cylinder is 9 in., the inner radius 5 in., and the height 2 in., find the volume of the flat ring.*

Volume of ring $= \pi(R + r)(R - r)h$

$= \{\frac{22}{7}(9 + 5)(9 - 5)2\}$ cub. in.

$= (\frac{22}{7} \times 14 \times 4 \times 2)$ cub. in.

$= 352$ cub. in. *Ans.*

The Right Circular Cone.

A Right Circular Cone is a solid produced by the revolution of a right-angled triangle about one of its sides containing the right angle while that side remains fixed.

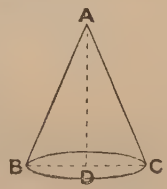
It is a "round pyramid" having a circle for its base.

To find the Area of the Surface of a Cone.

1. *The Area of the Slant Surface.*—Multiply the circumference of the base by half the *slant* height.

(1) The circumference of the cone and the slant height may be regarded as corresponding to the base and perpendicular height of a triangle; hence the above rule is analogous to that for finding the area of a triangle.

(2) Care must be taken to distinguish between the *slant* height and the *perpendicular* height of a cone. AC is the *slant* height, AD the *perpendicular* height.



2. *The Total Area.*—Add the area of the circular base to the area of the slant height.

Let radius of base of cone $= r$,
and slant height $= h$.

Then circumference of base $= 2\pi r$
and area of slant surface $= 2\pi r \times \frac{1}{2}h = \pi rh$
Area of base $= \pi r^2$
Then total area $= \pi rh + \pi r^2 = \pi r(h + r)$.

Rule.—Find the product of the three following factors:—
(1) The sum of the height and the radius; (2) π; (3) the radius.

To find the Volume of a Right Circular Cone.—A cone may be considered a pyramid with an infinite number of sides; hence the perimeter of the pyramid eventually becomes the circumference of the cone, and the formulæ for the two become alike.

$\therefore$ volume of cone $=$ base $\times \frac{1}{3}$ perpendicular height
$= \pi r^2 \times \frac{1}{3}$ perpendicular height.

Rule.—Multiply the area of the base by a third of the perpendicular height.

It will be observed that this formula applies also to the volume of a pyramid.

Example.—*Find the curved surface, the total surface, and the volume of a cone whose base diameter is* 6 *in., and whose perpendicular height is* 4 *in.*

Referring to the cone above,

$$AB^2 = BD^2 + DA^2 \text{ (Euc. I. 47)}$$
$$= 3^2 \text{ in.} + 4^2 \text{ in.} = 25 \text{ in.}$$
$$\therefore AB = 5 \text{ in.} = \text{height of cone}$$
$$\textit{Area of curved surface} = \pi rh = \frac{22 \times 3 \times 5}{7} \text{ sq. in.} = 47\tfrac{1}{7} \text{ sq. in.}$$
$$\textit{Total surface} = \pi r(r + h) = \frac{22 \times 3 \times (3 + 5)}{7} \text{ sq. in.}$$
$$= 75\tfrac{3}{7} \text{ sq. in.}$$
$$\textit{Volume} = \pi r^2 \times \tfrac{1}{3} \text{ perpendicular height}$$
$$= \frac{22 \times 3^2 \times 4}{7 \times 3} \text{ cub. in.}$$
$$= 37\tfrac{5}{7} \text{ cub. in.} \quad \textit{Ans.}$$

EXERCISE LXXIII. 73

Circles.

1. Find to 3 places of decimals the radius of the circle whose area is the sum of the areas of the two triangles whose sides are 35, 53, 66, and 33, 56, 65. ($\pi = 3\frac{1}{7}$.) (C.M.)

2. Find the perimeter and the radius of a circle, the area of which is 5·309304 sq. ft. (R.M.C.)

3. Taking $\pi = 3{\cdot}14159$, calculate approximately the area and the perimeter of the circle inscribed in a square, the side of which is 359·5678 ft. (R.M.A.)

4. I bend a piece of wire into the shape of an equilateral triangle, with a circle described about it. If the length of the side of the triangle is 14 in., how long is the wire? (C.M.)

5. ABC is an equilateral triangle; two circles are described, each passing through A, one of them touching BC at its middle point, and the other passing through B and C; and each side of the triangle is 14 in. long. Find the areas of the circles. (C.S.)

6. The area of an equilateral triangle is 17320·5 sq. ft. About each angular point, as centre, a circle is described with radius equal to half the length of a side of the triangle. Find the area of the space included between the three circles. ($\sqrt{3} = 1{\cdot}73205$.) (R.M.C.)

7. A halfpenny piece is 1 in. in diameter. Six halfpennies are placed so that each coin touches two others, their centres being all on the circumference of a circle. Find the area which they enclose. (R.M.C.)

8. The area of a rectangular field is $\frac{3}{5}$ ac., and its length is double its breadth. Determine the length of its sides approximately. If a pony is tethered to the middle point of one of the longer sides, find the length of the tether in yards (correct to 2 places of decimals), in order that he may graze over half the field. (R.M.C.)

9. One hundred and five halfpenny pieces lying on a flat surface with their edges in contact are just contained by a frame in the form of an equilateral triangle. The diameter of a halfpenny being 1 in., show that the side of the triangle is $(13 + \sqrt{3})$ in., and calculate its area approximately. (R.M.C.)

10. Two tangents, AB, AC, to a circle of radius 10 ft. contain an angle of 60°. Find to the nearest square foot the area included between them and the circle. ($\pi = 3\frac{1}{7}$.) (O.S.L.)

Chords, etc., of Circles.

11. How many panes of glass, each $4\frac{3}{8}$ by $3\frac{3}{8}$ in., will be required for a window with a semicircular head, whose extreme height is 35 ft. and width 14 ft.? ($\pi = \frac{22}{7}$.) (C.M.)

12. Prove that the area of a sector of a circle is equal to half the product of the length of the arc and the radius. If the area is 2240·567 sq. ft., and the radius 33·495 ft., calculate the length of the arc. (R.M.A.)

13. A pendulum swings through an angle of 30°, and the end describes an arc of $13\frac{2}{21}$ in. Find the length of the pendulum. ($\pi = 3\frac{1}{7}$.) (O.S.L.)

14. The minute-hand of a clock is 10 in. long. Find the area on the clock face which it describes between 9 a.m. and 9.35 a.m. (R.M.C.)

15. It is proposed to add to a square lawn measuring 58 ft. on a side two circular ends, the centre of each circle being the point of intersection of the diagonals of the square. How much turf will be required for the purpose? (R.M.A.)

16. Find the area of a segment of a circle, of which the arc is one-third of the circumference, the radius being $7\frac{1}{2}$ in. (Take $\pi = 3\frac{7}{50}$, $\sqrt{3} = 1\frac{11}{13}$.) (O.S.)

Polygons.

17. The area of a regular hexagon is $1944\sqrt{3}$ sq. in. Find the length of a side. (O.S.)

18. The side of a regular pentagon being 10, find its area. (O.S.)

19. A chain is taken along the line AD (920 lks.) in the six-sided field ABCDEF. The distance of the points B, C, E, F from AD are 182, 250, 190, 136 lks., meeting it at points 120, 560, 750, 95 lks. from A respectively. Find the acreage of the field. (O.S.)

20. In the pentagonal field ABCDE, the length of AC is 50 yds., and the perpendiculars from B, D, and E upon AC are 10, 20, and 15 yds., the distances from A of the feet of the perpendiculars from D and E being 40 and 10 yds. Find the area. (R.M.A.)

21. Find the area enclosed by 200 hurdles placed so as to form a regular polygon of 200 sides, the length of each hurdle being 6 ft. (R.M.A.)

22. Find in yards, to 4 places of decimals, the difference between the area of a regular hexagon, each of whose sides is 72 yds., and the area of the circle inscribed in it. ($\pi = 3\frac{1}{7}$.) (C.M.)

23. Find the expense of paving a circular court 80 ft. in diameter at 3*s.* 4*d.* per square foot, leaving in the centre a space for a fountain in the shape of a hexagon, each side of which is a yard. (R.M.A.)

Prisms.

24. The perpendicular height of a square chimney is 150 ft. 3 in. The side of the base measures 12 ft. 6 in., and the side at the top measures 6 ft. 3 in. The cavity is a square prism whose side measures 3 ft. 9 in. How many cubic feet and inches of solid brickwork are there in the building? (C.P.)

25. The sides of the base of a triangular prism are 17, 25, and 28 ft., and its height is 20 ft. Find its volume. (C.M.)

26. The solid content of a triangular prism is $4{\cdot}1\dot{6}$ cub. ft., and the sides of the triangular end are 2 ft. 1 in., 1 ft. 8 in., and 1 ft. 3 in. Find the length of the prism. (C.P.)

27. Determine the number of cubic yards in a bank of earth on a horizontal rectangular base 60 ft. long and 20 ft. broad, the four sides of the bank sloping up to a ridge at an angle of 40° to the horizon. (R.M.A.)

28. Find the area of the surface (including the ends) of a hexagonal prism whose height is 18 ft., the base being a regular hexagon with a side of length 3 ft. (R.M.A.)

Pyramids.

29. Find the volume of a pyramid, the height of which is 12 in., and the base an equilateral triangle, each side of which is 10 in.

30. Find the solid content of a pyramid whose vertical height is $14\frac{2}{3}$ ft., and which has for its base a triangle whose sides measure 72, 58, and 50 in. (C.P.)

31. A right pyramid stands on a regular hexagonal base, whose side is $20\sqrt{3}$ ft. The total area of the triangular faces is $3000\sqrt{3}$ sq. ft. What is the height? (O.S.L.)

32. A crystal consists of a cube capped at two opposite faces by pyramids, the sides of which are equilateral triangles. Show that the whole surface is equal to that of a square on the sum of the diameters of a rhombus of which one of the equilateral triangles is a half. (C.M.)

33. A square tower 21 ft. on each side is to have either a flat roof covered with sheet lead, which costs 6*d.* per square foot, or a pyramidal roof, whose vertical height is 10 ft., covered with slates, which cost 18*s.* 9*d.* per 100, and each of which has an exposed surface of 12 in. by 9 in. Find the cost in each case. (O.J.)

Cylinders.

34. The outer diameter of a water-pipe is 2 ft., the inner diameter is 1 ft. 8 in., and the length of the pipe is 40 ft. Find the number of solid inches of metal used in the construction of the pipe. (C.P.)

35. What will it cost to make a cylindrical barrel closed at both ends, 3 ft. high and 3 ft. in diameter, the price of wood being 6*d.* per square foot, and the charge for labour being made at the rate of 3*d.* a gallon on the contents of the barrel? (Take a gallon = 270 cub. in.) (O.J.)

36. A well 5 ft. in diameter and 30 ft. deep is to have a lining of bricks fitting close together without mortar, 9 in. thick. Required approximately in tons the weight of the bricks, supposing a brick $9 \times 4\frac{1}{2} \times 3$ in. to weigh 5 lbs. (R.M.A.)

37. If a bushel of oats measure $2169{\cdot}6\dot{4}2857\dot{1}$ cub. in., find the depth of a cubical corn-bin which will just contain 42 bush., and also the smallest whole number of inches in the diameter of a cylindrical bin of the same depth which will hold as much. (C.J.)

38. A right cylinder of wood, 10 ft. long and 2 ft. in diameter, is sawn along two planes parallel to each other and parallel to the axis of the cylinder, each plane being at a distance from the axis equal to one-half of the radius of the cylinder. Compare the volumes of the logs so formed. (C.S.)

39. Two cylindrical vessels are filled with water; the radius of one vessel is 6 in., and its height 1 ft., and the radius of the other is 8 in., and its height $1\frac{1}{2}$ ft. Find the radius of a cylindrical vessel 11 in. in height which will just contain the water in the two vessels. (R.M.A.)

40. Determine the diameter of a cylindrical gasholder to contain 10000000 cub. ft. of gas, supposing the height to be made equal to the diameter; and determine in tons the weight of iron plate, weighing $2\frac{1}{2}$ lbs. per square foot, required in the construction of the gasholder, supposing it open at the bottom and closed at the top by a flat top. ($\pi = 3{\cdot}1416$.) (R.M.A.)

41. One quarter of the volume of a cylindrical boiler 12 ft. long and 5 ft. in diameter internally is fitted with cylindrical heating tubes fitted longitudinally, and $7\frac{1}{2}$ in. in diameter. Find the number of tubes. (C.M.)

42. A cylindrical boiler, terminated by plane ends, is internally 15 ft. long and 4 ft. in diameter, and is traversed lengthwise by 50 cylindrical fire tubes, each 3 in. in external diameter. Determine the volume of water the cylinder could contain, taking π to be $\frac{22}{7}$. (R.M.A.)

43. Water is flowing steadily through a pipe, which consists of two parts. The cross-section of the first part is an equilateral triangle of side 6 in., and that of the second a circle of radius 2 in. Find (approximately) the rate of flow of water in the second part, that in the first being 31·7552 ft. per minute. ($\pi = \frac{22}{7}$; $\sqrt{3} = 1{\cdot}732$.) (R.M.A.)

Flat Rings.

44. Find the volume of a flat ring whose outer diameter is 4 ft., inner diameter 3 ft., and whose thickness is 5 in.

45. The cross-section of a flat ring is 3 in. high and 4 in. wide, and the extreme width of the ring is 30 in. Find its volume.

46. The area of a disc is 4900 sq. in., and a smaller disc 3600 sq. in. in area is cut out of it. If the thickness of the disc be 4 in., find the volume of the flat ring remaining.

Cones.

47. How many square feet of canvas are necessary for a conical tent whose perpendicular height is 10 ft. and diameter of its base 13 ft.? (C.S.)

48. The height of a conical tent is $7\frac{1}{2}$ ft., and it is to enclose 200 sq. yds. of ground. Find how much canvas will be required. ($\pi = \frac{22}{7}$.) (R.M.A.)

49. A right-angled triangle, whose sides are 4 and 3 in., is made to turn round its hypotenuse. Find the volume and surface of the double cone so formed. (C.M.)

50. Calculate the diameter of a circle equal in area to the curved surface of a right cone, whose vertical angle is 90° and vertical height 1·4142 ft. (R.M.A.)

51. The interior of a building is in the form of a cylinder of 15 ft. radius and 12 ft. altitude, surmounted by a cone whose vertical angle is a right angle. Find the area of the surface and the cubical content of the building. (R.M.C.)

52. A cylindrical stick $\frac{1}{3}$ in. thick is sharpened at one end into the

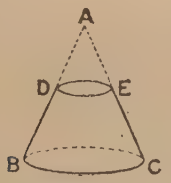
shape of a cone, the slant side of which is $\frac{5}{16}$ in. long. How much wood is cut away in doing this? (O.J.)

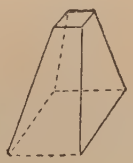
53. A vessel in the shape of a circular cylinder, open at the top, the height of which is double its diameter, stands on a horizontal table, and is filled with water. A solid circular cone, of the same base and the same height, is pushed into the water until its vertex reaches the base of the cylinder, and is then taken out. Find the height at which the water afterwards stands in the cylinder. (R.M.A.)

CHAPTER LII.

FRUSTA OF CONES AND PYRAMIDS.

The Frustum of a Cone (or of a pyramid) is that part which remains after a smaller cone (or pyramid) has been cut off at the top by a plane parallel to the base.





To find the Area of the Slant Surface of the Frustum of a Cone or of a Pyramid.

Rule.—Multiply half the sum of the perimeters of the ends by the slant height.

In the case of the cone, the perimeters will be the circumferences of the circle. Hence, area of curved surface of cone $= \pi l(R + r)$.

To find the Volume of the Frustum of a Cone or of a Pyramid.

Rule.—Add to the areas of the two ends the square root of the product of the areas; multiply this sum by a third of the perpendicular height of the frustum.

Let A and a = areas of the two ends;
and P = the perpendicular height of the frustum.

Then volume of frustum of a cone or pyramid $= \frac{1}{3}P(A + a + \sqrt{Aa})$

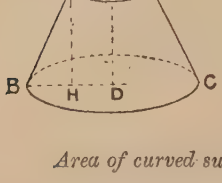
Or, in the case of the frustum of a cone, we may re-write in terms of the radius, thus—

$$\text{Volume of frustum of cone} = \tfrac{1}{3}P(\pi R^2 + \pi r^2 + \sqrt{\pi R^2 \times \pi r^2})$$

$$= \tfrac{\pi}{3}P(R^2 + r^2 + Rr)$$

Example.—*Find the curved surface, the total surface, and the volume*

of the frustum of a cone, the circumferences of whose bases are respectively 44 *and* 22 *in., and whose slant height is* 5 *in.* (C.S.)



Let R and r = radii of large and small circles respectively.

Circumference of large circle = 44 in. = $2\pi R$

$$\therefore R = \frac{44 \times 7}{2 \times 22} \text{ in.}$$

$$= 7 \text{ in.}$$

Similarly, $r = 3\frac{1}{2}$ in.

$$\textit{Area of curved surface} = \pi h(R + r) = \frac{22 \times 5(7 + 3\frac{1}{2})}{7} \text{ sq. in.}$$

$$= 165 \text{ sq. in.}$$

$$\textit{Area of whole surface} = \text{area of ends} + \text{area of curved surface}$$

$$= \{\pi(R^2 + r^2) + 165\} \text{ sq. in.}$$

$$= \frac{22 \times \{7^2 + (3\frac{1}{2})^2\}}{7} \text{ sq. in.} + 165 \text{ sq. in.}$$

$$= 357\tfrac{1}{2} \text{ sq. in.}$$

$$\textit{Volume of frustum} = \frac{\pi P}{3}(R^2 + r^2 + Rr)$$

To find value of P or of FD

$$FD^2 = HA^2 = AB^2 - BH^2 = 5^2 - (3\tfrac{1}{2})^2 \text{ in.} = \tfrac{51}{4} \text{ in.}$$

$$\therefore \text{P or FD} = \sqrt{\tfrac{51}{4}} \text{ in.} = \tfrac{1}{2}\sqrt{51} \text{ in.}$$

Substituting in the equation above—

$$\textit{Volume of frustum} = \frac{22 \times \sqrt{51}}{7 \times 3 \times 2}\{7^2 + (3\tfrac{1}{2})^2 + 7 \times 3\tfrac{1}{2}\} \text{ cub. in.}$$

$$= \frac{22 \times \sqrt{51} \times 85\frac{3}{4}}{7 \times 3 \times 2} \text{ cub. in.}$$

$$= 302{\cdot}8 \text{ cub. in. approximately.}$$

Ans. 165 sq. in.; $357\frac{1}{2}$ sq. in.; 302·8 cub. in.

The Sphere.

A sphere is a solid every point of whose surface is equidistant from a point within the solid called the centre; or it may be described as a solid produced by the revolution of a semicircle about a diameter which remains fixed.

To find the Surface of a Sphere.

Rule.—Multiply the square of the diameter by 3·1416.

Surface of sphere = $4\pi r^2$ square units.

To find the Volume of a Sphere.—The sphere may be considered as the sum of an infinite number of small cones or pyramids, whose height is the radius of the sphere, and the sum of whose bases is the area of the sphere.

Now, volume of one cone = area of base $\times \frac{h}{3}$ (p. 483)

$= \text{area of base} \times \frac{r}{3}$

But bases of all the cones = surface of sphere

$\therefore$ volume of sphere = area of surface of sphere $\times \frac{r}{3}$

$= 4\pi r^2 \times \frac{r}{3} = \frac{4\pi r^3}{3}$

Rule.—Multiply the cube of the radius by 4π, and divide by 3.

Example.—*Find the surface and volume of a sphere whose radius is 7 ft.*

$$\text{Surface} = 4\pi r^2 = \frac{4 \times 22 \times 7^2}{7} \text{ sq. ft.} = 616 \text{ sq. ft.}$$

$$\text{Volume} = \frac{4\pi r^3}{3} = \frac{4 \times 22 \times 7^3}{3 \times 7} \text{ cub. ft.} = 1437\tfrac{1}{3} \text{ cub. ft.}$$

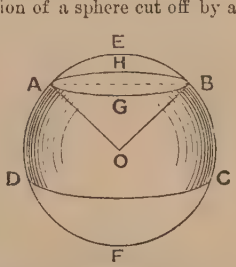
Ans. (1) 616 sq. ft.; (2) $1437\frac{1}{3}$ cub. ft.

The Zone and Segment of a Sphere.

A zone of a sphere is that portion of a sphere which lies between two parallel planes which cut the sphere; *e.g.* the solid ABCD.

A Segment.—A segment of a sphere is that portion of a sphere cut off by a plane passing through it; *e.g.* the plane AGBH cuts the sphere into two spherical segments, EAGB and FAGB.

A sector of a sphere consists of a spherical segment plus a cone standing on the cutting plane as a base, the apex of the cone being at the centre of the sphere; *e.g.* AOBE.



To find the Surface and Volume of the Segment of a Sphere.—A segment of a sphere is a particular form of the zone of a sphere. This is obvious if one of the cutting planes (*e.g.* AGB) is drawn to *touch* the sphere instead of cutting it. Hence the formulæ for the segment may be obtained from that of the zone.

$$\text{Area of segment} = 2\pi rh$$

where r is the radius of the sphere, and h the height of the segment.

$$\text{Volume of segment} = \frac{\pi h}{6}(3r_1^2 + h^2)$$

where r_1 is the radius of the base of the segment.

To find the Surface and Volume of the Zone of a Sphere.

$$\text{Curved surface of zone} = 2\pi rt$$

where t is the thickness of the zone.

$$\text{Volume of zone} = \frac{\pi t}{6}\{3(r_1^2 + r_2^2) + t^2\}$$

where r_1 and r_2 are respectively the radii of the two ends, or the top and bottom of the zone.

Note.—When the zone becomes a segment r_2 disappears, and the formula becomes $\frac{\pi t}{6}(3r_1^2 + t^2)$, the formula for a segment.

To find the Volume of a Spherical Shell.—The volume of a spherical shell may be regarded as the difference between the volumes of two spheres. The spheres may be regarded as being concentric, but whether they are or not, the volume is the same.

Let R be the radius of one sphere, and r that of the other.

$$\text{Then volume of shell} = \tfrac{4}{3}\pi R^3 - \tfrac{4}{3}\pi r^3 = \tfrac{4}{3}\pi(R^3 - r^3)$$

In terms of the two diameters, this formula becomes—

$$\tfrac{4}{3}\pi\left(\frac{D^3}{2^3} - \frac{d^3}{2^3}\right) = \tfrac{4}{3}\pi\left(\frac{D^3}{8} - \frac{d^3}{8}\right) = \frac{\pi}{6}(D^3 - d^3)$$

$$= \frac{\pi}{6}\{(D - d)(D^2 + Dd + d^2)\}$$

$$= \frac{\pi}{3}\left(\frac{D - d}{2}\right)(D^2 + Dd + d^2)$$

The student should compare this formula with that for the frustum of a cone (p. 489).

Example.—*The outer diameter of a shell is* 10 *in., and the inner diameter is* 9 *in. Find the volume of the metal contained in the shell.* (C.S.)

$$\text{Volume} = \frac{\pi}{3}\left(\frac{D - d}{2}\right)(D^2 + Dd + d^2)$$

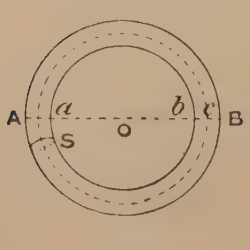
$$= \frac{22}{7 \times 3}\left(\frac{10 - 9}{2}\right)(100 + 90 + 81) \text{ cub. in.}$$

$$= \left(\frac{22}{21} \times \tfrac{1}{2} \times 271\right) \text{cub. in.}$$

$$= 141{\cdot}95 \text{ cub. in.} \quad \textit{Ans.}$$

To find Volume of a Cylindrical or Anchor Ring.

This should not be confused with the *flat* ring (p. 482).



The anchor ring may be regarded as a long narrow cylinder bent round into a circle until the two circular ends touch each other.

Volume of cylinder = area of circular end × height

Volume of cylindrical ring = area of cross-section × mean circumference.

Let R = OB, and r = Ob.

Then mean circumference (indicated by dotted circle) $= \pi(R + r)$

and area of cross-section $= \pi\left(\frac{R - r}{2}\right)^2$

$\therefore$ volume of ring $= \pi\left(\frac{R - r}{2}\right)^2 \pi(R+r)$

$= \frac{\pi^2}{4}(R - r)^2(R + r)$

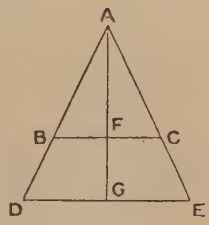
Similar Figures.

These may be defined as figures of the same *shape*, but not necessarily of the same size.

Thus *squares* of all sizes are similar figures; so are all *circles* and all *equilateral triangles*.

The triangles ABC, ADE are similar triangles.

Then AB : AD :: AC : AE
and AB : BD :: AE : CE
,, AG : GD :: AF : FB, etc.



The Areas of Similar Figures are to one another as the *squares* of their corresponding dimensions (Euc. VI. 20).

The Volumes of Similar Figures are to one another as the *cubes* of their corresponding dimensions.

Conversely, the corresponding dimensions of similar figures vary as the $\begin{cases}\text{square}\\\text{cube}\end{cases}$ roots of their $\begin{cases}\text{areas.}\\\text{volumes.}\end{cases}$

Example.—*The volume of a body of definite form varies directly as any one of its three axes varies, the other two being considered constant; if the weight of such a solid, whose axes are 5 in., 3 ft., and 4 in., be 1728 oz. when constructed of one material, what is the weight of one whose axes are 4 ft., 2 in., and 30 in., when made of a substance the density of which is to the density of that composing the first as 3 : 4?* (C.S.)

Let V and v be the volumes of the two bodies respectively; let D and d be their densities.

Then V : v	::	$5 \times 36 \times 4$	:	$48 \times 2 \times 30$
	::	720	:	2880
	::	1	:	4
and D : d	::	4	:	3
$\therefore$ DV : dv	::	4×1	:	4×3
i.e. W : w	::	4	:	12 (where W and w = weights of bodies)
	::	4×1728 oz.	:	12×1728 oz.
	::	4×108 lbs.	:	12×108 lbs.
	::	108 lbs.	:	324 lbs.

Ans. 324 lbs.

EXERCISE LXXIV.74

Frusta of Pyramids.

1. Find the volume of a frustum of a square pyramid, each side of one of its ends being 6 ft., each side of the other end 4 ft., and the perpendicular height 5 ft. (C.M.)

2. A frustum of a regular pyramid has square ends; the edge of the lower end is 10 in., and that of the upper end 5 in.; and the height of the frustum is $7\frac{1}{2}$ in. Find the length of a slant edge of the frustum, and the area of the slant faces. (R.M.A.)

3. If a pyramid is cut into two pieces by a plane parallel to the base, halfway between the vertex and the base, show that one piece is equal to 7 times the other. (C.S.)

4. A right pyramid 12 ft. high, standing on a square base, each side of which is 10 ft. long, is divided into 4 parts of equal thickness by planes parallel to the base. Determine the volume of each part in cubic feet. (O.S.L.)

5. An obelisk $68\frac{1}{2}$ ft. high has a square section throughout; it is $7\frac{1}{2}$ ft. wide at the base, and gradually tapers to a width of 5 ft., the summit being in the form of a pyramid $7\frac{1}{2}$ ft. high; it is made of granite, of which a cubic foot weighs 156 lbs. Find the weight of the obelisk. (R.M.A.)

6. Find the number of cubic feet of stone in Cleopatra's needle, taking the form of the obelisk as the frustum or portion of a square pyramid 61 ft. high, 7 ft. 8 in. square at the base, and 4 ft. 6 in. square at the top, and capped at the top by a square pyramid 7 ft. 6 in. from the base to the apex. (C.S.)

Frusta of Cones.

7. Find the curved surface of the frustum of a cone whose top and bottom diameters are 4 ft. and 6 ft., and slant side 8 ft. ($\pi = 3\frac{1}{7}$.) (C.M.)

8. Express the volume of a cone in terms of the radius of the base and the vertical height. If the diameters of the circular ends of a frustum of a cone be 4 in. and 6 in., and the volume of the frustum be 209 cub. in., find the height of the cone. ($\pi = 3\frac{1}{7}$.) (R.M.A.)

9. The height of a cone is 30 ft. A small cone is cut off the top by a plane parallel to the base. If its volume be $\frac{1}{27}$ of the volume of the given cone, at what height above the base is the section made? ($\pi = 3\frac{1}{7}$.) (O.S.L.)

10. A vessel in the form of a conic frustum is required to hold $3\frac{1}{2}$ gals. of water. If the diameters of its circular ends be 10 in. and 6 in., what must be its length? (Given the contents of gallon = 277·274 cub. in., circumference of circle = $\frac{22}{7}$ diameter.) (C.S.)

11. The cubical contents of a certain cask are equal to twice that of a frustum of a right cone $2\frac{1}{2}$ ft. high, the end diameters of which measure respectively 3 ft. and 3 ft. 6 in. How many such casks will hold as much as 508 cylindrical casks, each measuring internally 5 ft. in height and 3 ft. 3 in. in diameter? (C.S.)

12. A tent is made in the form of a conic frustum, surmounted by a cone of different angle. The diameters of the base and top of the

frustum are 14 ft. and 7 ft., its height 8 ft., and the height of the tent 12 ft. Find the quantity of canvas required for it. (C.S.)

Sphere.

13. Find approximately the diameter of a sphere whose volume is 5946 cub. in. (C.M.)

14. If the number of square feet on the surface of a sphere equals the number of cubic feet of its volume, what number represents its diameter? (C.S.)

15. A leaden sphere 1 in. in diameter is beaten out into a circular sheet of uniform thickness equal to $\frac{1}{100}$ in. Find the radius of the sheet. (R.M.A.)

16. A pyramid of lead is 14 in. high, and stands on a square base 6 in. long on each side. How many cubic inches of lead does it contain, and how many spherical bullets ·75 in. in diameter can be made out of it? (O.J.)

17. A circular disc of lead, 3 in. in thickness and 12 in. in diameter, is wholly converted into shot of the same density, and of ·05 in. radius each. How many shot does it make? (R.M.A.)

18. A cubical box 5 ft. deep is filled with layers of spherical balls, whose diameters, where they touch, are in vertical and horizontal lines. Find what portion of the space in the box would be left vacant if the diameter of a ball is half an inch. (R.M.A.)

19. A sphere, whose radius is $2\frac{1}{3}$ in., is enclosed in a hollow cylinder of the same radius, whose length is equal to its circumference. How many cubic inches are there in the remaining part of the cylinder? ($\pi = 3\frac{1}{7}$.) (C.M.)

20. A sphere of 1 ft. radius rests on a table. Find the volume of the right hollow cone which can just cover it, the section of the cone through the axis being an equilateral triangle. (R.M.A.)

21. Assuming a drop of water to be spherical and $\frac{1}{10}$ in. in diameter, to what depth will 500 drops fill a conical wine-glass, the cone of which has a height equal to the diameter of its rim? (R.M.A.)

22. A conical hole is bored in a sphere of 5 in. radius, so that the vertex of the cone is at the centre of the sphere, and that the portion of the spherical surface removed is one-quarter of the whole surface of the sphere. Find the area of the surface of the conical hole. (C.S.)

23. From a cubic foot of lead is cut out a pyramid whose base is one face of the cube, and whose vertex lies in the face opposite the base. If the remainder of the lead is melted and cast into a sphere, find its diameter. (R.M.C.)

24. A right prism on a triangular base, each of whose sides is 21 in., is such that a sphere, described within it, touches its five faces. Find the volume of the sphere, and of the space between it and the surface of the prism. ($\pi = \frac{22}{7}$, $\sqrt{3} = 1{\cdot}732$.) (R.M.A.)

25. A solid iron cube, the edge of which is 2 ft. in length, and a solid iron sphere, the radius of which is 1 ft., are thrown into a cubical tank which is 6 ft. across, and is half filled with water. Find the rise of the surface of the water in inches to 5 places of decimals, it being taken for granted that the cube and the sphere are both completely submerged. (R.M.A.)

26. A hollow right prism stands upon a base which is an equilateral triangle; the vertical faces of the prism are squares, the side of a square being 1 ft. The prism is filled with water, and the largest possible sphere is then submerged in it. Find (to the nearest cubic inch) the amount of water remaining in the prism. ($\pi = \frac{22}{7}$, $\sqrt{3} = 1{\cdot}73$.) (R.M.A.)

27. Within a hollow sphere of 1 ft. radius is placed a right prism, the ends of which are equilateral triangles. The side of one of these being 1 ft. in length, and the surface of the sphere being in contact with all the six angular points of the prism, find in cubic inches the volume of the latter. (R.M.A.)

28. A cone and a hemisphere have equal bases and equal volumes. Find the ratio of their heights. (C.M.)

29. Find the number of square inches in the surface of a glass shade in the shape of a cylinder with a hemispherical top, the diameter of the shade being 12 in., and the total height to the top of the dome 2 ft. ($\pi = 3{\cdot}1416$.) (C.S.)

30. An iron boiler is constructed in the form of a cylinder with hemispherical ends. If the radius of the boiler is 1 ft., and its extreme length (inside) 4 ft., calculate, as a fraction of a ton, the weight of water which it will hold, assuming that 1000 oz. is the weight of 1 cub. ft. of water. (R.M.A.)

31. Find the number of square feet in a buoy in the form of a cone and hemisphere, united by the rim of their common base, the height of the cone and the diameter of the base being 8 ft. each. (C.S.)

32. A cathedral has two spires and a dome; each of the former consists, in the upper part, of a pyramid 60 ft. high standing on a square base, of which a side is 20 ft. The dome is a hemisphere of 40 ft. radius. Find the cost of covering the three with lead at $7\frac{1}{2}d.$ per square foot. (Take $\pi = \frac{22}{7}$.) (R.M.A.)

33. Find the area of the convex surface of the segment of a sphere, the height being 8 in., and the diameter of the sphere $10\frac{1}{2}$ in. (C.M.)

34. A circular room, surmounted by a hemispherical vaulted roof, contains 5236 cub. ft. of air, and the internal diameter of the building is equal to the height of the crown of the vault above the floor. Find the height. (R.M.A.)

Spherical Shell.

35. What quantity of metal will be required to make a hollow spherical ball, the external diameter being 18 in. and the thickness $4\frac{1}{2}$ in.? (C.S.)

36. A cylindrical cup of external diameter $4\frac{1}{2}$ in., height $6\frac{1}{4}$ in., and thickness $\frac{1}{4}$ in., weighs 24 oz. 9 dwts. What would be the weight of a hemispherical lid, of the same material and thickness, to fit the top?

37. Write down the expression for the volume of a spherical shell, the external and internal radii of which are 5 in. and 4 in. respectively. If this shell is made of lead, and if it is filled with water, find to the nearest ounce its total weight, it being given that 1 cub. ft. of water weighs 1000 oz., and that lead is 11·5 times as heavy as water. (R.M.A.)

38. The radii of the internal and external surfaces of a hollow spherical shell of metal are 3 ft. and 5 ft. respectively. If it be melted down and the material formed into a cube, find an approximate value for the length of an edge of the cube. (R.M.A.)

39. A hollow shell 12 in. in diameter is placed in a conical vessel whose vertical angle is 60°, and water is poured into it until it just covers the shell and fills the cavity in it. When the shell, emptied of the water in it, is removed, and a solid ball of the same diameter substituted for it, the water stands $\frac{1}{2}$ in. above it. Find approximately the thickness of the shell. (R.M.A.)

Cylindrical Ring.

40. Required the solid content of a cylindrical ring whose thickness is 27 in. and inner diameter 96 in. (C.P.)

41. Find the content of a ring whose section is 9 in. square and outside diameter 6 ft. (C.S.)

42. Find the solid and superficial contents of a cylindrical ring whose thickness is 9 in. and inner diameter 32 in. ($\pi = 3\frac{1}{7}$.) (C.S.)

43. A circular ring fits closely round a cylinder of the same diameter ($2d$). Find the height of the cylinder (i.) if their volumes are equal, (ii.) if the total surface of the cylinder is equal to the surface of the ring. (C.M.)

Similar Figures.

44. The sides of a triangle are respectively 6, 4, and 4 yds. Compare the area of the original triangle with that of the triangle formed between one of the shorter sides and the perpendicular let fall on the other from the opposite angle to it. (K.S.)

45. Assuming that the volume of a cone is one-third of the base × height, find an expression for the difference between the volumes of two *similar* cones, of which the height and radii of the base are H, R and h, r respectively. Hence deduce the formula for the volume of a frustum of a cone. (C.M.)

46. A ball of lead 4 in. in diameter is covered with gold. Find the thickness of the gold in order that (1) the volumes of the gold and lead may be equal; (2) the surface of the gold may be twice that of the lead. (R.M.A.)

47. If a right cone on a circular base be divided into three portions by two sections parallel to the base at equal distances from the base and vertex, compare the three volumes into which it is divided. (R.M.A.)

48. Two pipes, one of lead and the other of tin, are respectively 49 in. and 61·6 in. long; they both have the same internal diameter, 1 in.; and the external diameter of the lead pipe is 1·2 in. If lead is 11 times and tin 7 times as heavy as water, what must be the external diameter of the tin pipe that both pipes may have the same weight? (R.M.A.)

49. The height of a solid right cone is 14 in., and its vertical angle is two-thirds of a right angle. If this cone is cut into two parts by a plane bisecting the axis at right angles, find to 4 places of decimals the volume and total slant surface of each part. (R.M.A.)

Miscellaneous.

50. A minute of latitude contains 6080 ft., and a metre is the ten-millionth part of a quadrant of the meridian. A kilogramme is the weight of the one-thousandth part of a cubic metre of water, and a cubic foot of water weighs 1000 oz. Express a kilogramme in ounces to 2 places of decimals. (C.M.)

51. Gold is 19·25 times as heavy as water, and a cubic foot of water weighs 997 oz. avoirdupois. Find (approximately) how many square feet

a cubic inch of gold will cover in the form of gold-leaf, given that one grain of gold will cover 56 sq. in. (R.M.A.)

52. A circular disc of cardboard 1 ft. in diameter is divided into six equal sectors by pencil lines drawn through the centre. In each sector there is described a circle touching the two bounding radii of the sector, and also the arc joining their ends at the middle point. If the circles are cut out from the six sectors, find the area of cardboard remaining. (R.M.A.)

53. Having given that the length of each edge of a regular tetrahedron is 4 in., determine to 3 places of decimals the number of square inches in the total surface of the tetrahedron. Also find the number of cubic inches in the volume of the tetrahedron. (R.M.A.)

54. Write down the formulæ for the volume of a cone of height a and base A, and for the volume of a sphere of radius r. Find to the nearest gramme the weight of a steel body consisting of a cone and a hemisphere with a common base, the height of the cone being equal to the diameter of the base—that is, 10 centimetres. (Given steel is 7·8 times as heavy as water, and one cubic centimetre of water weighs one gramme.) (R.M.A.)

55. Find in feet to 3 places of decimals the radius of a circle, the area of which is equal to the area of a regular hexagon, the side of which is 2 ft. Also find the radius of a sphere, the volume of which is equal to the volume of a solid circular cylinder of height 1 ft. and radius 9 in. (R.M.A.)

56. A frustum of a pyramid has rectangular ends, the sides of the base being 25 ft. and 36 ft. If the height of the frustum be 60 ft., and its volume 50480 cub. ft., find the area of the top. Find to the nearest foot the radius of the sphere whose volume is equal to the volume of the frustum. ($\pi = 3\frac{1}{7}$.) (R.M.A.)

57. The silk covering of an umbrella forms a portion of a sphere of $3\frac{1}{2}$ ft. radius, the area of the silk being $14\frac{2}{3}$ sq. ft. Find the area of the ground sheltered from vertical rain when the stick is held upright. ($\pi = \frac{22}{7}$.) (R.M.A.)

58. A solid sphere fits closely into the inside of a closed cylindrical box, the height of which is equal to the diameter of the cylinder. Having given the radius of the sphere, write down the expressions for the volume of the sphere, the surface of the sphere, and the volume of the empty space between the sphere and the cylinder. If the volume of this empty space is 134·0416 cub. in., what is the radius of the sphere? (R.M.A.)

59. A hollow cone, the length of whose slant side is twice the radius of its base, is held with its vertex vertically downwards and completely filled with water. A sphere of greater density than water is gradually immersed, and it is found that, when it rests upon the sides of the interior of the cone, it is just submerged. Find the amount of water displaced by the sphere, and also the amount contained between the sphere and the vertex of the cone. Consider the radius of the base of the cone as 1·73205 in. (R.M.A.)

60. A railway tunnel consists of a hollow semi-cylindrical top, terminating below in a trough with slanting sides and flat base. The radius of the former being 12 ft., the base and height of the latter being 20 ft. and 18 ft. respectively, and the length of the tunnel 1200 yds., find the cost of facing the sides and roof with brick at 1s. 6d. per square foot. ($\pi = \frac{22}{7}$.) (R.M.A.)

EXERCISE LXXV.

Test Papers.

1. Divide five millions seven hundred and ninety-one thousand seven hundred and fifty-five by one hundred and twenty-one, employing the method of short division. Express the result in words, and explain the process by which the remainder is obtained. (C.PR.)

2. A boy running up a staircase finds that when he goes up two steps at a time there is one step over; when he goes up three at a time there are two over; and when he goes up four at a time there are three over. Find the number of stairs, which is somewhere between 40 and 50.

3. Multiply ·208 by ·058, and ·02$\dot{7}$ by ·458$\dot{3}$; also find the difference, in a decimal form, between the two products. (C.P.)

4. A cistern partly full, into which a steady stream of water is flowing, has a number of equal holes in the bottom, which can be opened or closed at will. If 10 were opened, the cistern would be emptied in 20 min.; if 8 were opened, it would be emptied in 35 min. Twelve are opened: how soon will it be emptied? (C.S.L.)

5. In mental arithmetic, what rules would you give (1) for multiplying by ·75; (2) for finding the value of any number of things at £1 3*s.* 4*d.* each; (3) for finding simple interest on a given principal for a given time at 2½ per cent. per annum; (4) for finding the cost for a year at 7*d.* per day? (K.S.)

6. Extract the square root of 893830609, and raise 1·05 to the fourth power. (O.S.L.)

7. A casket contains diamonds and pearls; the diamonds are worth as many times the pearls as the pearls are worth the casket, and the values of the diamonds and the casket are £3411 9*s.* and 1 guin. respectively. Find the value of the pearls. (K.S.)

8. Of the boys in a school one-third are over 15 years of age, one-third between 10 and 15. A legacy of £100 can be exactly divided amongst them by giving 10*s.* to each boy over 15, 6*s.* 8*d.* to each between 10 and 15, and 3*s.* 4*d.* to each of the rest. How many boys are there in the school? (C.S.L.)

9. The price of gold is £3 17*s.* 10½*d.* per ounce. What ought 539 sovereigns to weigh, supposing $\frac{11}{12}$ of each to be pure gold, and that the value of a sovereign is equal to the gold it contains? (C.S.)

10. Between two railway stations the first, second, and third class fares were fixed at first in the ratio of 8 : 6 : 3, but afterwards the first-class fare was reduced by 2*d.* in the shilling, and the second by 1*d.* in the shilling. In a year the numbers of first, second, and third class passengers were respectively as 9 : 12 : 26, and the money taken at the booking-offices was £1088. How much was paid by first-class passengers? (O.S.L.)

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11. Obtain the weight in tons of two thousand one hundred million post-cards at 6$\frac{2}{3}$ lbs. a thousand. What will the cost of them be at the rate of 40 cents per thousand, there being 100 cents in a dollar, and a dollar being worth four shillings and twopence halfpenny? (C.P.)

12. Simplify $\dfrac{5\cdot75 - \frac{3}{4} \times 15\frac{3}{4} + 2\frac{2}{35} \div 1\cdot44}{\frac{3}{4} \text{ of } 7\frac{1}{7} - 5\cdot6 \div 3\frac{4}{15}}$. (K.S.S.)

13. One side of a rectangular field is ·05$\dot{4}$ of a mile, and the adjacent

side is $\dot{1}\dot{3}$ of a furlong. Find the length of each side in yards and feet, and express the area of the field as a fraction of an acre. (R.M.A.)

14. If either 10 men or 17 boys could finish a piece of work in 27 days, working 8 hrs. a day, how long would it take 8 men and 8 boys, working together for 10 hrs. a day, to finish the same work? (C.H.L.)

15. Show the simplest methods by which the following exercises may be worked by mental arithmetic:—

(*a*) 5 lbs. at $3\frac{3}{4}d.$ per ounce.
(*b*) 503×21.
(*c*) 864 yds. at $\frac{3}{4}d.$ a yard.
(*d*) 243 slates at $2\frac{1}{4}d.$ each.
(*e*) 1*s.* $9d. \times 19$.
(*f*) $\frac{16}{48} + \frac{1}{21}$.
(*g*) 1440 eggs at 5 for $6d.$
(*h*) 100 books at 6*s.* $3d.$ for 60. (P.T.)

16. Find the number whose square is equal to the difference between the squares of 6467 and 4683. (C.S.)

17. A watch which is set correctly at noon indicates 10 min. after 9 the same evening when the true time is 10. What is the true time when the watch indicates 11 on the same evening? (C.M.)

18. A bag contains a certain number of half-crowns, twice as many florins, and three times as many shillings, and the whole sum in the bag is 19 guins. Find how many of each coin it contains. (P.T.)

19. When the $2\frac{1}{2}$ per cents. are at $92\frac{1}{2}$, and $2\frac{3}{4}$ per cents. at $97\frac{1}{2}$, a person has a sum to invest which will produce £100 more of the former stock than of the latter. Find the difference in the income he can obtain by investing in either of the stocks. (Neglect brokerage.) (C.W.)

20. A grocer makes a mixture of chicory and coffee in the proportion of 1 to 6, and sold it at 1*s.* $6d.$ per pound, thereby making a profit of 20 per cent. If he had paid $1d.$ per pound more for the chicory, and $2d.$ per pound less for the coffee, then mixed them in the proportion of 1 to 5, and sold the mixture at 1*s.* $8d.$ per pound, his profit would have been 50 per cent. Find the price per pound paid for the chicory and the coffee. (O.S.L.)

21. Make out a bill for the following:—

1 cwt. of indigo at 14*s.* $6d.$ per pound.
1 ton of cloves at 1*s.* $2d.$ per pound.
5 cwt. 3 qrs. 18 lbs. spelter at $4\frac{1}{2}d.$ per pound.
7 cwt. 1 qr. 14 lbs. block tin at £64 per ton.

Subtract 10 per cent. discount for cash. (S.K.)

22. What numbers of 4 digits can have 119 as their G.C.M. and 13923 as their L.C.M.? (C.S.)

23. Which is greater, and by how much—

$$\frac{\cdot 2 \times \cdot 02}{1\frac{1}{2} \times (\frac{1}{3} + \frac{1}{4})} \text{ of 7 gals., or } \frac{1}{4 + \dfrac{1}{4 + \dfrac{1}{1 + \frac{1}{2}}}} \text{ qt.?}$$ (P.T.)

24. A cubic yard of earth weighs $3\frac{1}{2}$ tons, and a cubic foot of water weighs 1000 oz. The earth is a sphere of 3956 miles radius. Calculate approximately the radius of a sphere of water of weight equal to that of the earth. (S.A.)

25. A body appears to weigh $5\frac{1}{8}$ lbs. in one scale of a false balance, and $5\frac{1}{9}$ lbs. in the other scale. What is the true weight, and what could a fraudulent tradesman who used such a balance gain per cent. in selling his goods? (P.T.)

26. A man buys one kind of tea at 8 guins. per cwt., and another kind at £14 per cwt. He mixes them in proportion of 5 lbs. of the former kind to 3 of the latter. At what price per cwt. must he sell the mixture so as to gain 20 per cent. on his outlay? (C.W.)

27. The dimensions of a rectangular box are 2 : 3 : 4; and the difference between the cost of covering it with sheet lead at 8*d.* and $8\frac{1}{2}d.$ per square foot is 4*s.* $10\frac{1}{2}d.$ What are the dimensions of the box? (C.S.)

28. In a journey of 9 hrs. a train travels $45\frac{1}{2}$ mi. an hour, and stops 30 min. on the road. How long will a train take which travels $39\dot{6}$ mi. an hour, and makes stoppages amounting to $\frac{3}{4}$ of an hour? (C.S.)

29. A person invested money in 3 per cent. stock at $95\frac{7}{8}$, and derived a net income, after paying income-tax at 5*d.* in the £, of £146 17*s.* 6*d.*; he afterwards sold the stock at $102\frac{1}{8}$, and invested the proceeds in $2\frac{1}{2}$ per cent. stock; and, the income-tax having meanwhile been raised to 6*d.* in the £, found that his net income was 12*s.* 6*d.* less than before. How much money did he invest; and at what price did he buy the $2\frac{1}{2}$ per cent. stock? N.B.—One-eighth per cent. brokerage is to be allowed for in each sale or purchase. (C.S.)

30. The volumes of two substances contained in a certain mixture are as 4 : 5, and the weights of equal volumes of these substances are as 5 : 6. The weight of the mixture is 15 oz. Find the weight of each substance contained in it. (C.M.)

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31. Complete division given below by filling in numbers omitted in first and second lines:—

$$\begin{array}{r|l} 5) & \\ \hline 9) & \text{remainder 2.} \\ \hline 1260 & \text{remainder 6.} \end{array}$$ (P.E.A.)

32. $$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{6}} - \frac{\frac{1}{5}-\frac{1}{6}}{\frac{1}{5}+\frac{1}{6}} \div \frac{60}{61}.$$ (P.T.)

$$\frac{\frac{1}{2}+\frac{1}{9}}{\frac{1}{5}-\frac{1}{6}} + \frac{\frac{1}{5}-\frac{1}{6}}{\frac{1}{5}+\frac{1}{\square}}$$

33. If a rectangle 43·86 yds. long and 49·02 yds. wide is divided into 129200 equal squares, how long (in inches and decimal of an inch) will the side of each square be? (C.P.)

34. If 6 men do $\frac{1}{7}$ of a piece of work in 14 days, how many extra men must be engaged in order that the work may be finished in 9 days more? (O.S.L.)

35. Show how to tell in each of the following cases, *without working the sum*, whether the result is right or wrong. Give sufficient explanation:—

(*a*) Of the fractions, $\frac{1}{2}$, $\frac{18176}{36351}$, $\frac{27351}{51750}$, the third is greatest.

(*b*) The cost of 1291 hats at 13*s.* $4\frac{3}{4}d.$ each is £860 16*s.* $9\frac{1}{2}d.$

(*c*) The simple interest on £1102 for $2\frac{1}{2}$ yrs. at $4\frac{1}{2}$ per cent. is £87 19*s.* $6\frac{11}{50}d.$

(*d*) $\frac{1}{17} = \cdot 058352.$

(*e*) 764321360 is divisible by 8. (P.T.)

36. Find the discount on a bill for £1507 10*s.* due in 57 days at 3 per cent. per annum. (I.I.)

37. Coals at 32*s.* a ton are mixed with coke at 24*s.* a ton in the proportion of 5 tons of coal to 3 tons of coke. Find the money saved by using 11 tons of the mixture instead of 12 tons of coal. (S.K.)

38. A four-sided field has one square corner, and the two sides that form this corner are 36 yds. and 15 yds. long respectively; each of the other two sides is $32\frac{1}{2}$ yds. long. Find the area of the field. (K.S.)

39. A person invests half of £1500 in 3 per cent. consols at 90, and the other half in Chilian 6 per cent. bonds at 75. Find his average rate of interest; and if he sells consols at par, and the bonds at $37\frac{1}{2}$, find the reduction in his capital. (C.S.)

40. Three trains, 200, 528, and 130 yds. long, are travelling on parallel lines at rates of 15, 24, and 30 mi. per hour respectively, the first two in the same direction, the third meeting them; the second reaches the rear of the first train at the same moment as the third reaches its front. Find the interval in time between the moments when each of the two trains clears the third. (C.M.)

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41. The wages of one man are 3*s.* 9*d.* a day, and of another 2*s.* 3*d.*; a piece of work costs £3 15*s.* if the first man does it, and £3 7*s.* 6*d.* if the other does it. What would it cost if they did it together? (O.S.L.)

42. On a rock are three revolving lights; one revolves in 2 min. 12 sec., another in 3 min. 7 sec., the third in 3 min. 24 sec. I saw them all bright at midnight. When will they next be so? (K.S.)

43. Find correct to 4 decimal places the value of—

$$1 - \frac{1}{1.2.3} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.4.5.6.7}. \quad \text{(C.P.)}$$

44. A room 24 ft. long, 20 ft. wide, and 14 ft. high is ventilated by a rectangular opening 16 in. by 10 in. At what rate per minute must the air pass through the opening that one-fifth of the air in the room may be changed every hour? (C.W.)

45. Give rules for rapid mental solution in the following cases: (1) Cost for a year at so much per day; (2) any number of things at £83 6*s.* 8*d.* each; (3) simple interest at $6\frac{1}{4}$ per cent. per annum. (C.M.)

46. A shilling consists of 3 dwts. $8\frac{1}{2}$ grns. of pure silver, together with $7\frac{1}{2}$ per cent. of its whole weight of alloy. Find the weight of the shilling, and the proportion of alloy to silver. (C.S.)

47. In a race of half a mile, A gives B 40 yds. and C 75 yds. start. If their respective speeds be uniform, and as 23 : 22 : 21, who will win, and by how much? (C.PR.)

48. A train consists of engine, tender, and eight carriages. The weight of the engine and tender equals $\frac{8}{25}$ of the total weight of the train, and is 72 tons less than the weight of the carriages. Find the weight of a carriage. (P.T.)

49. A four-sided field ABCD is right angled at A. AB = 32 yds., AD = 24 yds., and BC = CD = 25 yds. Find the area of ABCD. (K.S.)

50. A man invests a certain sum in the ordinary stock of a railway at 80, which pays a dividend of $3\frac{1}{4}$ per cent. at the end of a year. He then sells out at 84, and invests the proceeds in a 5 per cent. stock at 125, and thus in the second year improves his income by £6 12*s.* How much did he originally invest, assuming that the prices of the stocks given above include brokerage, etc.? (C.W.)

51. Divide the difference of the squares of 9604 and 1404 by the sum of the squares of 98 and 102. (C.PR.)

52. Sixty-four yards of cloth are sold, part at 5*s.* a yard, and the remainder at 3*s.* 8*d.* a yard, the whole bringing in £14 12*s.* How many yards were sold at each rate? (S.A.)

53. Simplify the following fraction:—

$$\frac{(\frac{5}{6}+\frac{1}{4})\text{ of }\frac{3}{4}-\frac{2}{3}}{\frac{5}{6}+\frac{1}{3}\text{ of }(\frac{3}{4}-\frac{2}{3})} \div \frac{\frac{5}{6}\text{ of }\frac{1}{5}-\frac{3}{4}\text{ of }\frac{2}{3}}{(\frac{1}{3}-\frac{1}{4})\text{ of }\frac{2}{3}}. \quad \text{(C.S.L.)}$$

54. Employ an abbreviated method to divide 33·82289 by 66·4684 to 6 places of decimals (so that the error is less than 0·0000005). (R.M.A.)

55. (*a*) Define fraction, complex fraction, recurring decimal, giving an example of each.

(*b*) Define ratio. If the first of three numbers be to the second in the ratio of 2 to 5, and the second to the third in the ratio of $7\frac{1}{2}$ to 11, what is the ratio of the first to the third?

(*c*) Define present worth, discount, and explain why £870 divided by 1·05 will give you the present worth of £870 due twelve months hence at 5 per cent. (C.W.S.)

56. If a reduction of 30 per cent. were made in the price of eggs, it would enable the buyer to obtain 54 more for a guinea. What is the present price? (C.W.)

57. A sum of money was left to A, B, and C in the proportions of 7, 9, and 5 respectively. A's share, invested at 5 per cent., brought in less than B's invested at 4 per cent. by $12s.$ a year. Find the total sum left, and the share each received. (C.S.)

58. A train, going 30 mi. an hour, passes a man walking 3 mi. an hour in the same direction in 10 sec.: what is the length of the train? If another train, 88 yds. long, going in the opposite direction, meets the man and goes by him in 8 sec., at what rate is this train going? (C.S.)

59. Find the length of the diagonal of a square which contains 24336 sq. yds. (C.S.)

60. A man invests two equal sums of money; the one bears interest at the rate of $2\frac{1}{2}$ per cent. per annum, the other at the rate of 3 per cent. per annum. At the end of $5\frac{1}{2}$ yrs. he has received £18 7*s.* $1\frac{1}{2}d.$ more from the latter investment than from the former. Find the sums of money invested. (S.K.)

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61. How many sums of £27 2*s.* 9*d.* may be taken out of £379 4*s.* 9*d.*? If any sum be left, how many children may have sixpence each from it? (C.P.)

62. A company making 6 dols. a ton on 100 tons per working day—a dollar being worth 4*s.* $1\frac{1}{2}d.$—earns £36000 a year. How many days' holiday are there in an ordinary year? (S.A.)

63. A courtyard is 97·29 ft. long and 47·61 ft. broad, and it is to be paved with square tiles of uniform size. What is the size of the largest tile that can be used, and how many tiles will be required? (C.M.)

64. Simplify $\dfrac{1\frac{1}{2}+\frac{1}{3}-\frac{1}{8}\times\frac{1}{4}+\frac{1}{16}}{\frac{1}{16}+\frac{1}{4}\div\frac{1}{6}-\frac{1}{3}+\frac{1}{12}} \times 4\frac{53}{43}$. (O.R.)

65. Perform the following calculations in writing, but so as to show the best way for a mental reckoner:—

(*a*) £239 18*s*. $4\frac{1}{2}d.$ − £229 18*s*. $5\frac{3}{4}d.$

(*b*) The value of 9600 articles at $7\frac{3}{4}d.$ each.

(*c*) $325 \times 235 - 325 \times 233 + 27\frac{1}{2} \times 94 - 28 \times 94.$

(*d*) $16\frac{1}{10} - 9\frac{1}{9}.$ (P.T.)

66. A, B, and C send 55, 70, and 75 sheep respectively to a field of grass, the rent of which is £30. What percentage of the rent should each pay? (P.T.)

67. A and B grind corn. A grinds 10 per cent. more corn than B in an hour, but works for 10 per cent. less time in a day. Will A grind more or less corn than B in a day, and by how much per cent.? (K.S.)

68. A privateer, running at the rate of 10 *mj.* an hour, discovers a ship 18 mi. off, making way at the rate of 8 mi. an hour. How many miles can the ship run before she will be overtaken?

69. Find the size of each face of a rectangular parallelopipedal box whose adjacent edges are $6{\cdot}8571\dot{4}2$, $2{\cdot}5714\dot{2}8$, and $3{\cdot}4285\dot{7}1$ in. (C.S.)

70. The capital of a trading company consists of 4000 A shares of £80 each and 2000 B shares of £25 each. In dividing the profits 5 per cent. of the amount of each share is first paid, and then the remainder, if any, is divided equally amongst the shareholders. The profits of the undertaking in one year were £34853 12*s*. $6\frac{1}{2}d.$ How much would be paid to the holder of an A share, and how much to the holder of a B share? (S.K.)

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71. It has been stated that the entire population of the world could stand on the Isle of Wight. Assuming the population to be 1576 millions, and the area of the island to be 145 sq. mi., calculate approximately how many square feet and inches each person would have to stand upon. (C.S.L.)

72. Reduce to the simplest form as a mixed number—

$$\frac{2\frac{1}{4} - \frac{2}{3} \text{ of } 1\frac{5}{6}}{\frac{1}{3} \text{ of } 3\frac{1}{3} + \frac{7}{8}} \div \frac{2\frac{1}{2} - \frac{1}{3} \div \frac{4}{13}}{1\frac{1}{7} \text{ of } 8\frac{1}{2}} + \frac{1}{1 + \frac{5}{16}}. \quad \text{(S.K.)}$$

73. Explain what are meant by 3, 4, 34, $\cdot 34$, $\cdot 3\dot{4}$, $\cdot\dot{3}\dot{4}$, 3^4, $\frac{3}{4}$, $3 \div 4$, 3/4, 3 : 4. Prove that $7^4 + 4 \times 6^4 = 9^4 + 4 \times 4^4$. (S.A.)

74. If a metre be taken as 40 in., and a cubic centimetre of water weighs one gramme, what, stated metrically, is the weight of a cubic inch of water? (P.T.)

75. (*a*) Prove that if the numerator and denominator of a fraction be divided by the same number, the value of the fraction is not altered.

(*b*) Prove that $\cdot 3\dot{2} = \frac{29}{90}$.

(*c*) How would you work the following sum before a class of children who have already had some lessons on stocks?—Which is the better investment: a $2\frac{1}{2}$ per cent. stock at $97\frac{1}{2}$, or a 4 per cent. stock at 130?

(*d*) Define ratio, proportion, average, discount, with examples. (C.W.)

76. A Paris shopkeeper buys cloth in Yorkshire which costs him, when delivered in Paris, 5*s*. per yard (of length). At what price in French money per metre must he sell the cloth in order to gain 8 per cent.? (Take a metre = 3·28 ft., and a sovereign = 25 fr.) (C.W.)

77. A tradesman's rate of profit is 35 per cent. If he now lowers his selling prices 2*d*. in the shilling, in what proportion must the quantity of goods sold increase if his whole profit becomes now $1\frac{2}{7}$ of what it was before? (C.H.L.)

78. A man buys goods, and finds that the cost of carriage is 4 per cent. on the cost of the goods. He is compelled to sell at a loss of 5 per cent. on his total outlay; if, however, he had received £3 5*s.* more than he did, he would have gained $2\frac{1}{2}$ per cent. What was the original cost of the goods? (C.S., R.M.A.)

79. Two boats start to row a race at 3 o'clock. The race is over at $6\frac{2}{5}$ min. past 3, the losing boat being 40 yds. behind at the finish. At 4 min. past 3 this boat was 700 yds. from the winning-post. Find the speed of each boat in miles per hour. (C.PR.)

80. One sum of money is to another sum as 37 is to 35. The former is invested in $2\frac{3}{4}$ per cent. stock at $101\frac{1}{8}$; the latter in 4 per cent. stock at $109\frac{1}{4}$. Determine the average rate of interest obtained. (Brokerage, $\frac{1}{8}$ per cent.) (C.P.)

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81. What number, when divided by 593, will give 626 as quotient and 367 as remainder? (C.H.L.)

82. Find the cost of 38 ac. 1 ro. 24 po. at £2 0*s.* 7$\frac{1}{2}$*d.* an acre. (K.S.)

83. Find the value of $\dfrac{(2\frac{3}{4} \text{ of } 3\frac{7}{9}) + 2\frac{1}{3}\frac{4}{5}}{1\frac{1}{2} - 2\frac{1}{10} - \frac{7}{90}} \div \dfrac{6\frac{3}{8} - (8\frac{1}{3} \div 1\frac{1}{9})}{5\frac{3}{4} - (\frac{8}{13} \text{ of } 8\frac{3}{4})}$. (O. and C.H.C.)

84. If a pond an acre in area is covered with ice 6 ft. thick, and a cubic foot of ice weighs 928 oz., find the whole weight of the ice in tons. If an ice merchant were to buy it at 2*s.* the cubic yard, and retail it at 1*d.* the cubic foot, what profit would he make? (C.S.)

85. Divide the square of 51 into parts proportional to the quantities 1·6, ·16, and 1·13; also express 13*s.* 11$\frac{1}{4}$*d.* as the decimal of £3 6*s.* 8*d.* (C.P.)

86. From 235 tons of ore from a mine, A, are attainable 12 tons 10 cwt. of metal, and from 164$\frac{1}{2}$ tons from a mine, B, 8 tons 9 cwt. of metal. How much higher is the percentage of metal in one case than in the other? (P.T.)

87. A person walked from Cambridge to Newmarket, a distance of 14 mi., and back in 7 hrs. 30 min. His rate going was to his rate returning in the proportion of 8 : 7. Find the rates. (S.L.C.)

88. Find the sum which must be invested in the 3 per cents. at 90 at simple interest payable half-yearly, to amount in $23\frac{1}{2}$ yrs. to £2317 sterling, the price of the funds remaining unchanged. If the funds rose to 96, in how many years sooner could the required amount be realized? (C.S.)

89. A wholesale dealer sells to a retail dealer at 10 per cent. profit, and the retailer to the consumer at 50 per cent. profit. What proportion of the price paid by the consumer is profit? (P.T., C.W.)

90. A person dies worth £31000, leaving his property to his son, his widow, and a cousin, so that the legacy, after payment of duty, to the son shall be twice that to his widow, and 20 times that to the cousin. How much does he leave to each, the duty payable by a son being 1 per cent. on the gross legacy, that by a cousin 10 per cent., while the legacy to the widow is duty free? (L.M.)

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91. How many quarters of barley, worth 2*s.* 3*d.* a bushel, must I give in exchange for 174 qrs. of wheat, worth 3*s.* 4$\frac{1}{2}$*d.* a bushel? (C.P.)

92. What is the L.C.M. of two numbers? Find the least sum of money that can be exactly distributed either in half-crowns or half-guineas. (P.E.A.)

93. Simplify and express as a decimal—

$$\frac{\left(1+\frac{1}{10+\frac{1}{10}}\right)\times\left(1+\frac{1}{10+\frac{1}{10}}\right)-\left(1-\frac{1}{10-\frac{1}{10}}\right)\left(1-\frac{1}{10-\frac{1}{10}}\right)}{\left(1+\frac{1}{10+\frac{1}{10}}\right)+\left(1-\frac{1}{10-\frac{1}{10}}\right)}$$ (P.T.)

94. A section of a stream is 10 ft. wide and 10 in. deep; the mean flow of the water through the section is 3 mi. an hour. Taking 25 gals. to equal 4 cub. ft., find how many gallons of water flow through the section in 24 hrs. (S.K.)

95. Show that $(3009)^2+(4012)^2=(5015)^2$. (P.P.)

96. When French $3\frac{1}{2}$ per cents. are sold at 102 fr. 55 c., and English $2\frac{3}{4}$ per cents. at $109\frac{7}{8}$, compare the rates of interest obtained by investments in France and England. (1 fr. = 100 c.) (C.W.)

97. Of two square fields one contains 10 ac., while the other is broader by 1 per cent. Find the difference in area, expressed in yards, feet, and decimals of a foot. (C.W.)

98. A man on the point of bankruptcy sees that he will be able to pay his creditors 9*s.* 6*d.* in the £. What are his liabilities if an immediate gift of £16000 stock in the $3\frac{1}{2}$ per cents. at $110\frac{1}{4}$ would just save him from bankruptcy? (K.S.)

99. How many pages of a magazine will a manuscript occupy which has 123 pages, each of 28 lines, with 7 words per line on an average, while a page of the magazine has 46 lines, with 10 words per line on an average? (S.L.C.)

100. The angle of inclination to the perpendicular of a bicycle moving round a circular track varies (approximately) as the square of the velocity, and inversely as the circumference of the track. If the inclination is 6° when the velocity is 20 mi. an hour, and the circumference of the track $\frac{1}{4}$ mi., what is it when the velocity is 2 mi. an hour less, and the circumference 440 ft. shorter? (C.S.)

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101. Multiply seven hundred and eighty-three millions twenty-four thousand and nine by eight hundred and seven. (C.J.)

102. Simplify the expression—

$$\frac{\frac{5}{14}-\frac{3}{7}\text{ of }\frac{1}{2}}{\frac{5}{16}+\frac{7}{12}\text{ of }3\frac{1}{4}-\left(\frac{7}{8}\text{ of }\frac{37}{21}-\frac{1}{3}\right)}\div\frac{\frac{1}{3}\text{ of }\frac{1}{2}\text{ of }\frac{3}{5}\text{ of }5}{9\frac{1}{8}-1\frac{2}{3}}.$$ (O.R.)

103. If 2·64 tons of water flow out of a pipe in 3 hrs. 47 min. 42 sec., in what time does 1 ton flow out, assuming that the water flows out at a uniform rate? (C.P.)

104. If a kilogramme be equal to 2·205 lbs., and a metre be 39·37 in., find the number of ounces in the mass of a cubic foot of water, it being given that the mass of a cubic centimetre of water is one gramme. (C.PR.)

105. What is the height of a steeple whose shadow was 112 ft. at the same time that a pole 6 ft. 4 in. long cast a shadow of 5 ft. 4 in.? (P.T.)

106. When malt cost 7*s.* 6*d.* a bushel, and 3 bush. of it were used to the barrel, beer could be sold at 1*s.* 3*d.* per gallon. What should be the

price of beer per gallon when 4 bush. of malt at 6*s.* per bushel are used to the barrel? (C.S.)

107. A contractor, having engaged to make a railway $58\frac{1}{2}$ mi. long in 40 wks., employed 2160 men upon the work. At the end of 13 wks. $19\frac{1}{2}$ mi. were finished. How many men had he then to pay off so that the work might not be completed before the stipulated time? (K.S.S.)

108. If the average cost of 20 oxen was £15 19*s.* 6*d.*, and 8 of them cost £15 each, what was the average cost of the remainder? (P.T.)

109. If the earth be a globe of 7920 mi. diameter, and the ocean having a mean depth of 4 mi. cover $\frac{5}{8}$ of the surface of the globe, what is the approximate ratio of the volume of the ocean to that of the earth? (C.S.)

110. A measurement of 40 ft. is known to be within a millimetre of the truth; it is required to find within how much per cent. it is right. Given that a metre is 39·3708 in., find first by a very rough approximation two limits within which the answer must lie, and then calculate it as accurately as the data permit. (S.A.)

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111. Two dozen of port and 16 bots. of sherry were bought for two equal sums of money. If the quantities of the wines had been reversed, the sherry would have cost £3 more than the port. What was the price of each wine per bottle? (P.T.)

112. If the gold in circulation amounts to 120 millions of sovereigns, and of these 25 in every 100 have lost $\frac{1}{30}$ of their weight, while 10 more in every 100 have lost $\frac{1}{50}$ of their weight, what would be the loss to the Mint if it recalled all light gold and issued full-weight sovereigns instead? (C.H.L.)

113. A rectangular sheet of metal, initially 7·9 ft. long and 5·3 ft. wide, became when expanded 7·93 ft. long and 5·32 ft. wide. Express the difference between the initial and final areas of the sheet as a decimal of the initial area, correct to 6 places of decimals. (C.P.)

114. Express as a circulator the sum of $\cdot 43\dot{1}\dot{8}$ and $\cdot 4\dot{5}5\dot{4}$. (C.J.)

115. The wine in a pipe when full is worth £19 9*s.* 9*d.* How much has leaked away if what is left is worth £9 16*s.* $7\frac{85}{254}d.$? (C.S.)

116. If 11 coins, each $\frac{1}{20}$ in. thick, weigh 2 oz. 17 dwt. 7 grns., find the weight of 18 similar coins, each $\frac{1}{25}$ in. thick. (O.J.)

117. What would be the simple interest of 649 fr. 75 c. for 5 yrs. 4 mths. at 3 c. interest per annum for each franc lent? (P.T.)

118. A hollow closed cubical box is made of iron, and is 2 in. thick, the external edge being 4 ft. long. It being given that iron weighs 7·112 times as much as water, and that a cubic foot of water weighs 1000 oz., find in tons the weight of the box. (L.M.)

119. If 50 qrs. of wheat are sold at 77*s.* 8*d.* per quarter, and 100 qrs. at 78*s.* 3*d.* per quarter, what is the *average* price per *bushel*? (P.T.)

120. A tradesman's stock-in-trade is valued on Jan. 1, 1898, at £8000; he has also £350 in cash, and owes £1870. During the year his personal expenses, £300, are paid out of the proceeds of his business, and on Jan. 1, 1899, his stock is valued at £7950, he has £570 in cash, and owes £1510. What is the whole profit on the year's transactions after deducting 5 per cent. interest on the capital with which he began the year? (C.S.)

121. If 2864 bricks are used per yard to build a tunnel which is 5926 yds. long, and 2345 bricks per yard to build another tunnel which is 6014 yds. long, state *in words* how many bricks are used for each tunnel, and how many more are used for the one tunnel than for the other. (C.W.)

122. A, B, C start on a journey, having £9, £7 10*s.*, £5 10*s.* respectively with them. At the end it is found that A has left 10*s.*, B £5, C £1 15*s.* For expenses common to B and C only, B has paid £1 10*s.*; for private expenses A has paid £1, B 10*s.*, C £1; all other payments have been made to defray expenses in common. How should they divide the money which remains over? (C.S.)

123. Reduce to its simplest form—

$$\frac{100\frac{1}{13} - 91\frac{89}{91}}{27\frac{3}{11} + 2\frac{117}{143}} - \frac{3\frac{3}{7} + 6\frac{3}{5} \text{ of } 1\frac{11}{39}}{1\frac{7}{11} \text{ of } 25\frac{3}{5} + 48\frac{1}{6}}. \quad \text{(S.K.)}$$

124. A square lawn is bordered by a path 4 ft. 6 in. wide, the path and lawn together occupying $\frac{1}{10}$ ac. Find the cost of paving the path with bricks at 7*s.* 6*d.* per square yard. (C.PR.)

125. If a farmer pay a rent of £100 19*s.* $10\frac{1}{2}d.$ for a farm of 53 ac. 2 ro. 20 po., and his landlord remit one quarter of the rent, what does the farmer then pay per acre? (K.S.)

126. If the rate at which a vessel moves is proportional both to the total area of sails set and to the velocity of the wind, and if it travels 8 knots an hour with a wind of 15 mi. an hour, and 400 sq. ft. of sails set, how many knots an hour will it travel if the wind freshens to 20 mi. an hour, and 50 sq. ft. of canvas be taken in? (P.E.A.)

127. If gold weighs 19·3 times as much as water, and copper 8·89 times as much, find how many times heavier than water is a coin which contains 11 parts of gold and 1 of copper. (C.M.)

128. The average daily number of persons passing a certain point on Sunday, Monday, Friday, and Saturday is 1765. The average daily number passing on Tuesday, Wednesday, and Thursday is 1541. What is the daily average for the whole week? (C.S.)

129. A bicycle manufacturer prints in his lists a price which would give him a profit of 60 per cent. on the cost of manufacture, and then allows for "cash down" a discount of 22 per cent. off the list price. What profit per cent. does he actually make? (C.J.)

130. Extract the square root of 0·08450649, and the cube root of 6434·856. (S.K.)

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131. The income of a family is £300 a year. They remain at home all the year, except that they take a holiday at the seaside. Rent, rates, and taxes are £55 9*s.*; clothing costs £44 11*s.* a year, and travelling to and from the seaside costs £6. Other expenses are: at home, £2 10*s.* a week; from home, £6 10*s.* a week. How long is their holiday if £40 is saved in the year? A year = 52 wks. (C.W.)

132. What will be the cost (to the nearest penny) of 15 tons 15 cwt. 3 qrs. 16 lbs. of goods at £14 3*s.* 6*d.* per cwt.? (C.P.)

133. Simplify the expression—

$$16\left(\frac{1}{5} - \frac{1}{5^3 \times 3} + \frac{1}{5^5 \times 5} - \frac{1}{5^7 \times 7}\right) - 3\frac{4}{39}. \quad \text{(C.S.)}$$

134. Write out the numbers from 1 to 50 expressed as they would be in the scale of 6. (C.M.)

135. A man swam across a lake 3 mi. wide in $2\frac{3}{4}$ hrs. If he made a stroke with his arms every other second, find how many inches he travelled between each stroke. (O.S.L.)

136. If a man walks 252 mi. at the rate of $3\frac{1}{2}$ mi. per hour for 8 hrs. a day, find how many hours a day another man must walk, at the rate of $4\frac{1}{2}$ mi. an hour, in order to walk 243 mi. in the same time. (C.M.)

137. 355·27 oz. Troy is the weight of 35 cub. in. of gold, of which the S.G. is 19·258. Find the weight of 49 cub. in. of silver, of which the S.G. is 10·474 (to 4 places of decimals). (P.T.)

138. A train travels at the rate of 40 mi. an hour for 10 min.; and then at 45 mi. an hour for 15 min. What is the average rate per hour for the whole time? (P.T.)

139. A silk mercer buys 2000 yds. of silk at 6*s.* $7\frac{1}{2}$*d.* per yard; he sells two-thirds of it at 7*s.* 5*d.* per yard, and the remainder at 6*s.* $10\frac{1}{2}$*d.* per yard. How much per cent. does he gain? (I.I.)

140. Determine (in yards) the sides of a square, the contents of which are 18352656 sq. in. Find the length of the edge of a block in the form of a cube, which contains 3 cub. yds. 20 cub. ft. 1088 cub. in. (C.P.)

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141. Two cog-wheels, one with 15 teeth, the other with 28 teeth, work together. If the former turns round 16 times in $7\frac{1}{2}$ sec., how many times will the latter turn round in 21 sec.? (C.P.)

142. Find the value of a mass of pure gold weighing 5 lbs. 10 oz. 12 dwt. 18 grs. at £4 5*s.* per ounce. (C.H.L.)

143. A contractor undertakes to do a piece of work in 38 days. By employing 60 men on it, he does $\frac{3}{7}$ of it in 22 days. How many additional men must he employ to finish it in time? (L.M.)

144. A rectangular vessel, 3·4 m. long and 2·6 m. wide, contained wine 7 dcm. deep. If as many casks as possible, each capable of holding 3 Hl., were filled from the vessel, how many casks were there? and what, to the nearest millimetre, was the depth of the wine left in the vessel? (A litre = a cubic decimetre, and a hectolitre is 100 litres.) (C.P.)

145. In a book on arithmetic, an example was printed thus: "If 11 ac. — ro. 8 po. be worth £2250, what will 44 ac. 3 ro. 1 po. cost?" (The number of roods in one being accidentally omitted.) The answer at the end of the book was £8718 15*s.*: required the missing number. (K.S.)

146. When the price of wheat was 45*s.* 2*d.* per quarter, and the yield 4 qrs. per acre, the produce of a certain piece of land was worth £153 11*s.* 4*d.* What must the yield be per acre if the produce of the same land is worth £157 5*s.* when the price of wheat is 30*s.* 10*d.* per quarter? (O.S.)

147. The external length, breadth, and height of a rectangular wooden closed box are 18 in., 10 in., and 6 in. respectively, and the thickness of the wood is $\frac{1}{4}$ in. When the box is empty it weighs 15 lbs., and when filled with sand 100 lbs. Compare the weight of equal bulk of wood and sand. (O.S.)

148. Of 100 men, 37 earn 3*s.* 10*d.* per day, 40 earn 3*s.* 6*d.* per day. What should be the daily pay of the remainder that the average weekly pay of the whole, at $5\frac{1}{2}$ days to the week, should be £1 2*s.*? (P.T.)

149. A builder sold a house for £945, thereby gaining 8 per cent. on his outlay. What did it cost him to build it? If the purchaser lets the house at £70 a year, find how much per cent. per annum he makes on the purchase money. (S.K.)

150. Find the length of the edge of a cubical block of stone containing 46 cub. yds. 513 cub. in., and the number of square inches in its entire surface. (R.M.A.)

EXERCISE LXXVI.

Miscellaneous Examples.

1. There is a certain whole number consisting of six digits, and having 7 in the units' place. If the digit to the extreme left be removed to the immediate right of the units' place, the new number is three times the original number. Find the number. (C.S.)

2. In the multiplication sum given below, the multiplier, multiplicand, and one row of figures in the working have been obliterated. (None of these contain a decimal quantity.) Complete the sum in all possible ways, explaining why more than one solution can be obtained:—

$$\begin{array}{r} \cdots\cdots \\ \cdot\cdot \\ \hline \cdots\cdots \\ 1661490 \\ \hline 19107135 \end{array}$$

(C.S.)

3. Calculate, to 7 places of decimals, the difference between $\sqrt{\frac{1}{2}}$ and $\frac{7}{10} + \frac{7}{985}$. (L.M.)

4. On one day as many hundreds of oranges could be bought for half a sovereign as dozens could be bought for a shilling on the previous day. A woman bought 75 on each day, and paid altogether for them 5*s.* 6*d.* How many oranges could be bought for sixpence on the first day? (C.P.)

5. A steel rod 1 ft. long and 1 in. square weighs $3\frac{1}{2}$ lbs., and will just support 50 tons. What is the greatest length of steel wire which, when hung up by one end, will just not break by its own weight? (C.PR.)

6. A person writes down all the numbers from 21 to 40, both included, and proceeds to form the product of every two of them. Without attempting to form the products, determine what percentage of the whole number of them will be divisible by 6. (C.S.)

7. Two casks, A and B, contain 8 and 3 gals. of wine, worth 18*s.* and 15*s.* per gallon respectively. One gallon is taken from A and put into B, and then one gallon is taken from B and put into A. Find the worth per gallon of the wine now in each cask. (C.M.)

8. (i.) Pepys says that out of £750000 coined by the Commonwealth, £100000 had disappeared from circulation in 12 yrs., and calculates that at the same rate, out of £10000000 coined by Charles I., £3888880 and odd pounds would have disappeared in 35 yrs. Assuming, as Pepys did, that the sum is one of compound proportion, find the exact sum that would have disappeared. (ii.) Point out the mistake in Pepys' method, and calculate by a more correct method what number of coins will have disappeared at the same rate in 36 yrs. (C.P.)

9. Two penny papers, A and B, are each issued 6 times a week, and have each a circulation of 200000 copies. A decimal coinage is introduced, which retains the pound and the florin, but divides the florin into 10 cents and the cent into 10 mils. A now adopts the retail price nearest in value to a penny, B the nearest but one. Find (1) the whole loss or gain in 52 wks. to the 200000 purchasers of A; (2) the same to the 200000 purchasers of B; (3) the total net loss or gain per cent. to a newsagent who purchases, as before, equal quantities of A and B at the rate of 12 papers for the retail price of 10, and sells all his papers and just as many of each as before. (C.M.)

10. A piece of copper weighing 3 lbs. 2 oz. is dropped gently into a vessel full of water. Find (correct to $\frac{1}{100}$ cub. in.) the volume of water displaced, assuming 1 cub. ft. of water weighs 1000 oz., and that copper is 8·915 times heavier than water. (P.E.A.)

11. One day a country postman went half his round at the rate of 3 mi. per hour, and the other half at the rate of 4 mi. per hour, and was 10 min. late. On the next day he went half his round at the rate of $3\frac{1}{3}$ mi. per hour, and the other half at the rate of $4\frac{1}{2}$ mi. per hour, and took 12 min. less than the allotted time. How far had he to go, and what time was he allowed? (C.P.)

12. Calculate to 4 places of decimals the value of $\sqrt{\frac{1+x}{1-x}}$, where $x = \cdot 2375$. (L.M.)

13. Find to 4 places of decimals the value of $2\sqrt{3} - \frac{1}{2}\sqrt{12} + \sqrt{27}$. (C.S.)

14. A contractor agrees to bore an artesian well at the rate of 1*s.* per foot depth for the first 40 ft., 1*s.* 2*d.* per foot for the next 40 ft., and so on, increasing his charge 2*d.* each 40 ft. He receives £20 altogether. Find the depth of the bore. How much would he have received if he had contracted to bore it at the uniform rate of 1*s.* 6*d.* per foot? What would have been the depth if either contract would have brought him in the same amount? (C.H.L.)

15. A man buys a bicycle. When he has ridden a certain distance, he reckons that it has cost him $2\frac{1}{2}d.$ a mile; he rides 252 miles further, and finds that it has cost 2*d.* a mile. How much further must he ride to reduce the cost to $1\frac{1}{2}d.$ per mile? (S.A.)

16. A gallon of spirit valued at 24*s.* weighs $\frac{1}{4}$ less than a gallon of water. Water is now added to the spirit till a gallon of the mixture weighs $\frac{1}{5}$ more than a gallon of the spirit. What is the value of this mixture per gallon? (C.S.)

17. Reduce the following fractions to the forms of decimals true to 3 places:—

$$\frac{3\frac{3}{11} \times 25{\cdot}15}{4\frac{6}{11} \times {\cdot}4} \text{ and } \frac{\sqrt{5} + \sqrt{10}}{{\cdot}067}. \quad \text{(S.K.)}$$

18. The area of one square is 5 per cent. greater than that of another. What two numbers, each less than 100, will most nearly represent the ratio of the sides of the two squares? (C.P.)

19. Two steamers travel between two ports by the same route, one in ten days, the other in five. The cost of coals per voyage varies as the square of the speed, and the general expenses as the duration of the

voyage. If on the faster voyage the expenditure for coals exceeds that on the slower by £1000, and the general expenses on the slower voyage exceed those on the faster by £250, find the cost of running the vessels per annum, the faster vessel making forty voyages, and the slower twenty. (C.S.L.)

20. Simplify $\frac{11}{10}$ of $\sqrt{\frac{(1\cdot345)^2-(\cdot905)^2}{1\cdot\dot{2}+\cdot\dot{2}\dot{3}}} \times \sqrt[3]{\cdot\dot{0}3\dot{7}}$. (I.I.)

21. Two men, A and B, begin at the same time to run opposite ways once round a circular course, starting from the same point. From the moment at which they first meet, A takes 63 sec. to finish the round, B 84 sec. But whereas A ran at a uniform rate the whole way, B mended his pace in the ratio of 4 : 3 from the moment at which they first met. Find the time each takes to run the whole course. (C.S.)

22. (*a*) Compute the value of $\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{2}-1}$ correct to 2 places of decimals.

(*b*) Compute the sum of the infinite series of terms—

$$1+\frac{1}{1}+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\frac{1}{1\cdot2\cdot3\cdot4}+\cdots$$

to 6 places of decimals. (L.M.)

23. Express $\sqrt[3]{44\cdot6}$ as a decimal of $\sqrt{167\cdot9619}$, obtaining the nearest value to two places of decimals. (C.S.)

24. Calculate the value of—

$$1-\frac{1}{1\times2}+\frac{1}{1\times2\times3}-\frac{1}{1\times2\times3\times4}+\frac{1}{1\times2\times3\times4\times5}-, \text{ etc.},$$

correctly to 9 places of decimals, the series being unending. (C.S.)

25. Assuming that the areas of circles are proportional to the squares of their radii, compare the areas traced out by the short and long hands of a clock in half an hour, the long hand being three times the length of the short hand. (C.M.)

26. The breadth of a room is half as much again as its height, and it costs 10 guins. to paint its walls at $2\frac{1}{2}d.$ per square foot. What are its dimensions? (S.A.)

27. The railway between two stations, A and B, ascends at a gradient of 1 in 275 from A to a place C, distant $16\frac{1}{4}$ mi. from A, and then descends at a gradient of 1 in 165 from C to B. If the station at B is 28 ft. lower than that at A, find the distance from A to B. (C.S.L.)

28. Express the atmospheric pressure of $14\frac{3}{4}$ lbs. per square inch, in grammes per square centimetre, taking 1 cm. $= \frac{2}{5}$ in., and 1 kg. $= 2\cdot2$ lbs. (C.S.)

29. An estate duty of 5 per cent. is charged on estates whose value is between £50000 and £75000, and a duty of $4\frac{1}{2}$ per cent. on estates whose value is between £25000 and £50000. It was found that after payment of estate duty on two estates, each of which was an exact number of pounds, and one of which was over £50000 and the other under £50000, equal sums were left. What was the value of each of the two estates on which the duty was paid? (C.H.L.)

30. Calculate the value of $\sqrt{\frac{(\cdot\dot{7}\dot{8})\times(\cdot00\dot{0}\dot{4})}{(\cdot0\dot{1}\dot{3})\times(\cdot\dot{0}\dot{1}\dot{5})}}$. (L.M.)

31. A room is $\frac{1}{3}$ as wide and $\frac{1}{2}$ as high as it is long, and $\frac{1}{12}$ of the wall-space is occupied by windows and fireplace. The rest of the walls and the ceiling are painted at a cost per square foot which is $\frac{1}{4}$ of the cost per square foot of carpeting the floor with carpet 2 ft. 3 in. wide, costing 3*s.* 4½*d.* per yard. The whole cost of carpeting and painting is £52 13*s.* Find the length of the room. (C.S.)

32. Find the value of—

$$\frac{10 \times \sqrt{2+\frac{3}{5}} \times \sqrt{1-\frac{3}{7}}}{13 \times \sqrt{1-\frac{3}{7}} \div \sqrt{2+\frac{3}{5}}} \times \frac{\sqrt{3+\frac{1}{4}}}{9\frac{3}{4}} \div 1\tfrac{1}{2}. \quad \text{(R.M.A.)}$$

33. Write down neatly, in a table, the powers of 2 up to the 30th power. By means of this table perform the following operations:—

(i.) 32768×16384. (iii.) $(1{\cdot}28)^4$.

(ii.) $26{\cdot}8435456 \div {\cdot}0524288$. (iv.) $\sqrt{1073{\cdot}741824} \div \sqrt[3]{134{\cdot}217728}$. (C.S.)

34. In a plantation of ash saplings of the same age, each sapling is allowed a space of 10 sq. yds. after five years' growth. It is then worth 1*s.* 6*d.*, and increases in value by 1*s.* 6*d.* each year. At the end of each year afterwards a certain number of saplings are cut down and sold, so that upon the average each tree has a further space of 2 sq. yds. for every additional year of growth. What is the largest amount per acre realized in any one year by the sale of the wood, after deducting one-fifth of the gross receipts for expenses? (C.S.)

35. What is the least exact number of pounds I must spend in buying some apples at 18 a shilling, and five times as many more apples at 35 a shilling, in order that I may be able to sell them all at 27 a shilling, in each case paying or being paid an exact number of shillings? (C.P.)

36. Calculate to 2 places of decimals the value of—

$$\cfrac{1}{\sqrt{7}+\cfrac{1}{\sqrt{7}+\cfrac{1}{\sqrt{7}+\cfrac{1}{\sqrt{7}}}}} \quad \text{(L.M.)}$$

37. A district, which is in shape a square of 10 mi. side, has been partly mapped to a scale of 6 in. to the mile, and partly to a scale of 4 in. to the mile. The area of these two portions together is 2500 sq. in. What area has been mapped to each scale? (C.S.)

38. Work out in a thoroughly neat form the value of—

$$\frac{\pi}{10} - \frac{1}{1\times 2\times 3}\times\left(\frac{\pi}{10}\right)^3 + \frac{1}{1\times 2\times 3\times 4\times 5}\times\left(\frac{\pi}{10}\right)^5 - \frac{1}{1\times 2\times 3\times 4\times 5\times 6\times 7}\times\left(\frac{\pi}{10}\right)^7 + \text{etc.}$$

correctly to 3 places of decimals, π being 3·14159265. (The working may be contracted as much as is thought fit.) (C.S.)

39. Prove that the values of $\cfrac{3}{2+\frac{3}{2}}$, $\cfrac{3}{2+\cfrac{3}{2+\frac{3}{2}}}$, etc., are $\frac{6}{7}$, $\frac{21}{30}$, etc.; and that these fractions are alternately less and greater than unity, and

that they continually approach that value. Find one of these fractions that differs from unity by less than ·00001. (C.S.)

40. The number of vibrations of air to produce a certain sound being 660 per second, and the number of vibrations of ether for a certain light being 4×10^{14} per second, calculate (roughly) the number of centuries this sound must continue in order to execute as many vibrations as the light in one second. (S.A.)

41. Simplify $\dfrac{\sqrt{\cdot 13 + \sqrt{\cdot 0625 + \sqrt[3]{\cdot 001331}}}}{\sqrt{\frac{8}{3} + \sqrt{12\frac{52}{81}} + \sqrt[3]{\frac{56 \cdot 8}{81}}}}$. (C.S.)

42. A square field has a shrub border 11 yds. wide running along all four sides, within the boundary of the field; this border contains 1 ac. Find the area of the field. (C.S.)

43. How much fresh water, to the nearest pint, must be added to 50 gals. of sulphuric acid that the liquid may weigh as much as an equal quantity of salt water? (The specific gravity of fresh water being 1, that of salt water 1·0263, and of sulphuric acid 1·85.) (C.S.)

44. If 13 men working 6 days, 8 women working 5 days, and 5 boys working 4 days, earn between them £13 15s., and the wages per day of a man, a woman, and a boy are as 5 : 3 : 2, what sum is each of them entitled to? (C.S.)

45. Find value correct to 2 places of decimals—

$$1 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \frac{1}{1 \times 2 \times 3 \times 4 \times 5} + \frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6}.$$ (P.T.)

46. A marble floats in mercury with ·269 of its bulk below the plane of immersion. Water is poured on to the mercury until the marble is fully covered, and the volume now below the plane of immersion in the mercury is ·211 of the whole bulk. Compare the densities of water and mercury. (The weight of a floating body is equal to the weight of the liquid which it displaces.) (C.S.)

47. A metre is 39·37079 in., and a second's pendulum 39·1393 in. Find how many miles there are in the least exact number of yards that is an exact number of metres and also an exact number of seconds' pendulums. (S.A.)

48. Two vessels holding 40 gals. each are both full, the one of two parts of wine mixed with three parts of water, the other of wine and water mixed in a different ratio. One-fourth of the liquid in each vessel is then transferred to the other. The contents of each vessel having been well mixed, one-third of each is again transferred to the other, after which it is found that there are $1\frac{1}{3}$ more gallons of water than of wine in the first vessel. What was the original mixture in the second vessel? (C.S.)

49. Find the value of the series $\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} +$, etc., to seven places of decimals. (I.I.)

50. The capacities of three vessels, A, B, and C, are 2 gals., 3 gals., and 5 gals. respectively. A is empty, B is filled with wine, C with water. The following operation is performed three times: A is filled from B, B

is replenished from C, and A is then emptied into C. How much wine and how much water are in B and C respectively at the end? (C.P.)

51. Under the old tariff a telegram of 20 words, exclusive of the names and addresses of the sender and recipient, could be sent for 1*s.*, and for each 5 words or fewer beyond 20 the charge was 3*d.* Under the new tariff the charge for each word, inclusive of names and addresses, is $\frac{1}{2}d.$ A certain telegram would have cost 15*d.* under either tariff, but if 9 words had been added to the body of the telegram (the addresses remaining the same) the cost under the new tariff would have been $1\frac{1}{2}d.$ greater than under the old. Find, by a clear process of reasoning, what was the number of words in the body of the telegram. (C.S.)

52. Two trains run in opposite directions with equal speeds round a circle, which they traverse in 70 min. Ten minutes after a meeting of the trains, the steam-pressure in one falls so that half its speed is lost. When and where does the next meeting occur? (O. and C.H.C.)

53. On a circular railway connecting three towns, A, B, C, two trains are constantly running in opposite directions. Every four hours they leave A simultaneously, and, travelling uniformly at the same rate, cover the distances A–B and B–C in 65 min. and 75 min. respectively. If the trains pass at a point which divides the distance from B to C in the ratio of 7 : 5, and stop the same time at each station, determine what this time must be, and also the time occupied in traversing the distance between C and A. (C.S.)

54. The diameters of four spheres are to one another as 3·75 : 5 : 6·25 : 7·5; and the volumes of spheres vary as the cubes of their diameters. Prove that the greatest of these four spheres is equal to the other three together. (C.S.)

55. Simplify $\frac{7}{13}\left(\frac{\sqrt{2-\frac{3}{5}}\times\sqrt{3+\frac{4}{7}}}{3\sqrt{2-\frac{3}{5}}\div\sqrt{3+\frac{4}{7}}}\right)$. (C.S.)

56. The cyclometer of a bicycle wheel 28 in. in diameter registers $3\frac{1}{10}$ mi. in a distance of 3 mi. If the error is due to the tyre not being pumped up tight, find how much the diameter falls short of 28 in. (S.A.)

57. Four candles of different diameters are arranged in sockets at different levels, and are lighted together; but when 1 in. of the longest candle has burned away, the flames are seen to be precisely on a level. One candle is 7 in. long above the socket, and would be consumed in 3 hrs.; the rest in succession are $6\frac{1}{2}$, 6, and $5\frac{1}{2}$ in. long, and would burn away respectively in $3\frac{1}{2}$, 4, and $4\frac{1}{2}$ hrs. What are the differences of level between the sockets? (C.S.)

58. Find what weights of the series 1 lb., 2 lbs., 4 lbs., 8 lbs., etc., must be used to weigh 1719 lbs. (C.S.)

59. A rectangular column of stone of square section is to be worked down to a cylindrical shape. If the final diameter be less than the side of the square by $\frac{1}{50}$ part, find by what fraction the weight of the column will be reduced. (The area of a circle is ·7854 of that of a circumscribed square.) Also calculate the loss of weight to the nearest ounce, if the original weight be $2\frac{1}{2}$ tons. (L.M.)

60. A cart-wheel 4 ft. 6 in. in diameter rolls, without slipping, through half a revolution. Find the horizontal distances traversed by the centre of the wheel and by the two points of the rim upon a horizontal diameter at the time when the wheel was set rolling. (The circumference

of a wheel is $\frac{22}{7}$ times the diameter.) Is there anything noticeable about the *ratios* of these distances? (C.S.)

61. The number of seconds in the beat of a pendulum may be obtained approximately by multiplying by ·55368 the square root of its length expressed in feet. Find, correct to $\frac{1}{100}$ in., the length of a pendulum whose beat is 1 sec. (P.E.A.)

62. A piece of gold of unknown fineness is mixed with double its weight of standard gold (22 carat), and to this mixture is added 1 lb. of copper, the result being 18-carat gold. Had 9 lbs. of copper been added instead of 1 lb., the result would have been 10-carat gold. What were the weight and fineness of the first-mentioned piece of gold? (N.B.—Standard gold contains 22 parts, by weight, of pure gold and 2 parts of copper; the fineness of gold is denoted (in carats) by the number of parts of pure gold out of 24.) (C.S.)

63. Calculate to 2 significant figures the number of square yards in a rectangular area 87 mi. long and 63 mi. wide. Write your result in words. (S.A.)

64. If the volume of a right cylinder is obtained by multiplying its height by $3\frac{1}{7}$ times the square of the radius of its circular end, find (using duodecimal notation) how many cylinders of copper, each $6\frac{6}{12}$ in. long and $2\frac{1}{2}$ in. in diameter, are equal in value to five cubes of iron, each of whose sides is 2 ft. 9 in. long, 1 cub. ft. of copper being worth $9\frac{3}{5}$ cub. ft. of iron. (C.S.)

65. Two fog signals are placed on a railway at distances of 1 mi. and $1\frac{1}{2}$ mi. from a station, and are exploded by a train coming at a uniform speed towards the station. If the second report is heard at the station 1 min. after the first, in how long a time after that will the train pass, and at what rate is the train moving, sound being assumed to travel at the rate of 1125 ft. per second? (C.P.)

66. Find in inches and the decimal of an inch the length at 60° Fahr. of a column of mercury which at 32° is ·76 metre in length, supposing the cross-section of the column to be invariable, and mercury to expand ·00018 of its bulk for each degree Centigrade; a metre being 39·37079 in., and 5° Cent. = 9° Fahr. (C.S.)

67. In making gold thread, a cylinder of silver weighing 7 lbs. avoirdupois, cased with 3 oz. avoirdupois of gold, is drawn through holes continually diminishing in diameter until 200 ft. of the thread weighs only 1 dr. Find the entire length of the thread in miles, etc. (I.I.)

68. Out of a circular disc of metal 35 equal circular holes are punched. The weight of the metal thus punched out is to the weight of the perforated disc as 45 : 67. Compare the diameters of the disc and of the holes. (The area of a circle varies as the square of the diameter.) (C.S.)

69. A man left a valley at 3 a.m. to climb to a mountain-top 5400 ft. above him, and to return by the same way. The way up consisted of three parts, (1) wood, (2) grass, (3) rock, their respective heights being 1500, 2700, 1200 ft. Exclusive of halts, the man's rates, in feet of height (or depth) per hour were thus: in the wood 1500 up, 3000 down; on the grass 1000 up, 1800 down; on the rock 500 (whether up or down). He rested, in all, $1\frac{1}{2}$ hr. At what hour did he regain the valley? (C.W.)

70. A wood is 3 fur. 8 ch. long by 2 fur. 8 ch. wide; the trees are in squares 5 ft. apart and 60 ft. high on the average. What fraction of the

wood must be cut down to make a fence that requires 5 times as many trees as placed end to end would go round the wood? (S.A.)

71. The area of a side of a cubical cistern is $12\frac{1}{4}$ ft.; it is furnished with 3 taps, one of which could fill the cistern in 10 min., the second could fill it in 12 min., and the third could empty it in 15 min. If all three be opened, the cistern being empty, find the weight of the water in the cistern at the end of 5 min. (1 cub. ft. = 1000 oz.) (C.S.)

72. A sub-contractor, who is unable to employ more than 9 hands, has undertaken to finish a piece of work by a given time. If he employed 6 men and 3 women, he would get the work done 5 days sooner than is necessary at a total cost of £52 10s.; if he employed 2 men and 7 women, the cost would only be £46 4s., but they would take 2 days too long over the work. Assuming that a man does half as much work again per day as a woman, what is the least sum for which the contractor can get the work done within the given time? (O.S.L.)

73. A mortar is in the form of a cube, 24 in. each way, with a cylindrical bore 18 in. in diameter perpendicular to one of its faces. The weight of the iron is estimated to be 7·5 times that of an equal volume of water, and a cubic foot of water to weigh 1000 oz.; and the mortar weighs 1 ton 2 cwt. 1 qr. 15 lbs. $7\frac{9}{16}$ oz. What is the depth of the bore? ($\pi = 3{\cdot}1416$.) (C.S.)

74. The following rule for finding the distance of the horizon is very nearly correct: Take the number of feet in the height of the station above the level of the sea, and increase it by half that number; the square root of this quantity will give the distance of the horizon in miles. Hence find to one place of decimals the distance of the horizon from the top of Kanchengjunga, whose height is 28156 ft. Find also the height of another mountain 350 mi. from Kanchengjunga, in order that its top may be just visible from that of the latter. (C.M.)

75. Express $(\cdot\dot{1})^2$ as a repeating decimal. (C.S.)

76. In leap year a number of women, of whom $\frac{1}{8}$ were widows, made offers of marriage; in consequence a number of men were married, of whom $\frac{1}{11}$ were widowers. Of the offers to widowers $\frac{1}{5}$ were declined, all the widows were accepted, and 1221 spinsters were refused by the bachelors; $\frac{35}{41}$ of the widows made offers to bachelors, and the number of bachelors who married spinsters was 7 times the number of widows who married bachelors. How many women made offers? (S.A.)

77. A pyramid on a square base, with altitude equal to $\frac{8}{9}$ of a side of the base, is of equal volume with a cube whose surface is 600 sq. ft. Find the area of the whole surface of the pyramid. (In the working assume $\sqrt{8425} = 91{\cdot}8$ approximately.) (C.S.)

78. If the volume of a gas at 41° Fahr. under a pressure of 15 lbs. to the square inch is 5 cub. ft., what will it be when the temperature is increased to 176° Fahr., and the pressure increased to 19 lbs. to the square inch? (It is given that the volume of a gas varies inversely as the pressure with an unchanged temperature, and that with the pressure unchanged the volume increases for each degree of temperature by $\frac{1}{465}$ part of the volume at 32° Fahr.) (C.H.L.)

79. In a hollow rectangular case of cast iron, the external dimensions of which are, length 14 ft. 4 in., breadth 2 ft., and height 2 ft., and the thickness of which is 2 in., are placed 4 equal hollow closed cylinders of steel, each 14 ft. long and 1 in. thick, and having an internal radius of

4 in. These cylinders are full of mercury, and the remaining spaces inside the case are full of water. If the area of a circle be $\frac{22}{7}$ × the square of its radius, and if cast iron be 7 times, steel 8 times, and mercury $13\frac{1}{2}$ times as heavy as water, and if a cubic foot of water be taken to weigh $62\frac{1}{2}$ lbs., find the weights of the water, the mercury, the steel, and the iron respectively, and their sum. (C.S.)

80. A piece of lead, in the shape of a peg-top (without the peg), consists of a hemisphere and a cone, the greatest length, 15·25 cms., being twice the greatest breadth. Calculate its weight to the nearest gramme, given that (1) the lead is 11·19 times as heavy as an equal volume of water; (2) the volume of a sphere is $\frac{4\pi}{3}$ × cube of radius; (3) the volume of a cone is $\frac{\pi}{3}$ × product of height and square of radius of base; (4) $\pi = 3{\cdot}1416$ approximately; (5) 1 cub. cm. of water weighs 1 grm. (C.S.)

81. A girl is told that 2 lbs. 2 oz. 15 dr. of sand is to be equally divided into 7 heaps, and that each heap is to be equally divided among 7 boxes; and she is asked to find how much sand there will be in each heap, and how much in each box. She divides accordingly, but neglects in each division the fraction of a dram. Of how much sand does she deprive each box? (P.T.)

82. In a division sum the remainder is equal to three times the quotient. If the quotient be now made the divisor, what will be the new remainder? And if 3071 were the original divisor, what will be the new quotient? Explain your answers. (C.S.)

83. A poulterer bought a number of geese, 3 times as many ducks, and 4 times as many fowls. Each goose cost 3*s.* 6*d.* more than a duck, and each fowl 6*d.* more than a duck. He sold each goose at a profit of 9*d.*, each duck at a profit of 4*d.*, and each fowl at a profit of 5*d.* Altogether they cost him £22, and his profit amounted to £2 14*s.* 8*d.* What price did he charge for a goose? (C.P.)

84. A boy playing at marbles lost in the first hour all he had, and got into debt for half a dozen more; in the second hour he gained double of all he had lost in the first, and 40 per cent. besides of the number he started with, which, after he had paid his debt, increased his store of marbles at the end of his play by 48. What number had he to start with? (P.T.)

85. Can you find any number which, if multiplied by 17, gives a result consisting only of threes? (C.S.)

86. The moon describes, relatively to the earth, a circle of 240000 mi. radius in $27\frac{1}{3}$ days. Find, to the nearest minute, the time she takes to travel over a space equal to her own diameter (2160 mi.), the ratio of the circumference of a circle to the diameter being taken as $3\frac{1}{7}$. (L.M.)

87. Multiply 268765 by 336567, the multiplication to be done in three lines. (C.M.)

88. A person has two casks, with wine in each. In order to have an equal quantity of wine in each, he pours out of the first, which contains the larger quantity, into the second as much as the second already contains; he next pours from the second into the first as much as the first then contains; and again pours from the first into the second as much as the second contains; each now has 16 gals. How much had each at first? (C.S.)

89. If the total annual yield of all the gold mines of the world is 4788000 oz., find the number of gold coins that could be coined from $\frac{1}{10}$ of the yield, so that 260 such coins can be made from $5\frac{1}{4}$ lbs. of gold. (K.S.)

90. The total yield of the wheat crop in the United Kingdom in 1888 was 74493133 bush., with an average yield of 27·97 bush. per acre. On what area (to the nearest acre) was wheat grown, and what was the value (to the nearest pound) of the wheat at 31*s.* 10*d.* per quarter? (O.P.)

91. If 1454170 ac. yield 31 bush. each at 60 lbs. per bushel, this is sufficient, allowing 2 bush. per acre for seed, to enable ·18 per cent. of the population to eat 120 4-lb. loaves per annum. Assuming that 280 lbs. of flour yield 360 lbs. of bread, what (within half a million) is the population? (S.A.)

92. If 3360000 people require 21000000 bush. of wheat and 700000 head of cattle for their yearly support, besides 660 lbs. of vegetables each, reckoning 3 ac. for each animal, 40 tons of vegetables or 30 bush. of wheat to an acre, find how many people may be fed from the produce of a square mile. (S.A.)

93. If the price of barley be 48*s.* per quarter, and the cost of malting a bushel of barley be $4\frac{1}{2}d.$, how much malt is made from 1242 qrs. when the maltster, after paying a tax of 3*s.* $1\frac{1}{2}d.$ per bushel of malt, makes 5 per cent. on his whole outlay by selling malt at 9*s.* 9*d.* per bushel? (C.S.)

94. A baker ordered from a mill a sack containing 10 st. $4\frac{1}{4}$ lbs. of flour. He intended to make this flour into 80 loaves. The sack left the mill at 12 hr. 10 min., but flour ran out of a hole in the sack at the rate of $\frac{2}{3}$ oz. per minute until 1 hr. 52 min., when the baker made what flour there was left into 80 loaves. How much flour (in pounds and ounces) was used for each loaf, and how much less for each loaf than the baker at first intended? (P.T.)

95. At an election 10 per cent. of the people on the voting list did not vote, and 60 votes recorded were rejected as illegal. The majority of the successful candidate was 308, and it was found that he had been supported by 47 per cent. of the whole number on the voting list. What was the number of legal votes recorded for each candidate? (O.J.)

96. A road passes through four towns, A, B, C, and D. From A to B is 9 mi., the road rising uniformly the whole way; from B to C there is a regular decline, the distance being 7 mi.; and from C to D there is another rise throughout the entire length of 10 mi. A man leaves A for D at 9 a.m., walking 3 mi. an hour uphill and 4 mi. an hour downhill. Another man leaves D for A at 9.30 a.m., and walks 4 mi. an hour uphill and 5 mi. an hour downhill. Find the time and place of their meeting. (C.S.)

97. A man spends a certain capital as follows: He buys 18000 ac. of uncultivated land, $\frac{1}{3}$ at 27*s.* an acre, $\frac{1}{3}$ at 33*s.* an acre, and $\frac{1}{3}$ at 38*s.*; he has the land cleared and ploughed at 8*s.* an acre, and builds farmhouses at a cost of £6400. He then leases 16000 ac. at a yearly rent of 7*s.* an acre, the farm buildings being let at the rate of 2*s.* 3*d.* per pound on their cost price; the remainder of the land he leases to a railway company at 8*s.* an acre. How much per cent. per annum does he make on the original outlay? (C.S.)

98. The price of land about London is every year rising 4 per cent. on previous prices, while the purchasing power of gold, as regards general

commodities, is falling $\frac{1}{2}$ per cent. per annum. A proprietor sells half his land this year for £8000, and invests the proceeds at 5 per cent. He proposes to sell the remainder 3 yrs. hence, and to invest the money in the same way. What income would now represent the purchasing power of the income he will have obtained when the sales are completed? (C.S.)

99. Prove (with the aid of a diagram, or in any arithmetical manner) that the difference of the squares of any two consecutive whole numbers is equal to the sum of the numbers. A sum of money is lent at compound interest, and the increase in its amount for two years which would result from an increase of 1 in the rate per cent. is 6*s.* 8*d.* more than the decrease would have been if the initial rate had been 1 less. Determine the sum, and show that the rate cannot be determined. (C.S.)

100. The assets of a bankrupt consist of (*a*) property valued at £250; (*b*) an undiscounted bill of £430 due 9 mths. hence, simple interest being reckoned at $4\frac{1}{2}$ per cent.; and (*c*) a sum of £320 invested at $96\frac{1}{2}$ in Government stock, which is now selling at 102. After deducting $\frac{1}{8}$ per cent. for brokerage in the last item, the bankrupt is able to pay his creditors 13*s.* 4*d.* in the £. What is the amount of his debts? (I.I.)

101. Two posts are offered at the same initial salary. In one the salary increases at the end of every year by 5 per cent. of that year's salary, in the other it increases at the end of every two years by 10 per cent. of the salary for either of those years. Which is the better paid post? Show that for the fifth year one salary is slightly more than $\frac{5}{11}$ per cent. better than the other. (C.S.)

102. A merchant who sold his goods at a profit of 10 per cent. found that, when he allowed $2\frac{1}{2}$ per cent. discount off his selling price, his business increased by $\frac{1}{3}$. Find whether his total profits were increased or diminished by adopting this plan, and in what proportion. (L.M.)

103. A sum of £6375 is to be divided between A and B, who are respectively 18 and 19 yrs. old, in such a way that, if their portions be invested at 4 per cent. per annum compound interest, they shall receive equal amounts on reaching 21 yrs. of age. How must the sum be now divided, and how much will each receive when 21 yrs. old? (I.I.)

104. The sum of £405 11*s.* 4*d.* is lent upon condition that the borrower shall owe to the lender, as interest, $\frac{1}{16}$ part of the sum borrowed for each half-year that he retains the money; and, if the money be not returned at the end of the first or any subsequent half-year, the interest due, instead of being paid, is to be added to the amount lent. How much is due from the borrower to the lender at the end of $1\frac{1}{2}$ yrs.? (L.M.)

105. A and B are twins, aged 26 yrs. A determines to invest £25 on his 27th and each succeeding birthday at 4 per cent. simple interest; B at the same time insures his life for a certain sum, paying an annual premium of £2, being $1\frac{1}{3}$ per cent. on the sum assured. Compare the amount which A and B respectively will have to bequeath, dying at the commencement of the 33rd year after A has made his sixth investment. (C.W.)

106. The weights of a penny, a halfpenny, and a farthing are in the ratio of 10 : 6 : 3, and their diameters are as 6 : 5 : 4. Find the ratio of their thicknesses, and find the smallest number of pennies which, piled on each other, will be equal in height to a pile made up of an equal number of halfpennies and farthings? (S.A.)

107. If A owes B £5000, payable at the end of a year, and an equal sum payable at the end of three years, and if he offers instead to pay him the whole at the end of two years, does B gain or lose by accepting the offer? If interest is calculated at $2\frac{1}{2}$ per cent. per annum, what payment at the end of the second year would be exactly equivalent to the original debt? (C.H.L.)

108. The value of a certain length of material A is $\frac{5}{6}$ of the value of the same length of another material B, and the weight of 17 yds. 2 ft. of A is $\frac{3}{7}$ of the weight of 15 yds. 1 ft. 9 in. of B. If the value of 3 cwt. 27 lbs. of A be £35 2*s.*, what is the value of 1 cwt. 2 qrs. 12 lbs. of B? (C.S.)

109. If 20 English navvies, each earning 3*s.* 6*d.* a day, can do the same piece of work in 15 days that it takes 28 foreign workmen, each earning three francs a day, to complete in 20 days; taking the value of the franc at 10*d.*, determine which class of workmen it is most profitable to employ. If a piece of work done by navvies cost £3000, what would be the cost of the same work done by foreign workmen? (R.M.A.)

110. A farmer had to keep 10 horses, 40 cows, and 100 sheep for 60 days. He had fodder sufficient to keep 300 sheep for 50 days. At the end of 40 days 5 cows died, and after this, when fodder was 20 per cent. dearer than at the beginning of the period, he had to expend £24 in providing for the cattle. If he had bought the requisite additional fodder before the price was increased, and the cows had not died, he would have had to pay the same sum. Assuming that a horse eats as much as 3 sheep, compare the quantities eaten in a day by a cow and a sheep. (O.S.L.)

111. In an orchard there are twice as many cherry trees as pear trees, as many plum as cherry and pear together, as many apple as the sum of the others, and there are 60 trees in all. When a pear tree produces 19*s.*, a plum tree 18*s.*, and an apple tree £1, the whole produce being £60, what do the cherry trees contribute? (C.S.)

112. A banker invests $\frac{1}{2}$ the money deposited by his clients in 4 per cent. preference stock at 150, and lends $\frac{1}{4}$ of it on mortgages at 5 per cent., keeping the remainder in cash at the bank. After paying 2 per cent. interest on deposits, he has an average annual balance of £1000 in his favour. Find the average amount of the deposits to the nearest pound.
(O. and C.H.C.)

113. If the number of different editions has increased 21 per cent., and the total number of books sold increased 19 per cent., while the cost of production of an edition has decreased 8 per cent., find the ratio of the receipts to the cost of production of an edition which formerly just paid its cost. (S.A.)

114. A merchant bought a number of eggs, of which 25 per cent. were broken in transit, and the remainder were sold at 1*s.* 4*d.* per dozen; the expenses were 10 per cent. of the receipts, and the net gain on the transaction £5. He calculated that if 10 per cent. of the eggs had been broken, and his expenses had been $16\frac{2}{3}$ per cent. of his receipts, his profit would have been $33\frac{1}{3}$ per cent. How many eggs were bought, and what sum did they cost? (O.S.L.)

115. If the duty on two kinds of foreign wine be changed from 3*s.* 6*d.* and 2*s.* per gallon respectively to a uniform rate of 2*s.* 6*d.*, and in consequence the consumption of the former, which was at the rate of

200000 gals. per annum, is increased 15 per cent., and that of the latter, which was 700000 gals., is diminished 25 per cent., while the cost of collecting the duty is reduced from 2*d.* to 1½*d.* per gallon, find the gain or loss to the revenue. (C.S.)

116. The national debt of a country was increased by ⅓ during a war. For a period of 10 yrs. after peace supervened £2000000 of the debt was annually paid off. At the end of the tenth year the rate of interest upon the debt was reduced from 6 to 5 per cent., and it was found that the interest became precisely the same as it was before the war. Find how much was added to the debt during the war. (L.M.)

117. A man in his will bequeathed £1000 to his widow, and £1500 in legacies to various institutions, and directed that ⅓ of the residue of his property should go to his widow, and that the remaining ⅔ should be equally divided amongst his 3 children. If the widow received altogether £4000 more than each child, what was the value of the property which the man left? (C.S.L.)

118. In an orphan asylum for a certain quarter of the year, in which the average number of boys was 104, and that of girls 72, the whole cost of keep was £669 12*s.*, while for a single month, in which the number of boys was 91, and that of girls 60, the cost was £191 17*s.* What is the ratio of the cost of a girl to that of a boy? (C.H.L.)

119. The number of paupers in a certain union was 336, the number of women being double that of the men, and the children being as many as the men and women together. If a man cost ⅓ more than a woman, and 3 children as much as a man and woman together, and the whole cost for the month be £83 6*s.*, how much is the daily cost of each man, woman, and child? (C.S.)

120. An exhibition has 17525 visitors. Of these ⅓ hold season tickets, each costing a certain number of half-sovereigns. Of the remainder ⅓ purchase day tickets, costing a like number of half-crowns, and the rest pay the same number of shillings for their evening tickets. The proceeds of the exhibition are £7991 8*s.* What were the three entrance charges? (C.P.)

121. A tradesman commences business with a certain capital. At the end of the first year he adds ⅙ to it; at the end of the second year he adds to it ⅓ of his new capital; at the end of the third year ⅛ of the capital with which he began that year; and at the end of the fourth year ⅙ of his last new capital. He then finds he has in his business £2737. What did he commence with? (C.S.)

122. A contractor employs a fixed number of men to complete a work. He may employ either of two kinds of workmen: the first at 26*s.* 6*d.* per week each, the second at 18*s.* 6*d.* per week each; the work of one of the former being to that of the latter as 5 to 4. If he finishes it as quickly as possible, he spends £270 more than he would have done if he had finished it as cheaply as possible, but takes 4 wks. less time. What would it have cost if he had employed equal numbers of the two kinds of workmen? (C.S.)

123. An income-tax levied at 3*d.* in the £ on the whole incomes of 400000 householders produces £2500000. If the tax be altered so as to produce the same amount when it is only levied on the excess of each

income over £200, what is the largest income the possessor of which will benefit by the change? (C.S.)

124. A building, which will let when finished for £7500 per annum, takes eight months to build by day labour alone; but if a second set of workmen be employed by night, it can be finished in four months. It is found that, although night labour costs 60 per cent. more than day labour, a net gain of £400 would result from employing the two sets of men. Find the total cost of labour in each case. (C.H.L.)

125. Suppose that when an income is less than £400, income-tax is not chargeable upon the first £120. A man having £13300 in the 3 per cents. sells out £300 at par, and invests it in a mortgage returning 4 per cent. His net income is now 1*s.* 6*d.* less than formerly. What is the amount of the income-tax in the pound? (C.S.)

126. A contractor agrees to sink an artesian well on the following sums: £5 per fathom for the first 30 fathoms (consisting of gravel only), £35 per fathom through the sand, £20 per fathom through clay, and £25 through chalk. He found the thickness of the chalk to be equal to half the depth of the well, and that the sand beds were three times as thick as the clay. At the end of the work he got £2700 for his trouble. What was the depth of the well? (C.S., C.M.)

127. A bought lambs at a certain price per lamb, and sold $\frac{1}{3}$ of them to B, and the rest to C. His actual profit on those sold to C was three times as great as that made on those sold to B, and his whole profit was at the rate of $10\frac{2}{3}$ per cent. The price paid per lamb by C exceeded that paid by B by 1*s.* How much did A pay for one lamb? (C.P.)

128. Three sums of money were divided between A and B; the first sum in the ratio of 7 to 13, and the second in the ratio of 13 to 23. The third sum was equal to the other two sums together, and B's share, in each case, exceeded A's share by the same amount. In what ratio was the third sum divided between A and B? (C.P.)

129. A candle, P, was lighted at 6.6 p.m., and a candle, Q, at 6.30 p.m. They were of the same length at 8.30 p.m.; also P was 1 in. long at 10.15 p.m., and Q 1 in. long at 11 p.m. Assuming that they burned away uniformly, find their original lengths, which differed by 2 in. (C.P.)

130. How many books 7 in. by 5 in. by $\frac{3}{4}$ in. can be packed in a box 2 ft. by 1 ft. by 1 ft.? and what will be the volume of the unoccupied space? (S.A.)

131. A's income was to B's income as 5 to 4. A's income was increased 5 per cent., and B's income increased at a different rate per cent. Later on their incomes were increased at the same rates, when A's income was to B's income as 1125 to 1024. At what rate per cent. was B's income increased? (C.P.)

132. A builder borrows money from three persons, A, B, C; from B he borrows £600 more than he does from A, and from C he borrows half as much again as he does from B. He agrees to pay interest to A at the rate of 10 per cent. per annum, to B at 5 per cent., and to C at 1 per cent. At the end of twelve months he finds that he owes them altogether exactly £14000. How much did he borrow from each? (O.S.L.)

133. £18, given to a school for prizes, is distributed among forms I., II., III. in the ratios 3 : 4 : 5. The number of prize-winners in forms I., II., III. are as 5 : 4 : 3. Find the total number of prize-winners, supposing that all in the same form receive the same sum, and as many prizes as

possible are given, consistent with each being an integral number of shillings. (S.A.)

134. A sum of £5000 is lent on the condition that if it is not repaid within a year compound interest will be charged. The option is given of 5 per cent. interest throughout, or $2\frac{1}{2}$ per cent. for the first year, 5 per cent. for the second year, $7\frac{1}{2}$ per cent. for the third year, and so on, the interest increasing by $2\frac{1}{2}$ per cent. every year. Show that, if the loan is repaid within 3 yrs., the latter system will be in favour of the borrower, and find the difference of the two systems if it be not paid off till the end of the fourth year. (C.S.)

135. After probate duty of £88 and lawyer's bill of £20 6*s.* 8*d.* are deducted from a personal estate of £3441 13*s.* 4*d.*, the remainder is divided into five equal parts: one to a widow, free of legacy duty; one to a sister, which pays 1 per cent. duty; and one to each of three nieces, paying 3 per cent. If the above probate and legacy duties be commuted for a probate duty of $4\frac{1}{2}$ per cent., and the lawyer's bill be thereby reduced to £3 15*s.* 10*d.*, how much ought to have been left to each legatee so that the net sums received should be in exactly the same proportion as before? (C.S.L.)

136. The sum of £405 11*s.* 4*d.* is lent upon condition that the borrower shall owe to the lender, as interest, $\frac{1}{40}$ part of the sum borrowed for each half-year that he retains the money; and, if the money be not returned at the end of the first or any subsequent half-year, the interest due, instead of being paid, is to be added to the amount lent. How much is due from the borrower to the lender at the end of $1\frac{1}{2}$ yr.? (L.M.)

137. If a dealer were to diminish the selling price of his wares 10 per cent., and thereby double his sale, he could make the same profit on the cost price as before. In what ratio would his profit be diminished if he were to increase the selling price by the same amount and thereby halve his sale? (S.A.)

138. A ship 78 mi. from the shore springs a leak which admits $2\frac{1}{3}$ tons of water in $6\frac{1}{2}$ min.; 68 tons would suffice to sink her, but the pumps can throw out 15 tons in an hour. Find the average rate of sailing that she may just reach the shore as she begins to sink? (C.M.)

139. A steamer A is in distress and stationary, and fires a gun which is heard on another steamer B coming direct towards her. A fires another gun 10 min. after the first, and this is heard on B 9 min. 48 sec. after the first. At this moment B is 4 mi. from A. How soon will the two steamers be alongside of each other? Give the answer to the nearest second, supposing sound to travel at the rate of 1130 ft. per second. (R.M.A.)

140. Using the theorem that the difference of the squares of any two numbers is equal to the product of the sum and difference of the numbers, find the difference between the sum of the squares of all the odd numbers, and the sum of the squares of all the even numbers up to 100. (C.S.)

141. A vessel, making for a harbour, fires a signal gun; the flash is seen from the harbour, and the sound follows in $22\frac{1}{2}$ sec. A tug puts off immediately, and steams in a straight course towards the vessel at the rate of 12 mi. an hour; and from the tug, 5 min. afterwards, the flash of a second gun is seen, the sound of which follows in 15 sec. If sound

travels 13 mi. per minute, at what rate is the vessel approaching the harbour, and how soon after starting will the tug meet her? (C.S.)

142. A ship sailing at the rate of 10 mi. an hour is observed to move $\frac{1}{4}$ mi. in the interval between the flash of a gun fired on board being seen, and the report heard on shore. The velocity of sound being 1140 ft. per second, and the flash being assumed to be seen instantaneously, how far is the ship from land? (C.S.)

143. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. a head. After being at sea 20 days she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ lb. Find the original number of the crew. (L.M.)

144. A ship is steaming directly towards a headland, from which a gun is fired. The interval between the sound and flash is noted to be 10 sec. Immediately the sound is heard the engines are reversed, and the ship becomes stationary after travelling 1070 ft. from the time that the flash is seen. The gun is then fired again, and the interval between the sound and flash is found to be $9\frac{3}{10}$ sec. Find the velocity of the ship at the time when the engines were reversed. (The velocity of sound is 1100 ft. per second; the velocity of light is practically infinite.) (C.S.)

145. A ship's company take a prize of £1001 19*s.* 2*d.*, which is divided according to their pay and time on board. The officers and midshipmen have been on board 6 mths., the sailors 3 mths.; the officers receive 40*s.*, the midshipmen 30*s.*, the sailors 22*s.* per month. There are 4 officers, 12 midshipmen, and 110 sailors. What will be each man's share?

146. A man leaves his estate to be divided among his 3 sons and 2 daughters in the following way: the eldest son is to have $\frac{1}{3}$ of the property, the second son a sum equal to $\frac{2}{3}$ of the elder's share, and the third a sum equal to $\frac{1}{2}$ of the second's. The elder daughter's share is to be equal to $\frac{2}{3}$ that of the second and third sons together, and the younger is to have the remainder. It is found by this division that the younger daughter will receive £520. Find the value of the estate, and what each of the others would receive. (C.S.)

147. A wine merchant buys two casks of whiskey, which differ in size and in the quality of their contents. He intends to sell the whiskey by the gallon, so as to make a profit of 20 per cent. on each cask, in which case the price of the better whiskey would have been 1*s.* 6*d.* per gallon more than that of the other; but by mistake the prices are interchanged, with the result that he receives £1 11*s.* 6*d.* less than he expected, and only makes a profit of 16 per cent. on the transaction. If the two casks together contain 59 gals., what was the cost of the whiskey in each cask? (O.S.L.)

148. Genuine milk contains 88·75 per cent. by bulk of water, 2·75 of fat, and 8·5 of non-fatty solids. A purchaser buys 7 gals. of milk at 3*d.* a quart; the milk on being analyzed is found to contain 90·84 per cent. of water, 2·24 of fat, the residue being non-fatty solids. Find whether anything besides water has been added to it; and find the sum of which the purchaser has been defrauded. (L.M.)

149. Zinc ore is purchased at £8 11*s.* per ton, and copper ore at £17 11*s.* per ton; 100 parts of the zinc ore contain 54 parts of metallic

zinc, and 100 parts of the copper ore contain 27 parts of metallic copper; 5 per cent. of the zinc and $2\frac{1}{2}$ per cent. of copper are lost in the smelting. If brass be made by mixing the zinc and copper in the proportion of 8 parts to 7, and the coals and labour employed in the brassfounding cost $\frac{9}{10}$ as much as the metal, what will be the net cost per ton of the brass? (C.S.)

150. An ore contains $4\frac{1}{2}$ dwts. of gold and $1\frac{1}{2}$ oz. of silver per ton, and $6\frac{3}{4}$ per cent. of copper. Allowing £2 per ton for expenses, taking gold at £3 15*s.*, silver at 2*s.* 1*d.* per ounce, and 9 cub. ft. of ore to the ton, the value of 1320000 cub. ft. of ore is £400000. What is the price of a ton of copper? (S.A.)

151. Three casks contain diluted spirits. From the first are taken 3 parts, from the second 5 parts, and from the third 7 parts, and on mixing these together it is found that they contain twice as much spirit as water. If the first part contains 7 parts of water to 11 of spirit, and the second 5 parts of water to 7 of spirit, find the percentage composition of a mixture formed by taking 3 parts from the first cask, 2 from the second, and 1 from the third. (C.S.)

152. It appeared from a daily paper one year that the annual expenses of the Suez Canal might be estimated at about 17800000 fr., exclusive of 10000000 fr. needed for the payment of the interest on the shares. Assuming that all the receipts over and above the sum of these amounts are to be divided as profits among the shareholders, the founders, and the Egyptian Government, the holder of shares taking 70 per cent., and each of the latter parties 15 per cent., what must the yearly receipts be in order that an English holder of 20 shares out of the 400000 representing the whole stock may receive £73 10*s.* as his proportion of the profits? (£1 = 25 fr.) (C.S.)

153. The specific gravities or relative weights of proof-spirit and water may be taken as 923 and 1000, disregarding temperature. From a gallon jar filled with proof-spirit a certain quantity is drawn off, and water added till the vessel is again full. The mixture is then found to have a specific gravity of 992. How much spirit has been abstracted? (C.S.)

154. A druggist takes a fluid ounce of a drug, and dilutes it with 9 times its quantity of water, supposed to be pure; he takes an ounce of the mixture and dilutes it again in the same proportion, and repeats the operation a third time. He afterwards finds that the water which he supposed pure had been previously mixed with the same drug in the proportion of 99 : 1. What difference will this make in the strength of the mixture? (C.S.)

155. A person shooting at a target at a distance of 500 yds. hears the bullet strike the target 4 sec. after he fired. A spectator equally distant from the target and the shooting-point hears the shot strike $2\frac{1}{2}$ sec. after he heard the report. Find the velocity of sound. (C.PR.)

156. If the children born in the United Kingdom consist of males and females in the ratio of 81 to 80, while the population always consists of males and females in the ratio of 27 to 28, find the average length of life of all the females born, assuming that 45 yrs. is the average length of life of the males, and neglecting the effects of immigration and emigration. (C.S.L.)

157. An elastic string of length ·477 m. is fastened at its upper

end, and is stretched to a length of ·504 m. by a suspended weight of 18 gr. Find to what length a piece of the same string, a yard long, is stretched by a ½-oz. weight (having given that a gramme = ·0022 lb., and that the ratio of the increase of length to the original length is always proportional to the suspended weight). (C.H.L.)

158. Two boys begin together to write out a book of Virgil's "Æneid," containing 817 lines. The first boy starts with the first line, writing at the rate of 200 lines an hour; the second boy starts with the last line, then writes line 816, and so on backwards, proceeding at the rate of 150 lines an hour. At what line will they meet? (P.E.A.)

159. A waggon leaves London for Oxford at 5.30 a.m. at the rate of 6 mi. per hour. A coach, whose rate is to that of the waggon as 5 to 3, leaves London some time after, also for Oxford, overtakes the waggon at 10.30 a.m. and proceeds to its destination, where, after remaining an hour, it commences its return journey, and meets the waggon 3¾ mi. from Oxford. Find the distance between London and Oxford, and the time (1) when the coach leaves London, (2) when it arrives in Oxford, (3) when it meets the waggon, (4) when it arrives in London, and (5) when the waggon arrives in Oxford.

160. Fifty cards, of which one is marked, are dealt round to form 5 heaps, and the heaps are then piled one on the other in such order that the heap containing the marked card is the middle one; the pack so made is dealt again in 5 heaps, and the heaps piled as before. If now the pack be dealt to 2 heaps, show that the marked card will be the thirteenth in the heap which contains it. (C.S.)

161. A bath can be filled by the cold-water pipe in 9 min., and by the hot-water pipe in 11¼ min. A person leaves the bath-room after turning on both pipes simultaneously, and returns at the moment when the bath should be full. Finding, however, that the waste-pipe has been open, he now closes it. In 3¾ min. the bath is full. In what time would the waste-pipe empty it? (C.S.L.)

162. The coach fare between two places is 5*s.*, and each passenger is allowed 56 lbs. of luggage free, excess luggage being paid for at a certain rate per pound. Two travellers, A and B, have more than 112 lbs. of luggage between them, but A has less than 56 lbs. If A alone had travelled, and taken with him the luggage of both, the coach proprietor would have received the same amount as he does when they both travel independently. If, however, they had divided their luggage equally between them, the proprietor would have received 2*s.* less. How much luggage had A? (C.J.)

163. In a certain household the consumption of coal in the winter half of the year was 50 per cent. more than in the summer half, and the price per ton of the coal consumed in the winter half was 20 per cent. more than the price per ton of the coal consumed in the summer half. The total cost for the year was £25 4*s.* Show that 9*s.* would have been saved if all the coal had been bought at a price 10 per cent. above the summer price. (C.H.L.)

164. Thirty stones are lying in a straight line at intervals of 1 yd. from one another. A man starts from a basket 20 yds. from the nearest stone, and in the same straight line, to fetch the furthest stone and place it in the basket; then to fetch the next, and so on. If the average rate for each journey to any stone and back is proportional to the distance

between that stone and the basket, and the man takes $7\frac{1}{2}$ min. to accomplish the task, find how long each journey takes. (C.S.)

165. Working at a certain uniform rate, it was estimated that some men could perform the coaling of a vessel by the time it was to start, but when $\frac{3}{5}$ of the time had elapsed only $\frac{1}{4}$ of the coaling was done; hence 10 per cent. more men were set to work, and, all working at another uniform rate, they just got the coaling done in time. Compare the latter with the former rate. (C.P.)

166. After driving some distance on the way from one place to another, a gentleman observed that $\frac{1}{3}$ of the distance he had driven was $\frac{3}{8}$ of the distance he then had to drive; also that, when he had driven $4\frac{1}{2}$ mi. farther, $\frac{1}{2}$ of his journey would be accomplished. How many miles had he to drive when he made the observation? (C.P.)

167. A section of a stream is 10 ft. wide and 10 in. deep; the mean flow of the water through the section is 3 mi. an hour. Taking 25 gals. to equal 4 cub. ft., find how many gallons of water flow through the section in 24 hrs. (S.K.)

168. If a man, his wife, and child consume 36 lbs. of bread in 9 days, when bread is at 5*d.* the 4-lb. loaf, how much bread a day would a family, consisting of a man and wife and 3 children, have to restrict each member to so as not to exceed the same rate of expenditure when the 4-lb. loaf costs 6*d.*? In each case assume that the rate of consumption of a man, woman, and child are as 4, 3, and 2. (C.S.)

169. A man went to market with £200. He bought 200 animals, consisting of sheep, pigs, and young geese. For each sheep he gave £5, for each pig £1, and for each goose 1*s.* How many of each did he buy? (C.W.)

170. When the duty on a commodity is reduced by 40 per cent., the consumption is increased by 60 per cent. By how much per cent. is the revenue from that commodity increased or diminished? With what increase of consumption would the revenue have remained unchanged? (L.M.)

171. On a certain farm the labour in one year cost £750, the rent was £350, other fixed expenses amounted to £550; the returns that year were 10 per cent. less than the outgoings. The landlord remitted 10 per cent. of that year's rent, and lowered the rent £50 per annum. The returns for the second year were better than those of the year before in the proportion of 7 : 4. On the two years the farmer gained $23\frac{3}{4}$ per cent. What was his labour bill in the second year? (C.H.L.)

172. A man buys a cottage for £150. He agrees to sell it for twelve quarterly payments of £15 each, the first payment to be due three months after he paid for the cottage. He also agrees to lend the aforesaid sums, as soon as he receives them, to a tradesman, who promises to repay all the money thus borrowed, together with simple interest at the rate of 4 per cent. per annum, on the day that the last quarterly payment becomes due. What rate of simple interest will the man receive for his £150? (C.S.L.)

173. A company of men can be formed either into a solid square, or into a hollow square 9 deep, having 970 men in the front rank of each side. How many men are there in the company, and how many in each side of the solid square? (O. and C.S.C.)

174. What must be the area to the nearest rood of the surface of the water in a reservoir with vertical boundaries, so that when no water

enters the reservoir the withdrawal of one million gallons per day may lessen the depth by one foot in a week. (A gallon contains 277·274 in.)
(S.A.)

175. A boy had to take a parcel to the middle house in a certain row. He assumed that there were 333 houses in the row, and that they were numbered 1, 2, 3, etc. He found No. 1, and followed the row till he reached what, calculating from the number-plate on its door, he took to be the middle house. The houses, however, were numbered 1, 3, 5, 7, etc., and there were 339 houses in the row. How many doors further had the boy to go? (K.S.)

176. A sells gold lace costing £375 per piece to B at £518 15*s.* per piece, and gives him 9 mths.' credit. At the same time B sells silk to A which has cost him £125 per piece, and gives him 6 mths.' credit. If the two invoices are made out for the same amount, at what price per piece must B invoice the silk, in order that at the moment of sale the total amount of A's profit may equal that of B, the present value of the invoices being determined upon the supposition that the interest of money is 5 per cent. per annum? (O.S.)

177. The first of a line of 10 sentries standing at equal distances from each other wishes to send a message of 20 words to the last. The men begin to walk at the same instant, and walk even distances right and left of their posts alternately, and each time they meet pass on 5 words. The transmission of the message occupies 48 min., and the men walk at the rate of 3 mi. per hour. How far is the first man from the last? (O.S.)

178. In climbing a mountain 14000 ft. high, it takes $\frac{1}{2}$ hr. to climb the first 1000 ft.; it takes $\frac{1}{6}$ as long again to climb the second 1000 ft. as it does to climb the first 1000; it takes $\frac{1}{6}$ as long again to climb the third 1000 as it does to climb the second 1000; and so on until a height of 11000 ft. is reached. After this it takes $\frac{1}{2}$ hr. longer to climb each 1000 ft. than it does to climb the preceding 1000 ft. How long does it take to climb the mountain? (L.M.)

179. On Monday the turnstiles recorded the entrance into the exhibition of 58682 persons. The money taken (in shillings) consisted of a number of pounds and 7*s.* over, and it was observed that the number of pounds was less by 295 than the number of persons who entered with season tickets. Find the number of persons who entered respectively by payment and by season tickets. (L.M.)

180. Sound travels at the rate of 1140 ft. a second. If a shot be fired from a ship moving at the rate of 10 mi. an hour, how far will the ship have moved before the report is heard at a place $14\frac{1}{4}$ mi. off? (C.PR.)

181. Tramcars start uphill from A to B (a distance of 10 mi.), and return downhill from B to A, at intervals of $7\frac{1}{2}$ min. in each case. The cars running in the same direction travel at a uniform rate, but their speed from B to A is greater than from A to B. A cyclist riding at a uniform speed from A to B passes tramcars at the rate of 8 in 3 hrs., and meets them at the rate of 30 in $1\frac{3}{4}$ hr. He returns at a uniform but different speed of 15 mi. an hour, and completes the whole journey from A to B and back in 115 min. Will the distance between the points at which he *passes* each car be greater on the outward or on the return journey, and by how much? Answer the same question in regard to the cars which he meets. (O.S.)

182. A cistern is filled by two taps, A and B, together in 1 hr. 12 min. For the tap A another tap C is substituted, which by itself would take 5 hrs. longer than A to fill the cistern; and for the tap B another tap D is substituted, which by itself would take 6 hrs. longer than B to fill the cistern. If C and D could together fill the cistern in 3 hrs. 56 min. 15 sec., find how long it would take to fill the cistern by A and by B separately. (L.M.)

183. A and B receive 1600 cabbages to plant, but of these 10 prove worthless. A plants 5 while B plants 4, and the rows are equal. After some hours they have not sufficient between them to complete another row, A having 45 to spare, and B 6. A then finds he has planted 3 rows more than B. How many did each plant, and how many cabbages in a row? (C.S.)

184. Ash saplings after 5 yrs.' growth are worth 1*s.* 3*d.*, and increase in value 1*s.* 3*d.* each year afterwards. For their growth they require each twice as many square yards as the number of years they are intended to grow before cutting. A plantation is arranged so that each year the same number may be ready for cutting. Find the greatest annual income which can be obtained per acre, allowing $\frac{1}{5}$ of the gross receipts for expenses.

185. A person set to calculate the decimal equivalent of $\frac{1390000}{837247}$ inadvertently transposes the 2 and 4 in the divisor, and in his third line of subtraction writes 20 . . . where he should write 19 . . . If he makes no other error, show (by the simplest approximations that will yield the result) that his answer will fall short of the true answer by more than ·00001 and less than ·0003. (C.S.)

186. A mother and two children start for a long voyage. At starting their united ages amount to 35 yrs., and were in the ratios of 365 : 39 : 25. On arriving the ages of the mother and the eldest child were in the ratio of 728 : 94. Find the length of the voyage, and the ratio of the youngest child's age to the mother's on arrival (reckoning 12 mths. to the year). (C.S.)

187. When a body is falling freely to the earth, the square of its velocity at any point of its fall is always proportional to the distance through which it has fallen. In falling through 16·1 ft. a body acquires a velocity of 32·2 ft. per second; and in falling through 2·503745 m. it acquires a velocity of 23 ft. per second. Find the number of feet in a metre, correct to 4 places of decimals. (C.H.L.)

188. Find by the contracted method the square root of 3121 correct to within a hundred-thousandth; and thereby obtain $(55)^2$, $(5586)^2$, and the first 7 figures of the square of 5586591. Show that $\sqrt{10}$ is less than $\frac{73}{22}\frac{1}{2}$ and greater than $\frac{1443}{1405}$; and that it differs from the latter by a quantity less than $\frac{1}{2 \times 228 \times 1405}$. (C.S.)

189. A piece of metal weighing 12 cwt. 60 lbs. has been formed by compounding three metals in quantities which, by measure, are as 5 : 3 : 2; but the weights of equal volumes of them would be as 7 : 11 : 13. What weight of each of the component metals has been used? (C.S.)

190. A crown weighing 64½ oz. was made of gold adulterated with silver. It was found by the water which it displaced to contain 8·5 cub. in. of metal; and a cubic inch of gold was found to weigh 10·25 oz., and of silver 5·75 oz. Find the weight of gold and silver in the crown. (C.S.)

191. Compare the weight of a solid brick of which the dimensions are 9 in., $4\frac{1}{2}$ in., and $2\frac{1}{4}$ in., with that of a brick of the same dimensions and of the same material, but perforated perpendicularly to the largest faces in 8 places, the perforations being circular and $\frac{3}{4}$ in. in diameter. (C.S.)

192. Any one of a number of men can dig 20 ft. (in length) of trench for the foundation of a wall, *or* build 9 ft. (in length) of the wall, *or* fill up 85 ft. of the trench after the wall is built, in a day of 10 hrs. Find how many out of 1058 men must be employed day by day in building the wall, so that the remainder may be just sufficient each day to dig for the next day's building, and to fill up after the preceding day's building the same length of trench. (C.S.)

193. A field of 70 ac. was rolled by a roller 11 ft. in circumference and 3 ft. 6 in. in width, and another field of 63 ac. was rolled by a roller 10 ft. 6 in. in circumference and 3 ft. 4 in. in width; the large roller revolved at the uniform rate of 1 revolution in 3 sec., and the smaller at the uniform rate of 9 revolutions in 25 sec. Assuming that all the ground was rolled only once, in how much less time did one roller do its work than the other? (C.S.)

194. The weight of a shilling being 87 grs., find the weight of a sovereign, the diameter of the sovereign being to that of a shilling as 57 to 60, and the thickness as 21 to 22; and gold being heavier than silver in the proportion of 32 to 19. (The area of a circle varies as the square of its diameter.) (C.S.)

195. The costs per ton of the two principal ingredients of a manufactured commodity are in the ratio 5 : 4, the quantities used are in the ratio 4 : 3; labour and other expenses are £6 per ton of the manufactured product, which is sold at a discount of 8 per cent. of the nominal selling price. The quantity sold is equal to that of the chief ingredient, the bye products being valueless. The expected profit is 25 per cent. But if the prices of the ingredients rise 10 per cent. and the selling price falls 10 per cent., the profit will be $3\frac{1}{2}$ per cent. What was the nominal selling price per ton? (S.A.)

196. Three points are kept moving at uniform rates of 5 ft. per second, 16 ft. in 3 sec., and 23 ft. in 4 sec. in concentric circles having diameters 32 in., 33 in., and $33\frac{3}{4}$ in. respectively. They start simultaneously from positions seen in the same direction from the centre, the last moving in the opposite direction to the other two. Find how long after the start they will next be seen in the same direction from the centre, and how many revolutions each will then have made. (C.S.)

197. The dimensions of a rectangular room are 25 ft. 4 in. by 18 ft. by 13 ft. 6 in. Find how many cubic feet of air must be supplied by the ventilating pipes in order that the whole atmosphere of the room may be renewed in an hour. Also find (in square inches) what must be the least total sectional area of the pipes in order that the current in them may not exceed $1\frac{1}{2}$ ft. per second. (L.M.)

198. The driving-wheel of a locomotive engine is $27\frac{1}{2}$ ft. in circumference, and the fore wheel 16 ft. Two particular spokes, one in each wheel, are observed pointing vertically upwards. How far will the engine travel before the same two spokes are again simultaneously in the same direction; and how often will this happen in 9 mi.? (C.S.)

199. A cask weighing 2 cwt. 12 lbs. 4 oz. floats in a cubical cistern

of water, whose side is 5 ft. On the removal of the cask, find how much the water will sink in the cistern, supposing a cubic foot of water to weigh 63 lbs. (C.S.)

200. Two similar logs of timber, each measuring 18 ft. in length and $8\frac{1}{8}$ in. square sectional area, were sent to be sawn up, each into 11 boards of equal thickness, to floor a room. A saw, which destroys $\frac{1}{8}$ in. of thickness in making its cut, was used, and in consequence there was a deficiency of flooring material. Had similar boards been cut with a saw which destroys only $\frac{5}{11}$ as much, there would have been a surplus of $22\frac{1}{2}$ sq. ft. What was the area of the room? (C.S.)

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